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TOP LOCAL COHOMOLOGY AND TOP FORMAL LOCAL COHOMOLOGY MODULES WITH SPECIFIED ATTACHED PRIMES

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Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring, M be a finitely generated R-module of dimension n and $\mathfrak{a}$ be an ideal of R. In this paper, generalizing the main results of Dibaei and Jafari [3] and Rezaei [8], we will show that if T is a subset of $\operatorname{Assh}_R M$, then there exists an ideal $\mathfrak{a}$ of R such that $\operatorname{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) = T$. As an application, we give some relationships between top local cohomology modules and top formal local cohomology modules.

1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, $\mathfrak{a}$ be an ideal of R and M be a finitely generated R-module of dimension n. For an R-module M, the i-th local cohomology module of M with respect to $\mathfrak{a}$ is defined as

$$\mathrm{H}^i_{\mathfrak{a}}(M) = \varinjlim_{n \geq 1} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

For the basic properties of local cohomology the reader can refer to [2]. Also, for each $i \geq 0$; $\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_t \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^t M)$ is called the i-th formal local cohomology module of M with respect to $\mathfrak{a}$. The formal local cohomology modules have been studied by several authors; see

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for example [1], [5] and [9]. Let M be a finitely generated R-module of dimension n, then $\mathrm{Max}\{i \in \mathbb{Z} : \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\} \leq n$ by [2, Theorem 6.1.2] and $\mathrm{Max}\{i \in \mathbb{Z} : \mathfrak{F}^i_{\mathfrak{a}}(M) \neq 0\} \leq n$ by [9, Theorem 4.5]. Recall that the module $\mathrm{H}^n_{\mathfrak{a}}(M)$ is called a top local cohomology module if $\mathrm{Max}\{i \in \mathbb{Z} : \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\} = n$ and the module $\mathfrak{F}^n_{\mathfrak{a}}(M)$ is called a top formal local cohomology module if $\mathrm{Max}\{i \in \mathbb{Z} : \mathfrak{F}^i_{\mathfrak{a}}(M) \neq 0\} = n$. For each Artinian R-module A, we denote by $\mathrm{Att}_R A$ the set of all attached prime ideals of A.

In section 2, we show that any subset T of $\mathrm{Assh}_R M$, where

$$\mathrm{Assh}_R M = \{\mathfrak{p} \in \mathrm{Ass}_R M \ : \ \dim(R/\mathfrak{p}) = \dim M\},$$

can be expressed as the set of attached primes of the top local cohomology module $\mathrm{H}^n_{\mathfrak{a}}(M)$ for some ideal $\mathfrak{a}$ of R. This generalizes a result of Dibaei and Jafari [3] to Noetherian local rings that are not necessarily complete.

We say that the top local cohomology module $\mathrm{H}^n_{\mathfrak{a}}(M)$ satisfies the property $(*)$, if

$$\mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) = \{\mathfrak{p} \in \mathrm{Ass}_R M \ : \ \dim(R/\mathfrak{p}) = n \text{ and } \sqrt{\mathfrak{a} + \mathfrak{p}} = \mathfrak{m}\}.$$

Rezaei in [8], showed that if $(R, \mathfrak{m})$ is a complete Noetherian local ring and M is a finitely generated R-module of dimension n then for each ideal $\mathfrak{a}$ of R there exists an ideal $\mathfrak{b}$ such that $\mathrm{H}^n_{\mathfrak{a}}(M) \cong \mathfrak{F}^n_{\mathfrak{b}}(M)$ and there exists an ideal $\mathfrak{c}$ such that $\mathfrak{F}^n_{\mathfrak{a}}(M) \cong \mathrm{H}^n_{\mathfrak{c}}(M)$. In section 3, we generalize this result. In fact, we show that over Noetherian local rings that are not necessarily complete, there exists an ideal $\mathfrak{c}$ such that $\mathfrak{F}^n_{\mathfrak{a}}(M) \cong \mathrm{H}^n_{\mathfrak{c}}(M)$ and if $\mathrm{H}^n_{\mathfrak{a}}(M)$ satisfies the property $(*)$ then there exists an ideal $\mathfrak{b}$ such that $\mathrm{H}^n_{\mathfrak{a}}(M) \cong \mathfrak{F}^n_{\mathfrak{b}}(M)$.

For any ideal $\mathfrak{a}$ of R, the radical of $\mathfrak{a}$, denoted by $\sqrt{\mathfrak{a}}$, is defined to be the set $\{x \in R : x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}$. Also, we denote $\{\mathfrak{p} \in \mathrm{Spec}\, R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $\mathrm{V}(\mathfrak{a})$ and $\mathrm{Min}\,\mathrm{V}(\mathfrak{a})$ by $\mathrm{Min}(\mathfrak{a})$. For an R-module M, we show the set of minimal members of associated primes of M by $\mathrm{mAss}_R(M)$. For any unexplained notation and terminology, we refer the reader to [2] and [6].

2. Top local cohomology modules with specified attached primes

In this section, we study the set of attached primes of top local cohomology modules.

Notation 2.1. Let $\mathfrak{a}$ be an ideal of R and M be a finitely generated R-module of dimension n. Let $0 = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R M} N(\mathfrak{p})$ be a reduced primary decomposition of the submodule 0 of M. Following [7], we set

$$\mathrm{Ass}_R(\mathfrak{a}, M) = \{\mathfrak{p} \in \mathrm{Ass}_R M \ : \ \dim(R/\mathfrak{p}) = n \textit{ and } \sqrt{\mathfrak{a}+\mathfrak{p}} = \mathfrak{m}\}.$$

Set $N^{\mathfrak{a}} = \bigcap_{\mathfrak{p}\in \mathrm{Ass}_R(\mathfrak{a},M)} N(\mathfrak{p})$. Note that $N^{\mathfrak{a}}$ does not depend on the choice of the reduced primary decomposition of zero because

$$\mathrm{Ass}_R(\mathfrak{a}, M) \subseteq \mathrm{mAss}_R M.$$

It is clear that $\mathrm{Ass}_R(\mathfrak{a}, M) = \mathrm{Ass}_R(M/N^{\mathfrak{a}})$ and

$$\mathrm{Ass}_R N^{\mathfrak{a}} = \mathrm{Ass}_R M \setminus \mathrm{Ass}_R(\mathfrak{a}, M).$$

For each integer $l \geq 0$ and any subset S of $\mathrm{Spec}\, R$ we define

$$S_l := \{\mathfrak{p} \in S : \dim(R/\mathfrak{p}) = l\}.$$

Lemma 2.2. *Let $N^{\mathfrak{a}}$ be defined as above. Then the following statements are equivalent:*

(i) $\mathrm{H}^n_{\mathfrak{a}}(N^{\mathfrak{a}}) = 0$*;*
(ii) $\mathrm{H}^n_{\mathfrak{a}}(M) \cong \mathrm{H}^n_{\mathfrak{a}}(M/N^{\mathfrak{a}})$*;*
(iii) $\mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) = \mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M/N^{\mathfrak{a}}) = \mathrm{Ass}_R(\mathfrak{a}, M)$*.*

Proof. By the exact sequence

$$\mathrm{H}^n_{\mathfrak{a}}(N^{\mathfrak{a}}) \to \mathrm{H}^n_{\mathfrak{a}}(M) \to \mathrm{H}^n_{\mathfrak{a}}(M/N^{\mathfrak{a}}) \to 0$$

it is enough for us to prove (iii)⇒(i). Suppose, on the contrary, that $\mathrm{H}^n_{\mathfrak{a}}(N^{\mathfrak{a}}) \neq 0$. Then there exists $\mathfrak{p} \in \mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(N^{\mathfrak{a}})$. By [4, Theorem A], $\mathfrak{p} \in \mathrm{Ass}_R N^{\mathfrak{a}}$ and $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = n$ and so $\mathfrak{p} \in \mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) = \mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M/N^{\mathfrak{a}})$. But by Notation 2.1, $\mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M/N^{\mathfrak{a}}) = \mathrm{Ass}_R(\mathfrak{a}, M)$, that means $\mathfrak{p} \in \mathrm{Ass}_R(\mathfrak{a}, M) = \mathrm{Ass}_R(M/N^{\mathfrak{a}})$, a contradiction. □

Definition 2.3. Let $\mathfrak{a}$ be an ideal of R, M be a finitely generated R-module of dimension n and $N^{\mathfrak{a}}$ be defined as in Notation 2.1. We say $\mathrm{H}^n_{\mathfrak{a}}(M)$ satisfies the property $(*)$, if one of the equivalent conditions of Lemma 2.2 holds.

Proposition 2.4. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathrm{H}^n_{\mathfrak{a}}(M)$ satisfies the property $(*)$. If $\mathrm{Att}_R \mathrm{H}^n_{\mathfrak{b}}(M) \subseteq \mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M)$, then there exists an epimorphism $\mathrm{H}^n_{\mathfrak{a}}(M) \to \mathrm{H}^n_{\mathfrak{b}}(M)$.*

Proof. Since $\mathrm{H}^n_{\mathfrak{a}}(M)$ satisfies the property $(*)$, we have

$$\mathrm{H}^n_{\mathfrak{a}}(M) \cong \mathrm{H}^n_{\mathfrak{a}}(M/N^{\mathfrak{a}}) \cong \mathrm{H}^n_{\mathfrak{m}}(M/N^{\mathfrak{a}})$$

and

$$\mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) = \mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M/N^{\mathfrak{a}}) = \mathrm{Ass}_R(\mathfrak{a}, M) = \mathrm{Ass}_R(M/N^{\mathfrak{a}})$$

where, $N^{\mathfrak{a}} = \bigcap_{\mathfrak{p}\in \mathrm{Ass}_R(\mathfrak{a},M)} N(\mathfrak{p})$. Now we show that $\mathrm{H}^n_{\mathfrak{b}}(N^{\mathfrak{a}}) = 0$. Suppose, on the contrary, that $\mathrm{H}^n_{\mathfrak{b}}(N^{\mathfrak{a}}) \neq 0$. Then there exists a prime ideal

$\mathfrak{p} \in \mathrm{Att}_R \mathrm{H}^n_{\mathfrak{b}}(N^{\mathfrak{a}})$ and therefore for this prime ideal, by [4, Theorem A] we have, $\mathfrak{p} \in \mathrm{Ass}_R N^{\mathfrak{a}}$ and $\mathrm{cd}(\mathfrak{b}, R/\mathfrak{p}) = n$. Since $\mathrm{Ass}_R N^{\mathfrak{a}} \subseteq \mathrm{Ass}_R M$, we have $\mathfrak{p} \in \mathrm{Att}_R \mathrm{H}^n_{\mathfrak{b}}(M)$ and therefore $\mathfrak{p} \in \mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M)$ that is a contradiction by Notation 2.1. So, $\mathrm{H}^n_{\mathfrak{b}}(M) \cong \mathrm{H}^n_{\mathfrak{b}}(M/N^{\mathfrak{a}})$. By [2, Proposition 8.1.2], for each $x \in \mathfrak{m} \setminus \mathfrak{b}$, there is a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^n_{\mathfrak{b}+Rx}(M/N^{\mathfrak{a}}) \longrightarrow \mathrm{H}^n_{\mathfrak{b}}(M/N^{\mathfrak{a}}) \longrightarrow \mathrm{H}^n_{\mathfrak{b}}((M/N^{\mathfrak{a}})_x) \longrightarrow \cdots$$

where $(M/N^{\mathfrak{a}})_x$ is the localization of $M/N^{\mathfrak{a}}$ at $\{x^i : i \geq 0\}$. Note that $\mathrm{H}^n_{\mathfrak{b}}(M/N^{\mathfrak{a}})$ is Artinian and $\mathrm{H}^n_{\mathfrak{b}}((M/N^{\mathfrak{a}})_x) \cong (\mathrm{H}^n_{\mathfrak{b}}(M/N^{\mathfrak{a}}))_x$. It follows that $\mathrm{H}^n_{\mathfrak{b}}((M/N^{\mathfrak{a}})_x) = 0$ and so there exists an epimorphism $\mathrm{H}^n_{\mathfrak{b}+Rx}(M/N^{\mathfrak{a}}) \to \mathrm{H}^n_{\mathfrak{b}}(M/N^{\mathfrak{a}})$. Repeating the argument with $\mathfrak{b} + Rx$ in place of $\mathfrak{b}$ and continuing gives an epimorphism $\mathrm{H}^n_{\mathfrak{m}}(M/N^{\mathfrak{a}}) \to \mathrm{H}^n_{\mathfrak{b}}(M/N^{\mathfrak{a}})$ and so we have the epimorphism $\mathrm{H}^n_{\mathfrak{a}}(M) \to \mathrm{H}^n_{\mathfrak{b}}(M)$. □

Corollary 2.5. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathrm{H}^n_{\mathfrak{b}}(M)$ and $\mathrm{H}^n_{\mathfrak{a}}(M)$ satisfy the property $(*)$. If $\mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) = \mathrm{Att}_R \mathrm{H}^n_{\mathfrak{b}}(M)$, then $\mathrm{H}^n_{\mathfrak{a}}(M) \cong \mathrm{H}^n_{\mathfrak{b}}(M)$.*

Proof. As in the proof of Proposition 2.4, since

$$\mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) = \mathrm{Att}_R \mathrm{H}^n_{\mathfrak{b}}(M),$$

we have $N^{\mathfrak{a}} = N^{\mathfrak{b}}$ and so

$$\mathrm{H}^n_{\mathfrak{a}}(M) \cong \mathrm{H}^n_{\mathfrak{m}}(M/N^{\mathfrak{a}}) \cong \mathrm{H}^n_{\mathfrak{m}}(M/N^{\mathfrak{b}}) \cong \mathrm{H}^n_{\mathfrak{b}}(M).$$

□

Dibaei and Jafari in [3], have shown that if R is a complete Noetherian local ring and M is a finitely generated R-module of dimension n, then any subset T of $\mathrm{Assh}_R M$ can be expressed as the set of attached primes of the top local cohomology module $\mathrm{H}^n_{\mathfrak{a}}(M)$ for some ideal $\mathfrak{a}$ of R (see [3, Theorem 2.8]). In the next theorem, we generalize this result to Noetherian local rings that are not necessarily complete.

Theorem 2.6. *Let M be a finitely generated R-module of dimension n and T be a subset of $\mathrm{Assh}_R(M)$, then there exists an ideal $\mathfrak{a}$ of R such that $\mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) = T$.*

Proof. Let $\mathrm{Assh}_R M = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ and $T = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$, where $r \leq k$. When $r = k$, the result is immediate from [2, Theorem 7.3.2], just take $\mathfrak{a} = \mathfrak{m}$. We therefore assume henceforth in this proof that $r < k$. So $\mathrm{Assh}_R M \setminus T = \{\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_k\}$. Since, for each $1 \leq i \leq k$, $\mathfrak{p}_i$ is a minimal associated prime of M, we have $\bigcap_{i=r+1}^k \mathfrak{p}_i \nsubseteq \bigcup_{i=1}^r \mathfrak{p}_i$. So we can choose an element $y \in \bigcap_{i=r+1}^k \mathfrak{p}_i \setminus \bigcup_{i=1}^r \mathfrak{p}_i$. Set $\overline{M} = \dfrac{M}{(\bigcap_{i=1}^r \mathfrak{p}_i)M}$, then $\mathrm{Assh}_R \overline{M} = T$ and $\dim(\overline{M}) = n$. Since $y \notin \bigcup_{i=1}^r \mathfrak{p}_i$, there are elements $x_1, \ldots, x_{n-1}$ such that $y, x_1, \ldots, x_{n-1}$ forms a system of parameters for R- module $\overline{M}$. Set $\mathfrak{a} = \langle y, x_1, \ldots, x_{n-1} \rangle$. It follows from [2, Independence Theorem 4.2.1 and Exercise 6.1.9] that

$$\mathrm{H}^n_{\mathfrak{a}}(M) \otimes \frac{R}{\bigcap_{i=1}^r \mathfrak{p}_i} \cong \mathrm{H}^n_{\mathfrak{a}}(M \otimes \frac{R}{\bigcap_{i=1}^r \mathfrak{p}_i}) \cong \mathrm{H}^n_{\mathfrak{a}}(\overline{M}) \cong \mathrm{H}^n_{\mathfrak{m}}(\overline{M}) \neq 0.$$

We can now use [2, Theorem 7.3.2 and Exercise 7.2.6] to deduce that

$$T = \mathrm{Att}_R \mathrm{H}^n_{\mathfrak{m}}(\overline{M}) \subseteq \mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M).$$

On the other hand, if $\mathfrak{p}_i \in \mathrm{Assh}_R M \setminus T$, then

$$\begin{array}{ll} \mathrm{H}^n_{\mathfrak{a}}(\frac{R}{\mathfrak{p}_i}) & = \mathrm{H}^n_{\langle y, x_1, \ldots, x_{n-1} \rangle}(\frac{R}{\mathfrak{p}_i}) \\ \text{[Since } y \in \mathfrak{p}_i] & \cong \mathrm{H}^n_{\langle x_1, \ldots, x_{n-1} \rangle}(\frac{R}{\mathfrak{p}_i}) \\ \text{by [2, Theorem 3.3.1]} & = 0. \end{array}$$

It follows from this observation and [4, Theorem A] that $\mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) \subseteq T$. Hence $\mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) = T$ and this completes the proof. □

Remark 2.7. Let M be a finitely generated R-module of dimension n and T be a subset of $\mathrm{Assh}_R M$. By Theorem 2.6, there exists an ideal $\mathfrak{a}$ of R such that $\mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) = T$. By the choice of this ideal in the proof of Theorem 2.6, one can see that, for each $\mathfrak{p} \in T, \sqrt{\mathfrak{a} + \mathfrak{p}} = \mathfrak{m}$. Therefore

$$\mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) = \{\mathfrak{p} \in \mathrm{Assh}_R M \ : \ \sqrt{\mathfrak{a} + \mathfrak{p}} = \mathfrak{m}\} = \mathrm{Ass}_R(\mathfrak{a}, M)$$

and so $\mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) = \mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M/N^{\mathfrak{a}})$. Hence $\mathrm{H}^n_{\mathfrak{a}}(M)$ satisfies the property $(*)$.

3. SOME RESULTS ON TOP FORMAL LOCAL COHOMOLOGY

In [8], Rezaei proved that if $(R, \mathfrak{m})$ is a complete Noetherian local ring and M is a finitely generated R-module of dimension n, then for each ideal $\mathfrak{a}$ of R there exists an ideal $\mathfrak{b}$ such that $\mathrm{H}^n_{\mathfrak{a}}(M) \cong \mathfrak{F}^n_{\mathfrak{b}}(M)$ and there exists an ideal $\mathfrak{c}$ such that $\mathfrak{F}^n_{\mathfrak{a}}(M) = \mathrm{H}^n_{\mathfrak{c}}(M)$. In this section we give a generalization of this result.

Lemma 3.1. (See [8, Theorem 2.2].) *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated R-module of dimension n. If T is a proper subset of $\mathrm{Assh}_R M$, then $\mathrm{Att}_R \mathfrak{F}^n_{\mathfrak{a}}(M) = T$ where $\mathfrak{a} := \bigcap_{\mathfrak{p}_i \in T} \mathfrak{p}_i$ is an ideal of R.*

Lemma 3.2. (See [8, Lemma 2.4].) *Let $\mathfrak{a}$ be an ideal of a Noetherian local ring R and M be a finitely generated R-module. If M is an $\mathfrak{a}$-torsion module, then $\mathfrak{F}^i_{\mathfrak{a}}(M) \cong \mathrm{H}^i_{\mathfrak{m}}(M)$ for all $i \geq 0$.*

By [5, Proposition 2.1], if $\mathfrak{a}$ is an ideal of R and M is a finitely generated R-module of dimension n, then $\mathfrak{F}_{\mathfrak{a}}^n(M)$ is an Artinian R-module and there exists an integer n_0 such that $\mathfrak{F}_{\mathfrak{a}}^n(M) \cong \frac{\mathrm{H}_{\mathfrak{m}}^n(M)}{\mathfrak{a}^{n_0}\,\mathrm{H}_{\mathfrak{a}}^n(M)}$. Now we can reduce the completeness assumption in [8, Theorem 2.5] to the assumption that $\mathrm{H}_{\mathfrak{a}}^n(M)$ satisfies the property $(*)$.

Theorem 3.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a Noetherian local ring $(R, \mathfrak{m})$ and M be a finitely generated R-module of dimension n such that $\mathrm{H}_{\mathfrak{a}}^n(M)$ satisfies the property $(*)$. If $\mathrm{Att}_R\,\mathrm{H}_{\mathfrak{a}}^n(M) = \mathrm{Att}_R\,\mathfrak{F}_{\mathfrak{b}}^n(M)$, then $\mathrm{H}_{\mathfrak{a}}^n(M) \cong \mathfrak{F}_{\mathfrak{b}}^n(M)$.*

Proof. Since $\mathrm{H}_{\mathfrak{a}}^n(M)$ satisfies the property $(*)$, by Notation 2.1 and Definition 2.3 we have $\mathrm{H}_{\mathfrak{a}}^n(N^{\mathfrak{a}}) = 0$ and

$$\begin{aligned}\mathrm{Att}_R\,\mathfrak{F}_{\mathfrak{b}}^n(M) &= \mathrm{Att}_R\,\mathrm{H}_{\mathfrak{a}}^n(M)\\ &= \{\mathfrak{p} \in \mathrm{Ass}_R\,M \;:\; \dim(R/\mathfrak{p}) = n \text{ and } \sqrt{\mathfrak{a}+\mathfrak{p}} = \mathfrak{m}\}\\ &= \mathrm{Ass}_R(\mathfrak{a}, M).\end{aligned}$$

Now we show that the Artinian module $\mathfrak{F}_{\mathfrak{b}}^n(N^{\mathfrak{a}})$ is zero. Suppose, on the contrary, that $\mathfrak{F}_{\mathfrak{b}}^n(N^{\mathfrak{a}}) \neq 0$. Therefore there exists a prime ideal $\mathfrak{p} \in \mathrm{Att}_R\,\mathfrak{F}_{\mathfrak{b}}^n(N^{\mathfrak{a}})$. By [5, Proposition 2.1], $\mathfrak{p} \in \mathrm{Ass}_R\,N^{a}$, $\dim(R/\mathfrak{p}) = n$ and $\mathfrak{b} \subseteq \mathfrak{p}$. Therefore $\mathfrak{p} \in \mathrm{Att}_R\,\mathfrak{F}_{\mathfrak{b}}^n(M) = \mathrm{Att}_R\,\mathrm{H}_{\mathfrak{a}}^n(M) = \mathrm{Ass}_R(\mathfrak{a}, M)$, a contradiction. Therefore $\mathfrak{F}_{\mathfrak{b}}^n(N^{\mathfrak{a}}) = 0$ and $\mathfrak{F}_{\mathfrak{b}}^n(M) \cong \mathfrak{F}_{\mathfrak{b}}^n(M/N^{\mathfrak{a}})$. On the other hand, since $\mathrm{Att}_R\,\mathfrak{F}_{\mathfrak{b}}^n(M) = \mathrm{Ass}_R(M/N^{\mathfrak{a}})$, we have $\mathfrak{b} \subseteq \bigcap_{\mathfrak{p}\in\mathrm{Ass}_R(M/N^{\mathfrak{a}})} \mathfrak{p}$. Therefore $M/N^{\mathfrak{a}}$ is a $\mathfrak{b}$-torsion R-module and by Lemma 3.2, we have $\mathfrak{F}_{\mathfrak{b}}^n(M/N^{\mathfrak{a}}) \cong \mathrm{H}_{\mathfrak{m}}^n(M/N^{\mathfrak{a}}) \cong \mathrm{H}_{\mathfrak{a}}^n(M/N^{\mathfrak{a}}) \cong \mathrm{H}_{\mathfrak{a}}^n(M)$. □

Corollary 3.4. *Let $\mathfrak{a}$ be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ such that $\mathrm{H}_{\mathfrak{a}}^n(M)$ satisfies the property $(*)$. Then $\mathrm{H}_{\mathfrak{a}}^n(M) \cong \mathfrak{F}_{\mathfrak{b}}^n(M)$, where $\mathfrak{b} = \mathrm{Ann}_R\,\mathrm{H}_{\mathfrak{a}}^n(M)$.*

Proof. Let $\mathfrak{b} = \mathrm{Ann}_R\,\mathrm{H}_{\mathfrak{a}}^n(M)$, then $\sqrt{\mathfrak{b}} = \bigcap_{\mathfrak{p}\in\mathrm{Att}_R\,\mathrm{H}_{\mathfrak{a}}^n(M)} \mathfrak{p}$. Since

$$\mathrm{Att}_R\,\mathrm{H}_{\mathfrak{a}}^n(M) \subseteq \mathrm{Assh}_R\,M,$$

it follows from Lemma 3.1 that $\mathrm{Att}_R\,\mathrm{H}_{\mathfrak{a}}^n(M) = \mathrm{Att}_R\,\mathfrak{F}_{\sqrt{\mathfrak{b}}}^n(M)$ and so by Theorem 3.3, we have $\mathrm{H}_{\mathfrak{a}}^n(M) \cong \mathfrak{F}_{\sqrt{\mathfrak{b}}}^n(M) \cong \mathfrak{F}_{\mathfrak{b}}^n(M)$. □

Now we can generalize [8, Theorem 2.6 (ii)] and [8, Corollary 2.7] to Noetherian local rings that are not necessarily complete.

Theorem 3.5. *Let $\mathfrak{a}$ be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and M be a finitely generated R-module of dimension n. Then there exists an ideal $\mathfrak{c}$ of R such that $\mathfrak{F}_{\mathfrak{a}}^n(M) \cong \mathrm{H}_{\mathfrak{c}}^n(M)$.*

Proof. Since $\operatorname{Att}_R \mathfrak{F}^n_{\mathfrak{a}}(M) \subseteq \operatorname{Assh}_R M$, it follows from Theorem 2.6 that there exists an ideal $\mathfrak{c}$ of R such that $\operatorname{Att}_R \mathrm{H}^n_{\mathfrak{c}}(M) = \operatorname{Att}_R \mathfrak{F}^n_{\mathfrak{a}}(M)$. By Remark 2.7, $\mathrm{H}^n_{\mathfrak{c}}(N^{\mathfrak{c}}) = 0$, where $N^{\mathfrak{c}}$ is defined as in Notation 2.1. Therefore $\mathrm{H}^n_{\mathfrak{c}}(M)$ satisfies the property $(*)$. Now by Theorem 3.3, $\mathfrak{F}^n_{\mathfrak{a}}(M) \cong \mathrm{H}^n_{\mathfrak{c}}(M)$. □

Corollary 3.6. *Let $\mathfrak{a}$ be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and M be a finitely generated R-module of dimension n. Then*

$$\mathfrak{F}^n_{\mathfrak{a}}(M) \cong \mathfrak{F}^n_{\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M)}(M).$$

Proof. By Theorem 3.5, there exists an ideal $\mathfrak{c}$ of R such that $\mathfrak{F}^n_{\mathfrak{a}}(M) \cong \mathrm{H}^n_{\mathfrak{c}}(M)$. As $\mathrm{H}^n_{\mathfrak{c}}(M)$ satisfies the property $(*)$, we have $\mathrm{H}^n_{\mathfrak{c}}(M) \cong \mathfrak{F}_{\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{c}}(M)}(M)$ by Corollary 3.4, and so $\mathfrak{F}^n_{\mathfrak{a}}(M) \cong \mathfrak{F}_{\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M)}(M)$, as required. □

Theorem 3.7. *Let $\mathfrak{a}$ be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and M be a finitely generated R-module of dimension n such that $\mathrm{H}^n_{\mathfrak{a}}(M)$ satisfies the property $(*)$. Then* $\mathrm{H}^n_{\mathfrak{a}}(M) \cong \dfrac{\mathrm{H}^n_{\mathfrak{m}}(M)}{(\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M))\,\mathrm{H}^n_{\mathfrak{m}}(M)}$.

Proof. By Corollary 3.4, we have $\mathrm{H}^n_{\mathfrak{a}}(M) \cong \mathfrak{F}^n_{\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M)}(M)$ and by [5, Proposition 2.1], there exists an integer t_0 such that

$$\mathfrak{F}^n_{\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M)}(M) = \frac{\mathrm{H}^n_{\mathfrak{m}}(M)}{(\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M))^t\,\mathrm{H}^n_{\mathfrak{m}}(M)} \quad \text{for all } t \geq t_0.$$

Hence $\mathrm{H}^n_{\mathfrak{a}}(M) \cong \dfrac{\mathrm{H}^n_{\mathfrak{m}}(M)}{(\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M))^t\,\mathrm{H}^n_{\mathfrak{m}}(M)}$ for all $t \geq t_0$ and so

$$\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M) = \operatorname{Ann}_R\Big(\frac{\mathrm{H}^n_{\mathfrak{m}}(M)}{(\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M))^t\,\mathrm{H}^n_{\mathfrak{m}}(M)}\Big) \quad \text{for all } t \geq t_0.$$

It follows that

$$(\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M))\,\mathrm{H}^n_{\mathfrak{m}}(M) \subseteq (\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M))^t\,\mathrm{H}^n_{\mathfrak{m}}(M) \quad \text{for all } t \geq t_0.$$

Hence $(\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M))^t\,\mathrm{H}^n_{\mathfrak{m}}(M) = \operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M)\,\mathrm{H}^n_{\mathfrak{m}}(M)$ for all $t \geq t_0$ and therefore

$$\mathrm{H}^n_{\mathfrak{a}}(M) \cong \frac{\mathrm{H}^n_{\mathfrak{m}}(M)}{(\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M))\,\mathrm{H}^n_{\mathfrak{m}}(M)}.$$

□

Theorem 3.8. *Let $\mathfrak{a}$ be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and M be a finitely generated R-module of dimension n. Then*

$$\mathfrak{F}^n_{\mathfrak{a}}(M) \cong \frac{\mathrm{H}^n_{\mathfrak{m}}(M)}{(\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M))\,\mathrm{H}^n_{\mathfrak{m}}(M)}.$$

Proof. By Corollary 3.6, $\mathfrak{F}^n_{\mathfrak{a}}(M) \cong \mathfrak{F}_{\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M)}(M)$. So an argument similar to the proof of Theorem 3.7 completes the proof. □

Corollary 3.9. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a Noetherian local ring $(R, \mathfrak{m})$ and M be a finitely generated R-module of dimension n.*

(i) *If $\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M) = \operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{b}}(M)$, then $\mathfrak{F}^n_{\mathfrak{a}}(M) \cong \mathfrak{F}^n_{\mathfrak{b}}(M)$;*

(ii) *If $\mathrm{H}^n_{\mathfrak{a}}(M)$ satisfies the property $(*)$ and*

$$\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M) = \operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{b}}(M),$$

then $\mathrm{H}^n_{\mathfrak{a}}(M) \cong \mathfrak{F}^n_{\mathfrak{b}}(M)$;

(iii) *If both $\mathrm{H}^n_{\mathfrak{a}}(M)$ and $\mathrm{H}^n_{\mathfrak{b}}(M)$ satisfy the property $(*)$ and*

$$\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M) = \operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{b}}(M),$$

then $\mathrm{H}^n_{\mathfrak{a}}(M) \cong \mathrm{H}^n_{\mathfrak{b}}(M)$.

Proof. All items are clear by Theorem 3.7 and Theorem 3.8. □

Theorem 3.10. *Let $\mathfrak{a}$ be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and M be a finitely generated R-module of dimension n. Then*

(i) *We have the equalities*

$$\begin{aligned}\operatorname{Att}_R \mathfrak{F}^n_{\mathfrak{a}}(M) &= \mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M)) \cap \operatorname{Assh}_R M \\ &= \operatorname{Min} \mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M)).\end{aligned}$$

(ii) *If $\mathrm{H}^n_{\mathfrak{a}}(M)$ satisfies the property $(*)$, then*

$$\operatorname{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) = \mathrm{V}(\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M)) \cap \operatorname{Assh}_R M = \operatorname{Min} \mathrm{V}(\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M)).$$

Proof. (i) Since for each Artinian R-module A,

$$\operatorname{Att}_R(A/\mathfrak{a}A) = \operatorname{Att}_R A \cap \mathrm{V}(\mathfrak{a}),$$

by Theorem 3.8, we have

$$\begin{aligned}\operatorname{Att}_R \mathfrak{F}^n_{\mathfrak{a}}(M) &= \operatorname{Att}_R \mathrm{H}^n_{\mathfrak{m}}(M) \cap \mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M)) \\ &= \operatorname{Assh}_R M \cap \mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M)) \\ &\subseteq \operatorname{Min} \mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M)).\end{aligned}$$

On the other hand

$$\begin{aligned}\operatorname{Min} \mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M)) &= \operatorname{Min} \operatorname{Att}_R \mathfrak{F}^n_{\mathfrak{a}}(M) \\ &\subseteq \operatorname{Assh}_R M \cap \mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M)).\end{aligned}$$

Therefore

$$\operatorname{Att}_R \mathfrak{F}^n_{\mathfrak{a}}(M) = \operatorname{Assh}_R M \cap \mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M)) = \operatorname{Min} \mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M)).$$

(ii) The proof is similar to the proof (i). □

Corollary 3.11. *Let* $\mathfrak{a}$ *be an ideal of a Noetherian local ring* $(R,\mathfrak{m})$ *and* M *be a finitely generated* R*-module of dimension* n*. Then*

(i) $\operatorname{Att}_R \mathfrak{F}^n_{\mathfrak{a}}(M) = \operatorname{Ass}_R(\dfrac{R}{\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M)})$.

(ii) *If* $\mathrm{H}^n_{\mathfrak{a}}(M)$ *satisfies the property* $(*)$*, then*
$\operatorname{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) = \operatorname{Ass}_R(\dfrac{R}{\operatorname{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(M)})$.

Proof. (i) Since $\mathfrak{F}^n_{\mathfrak{a}}(M)$ is Artinian, it follows from [10, Theorem 3.1 and Theorem 3.3 (b)] that $\operatorname{Ass}_R(R/\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M)) \subseteq \operatorname{Att}_R(\mathfrak{F}^n_{\mathfrak{a}}(M))$. But the sets $\mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M))$ and $\operatorname{Ass}_R(R/\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M))$ have the same minimal elements, by [10, Theorem 3.3 (c)]. Thus, by Theorem 3.10, $\operatorname{Att}_R(\mathfrak{F}^n_{\mathfrak{a}}(M)) \subseteq \operatorname{Ass}_R(R/\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M))$. Therefore

$$\operatorname{Att}_R(\mathfrak{F}^n_{\mathfrak{a}}(M)) = \operatorname{Ass}_R(R/\operatorname{Ann}_R \mathfrak{F}^n_{\mathfrak{a}}(M)).$$

(ii) The proof is similar to the proof (i). □

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TOP LOCAL COHOMOLOGY AND TOP FORMAL LOCAL COHOMOLOGY MODULES WITH SPECIFIED ATTACHED PRIMES

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بالاترین مدول‌های کوهمولوژی موضعی و کوهمولوژی موضعی صوری با ایده‌آل‌های اول چسبیده‌ی مشخص

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فرض کنید $(R, \mathfrak{m})$ یک حلقه‌ی موضعی نوتری، M یک Rـ مدول متناهی‌مولد از بعد n و $\mathfrak{a}$ ایده‌آلی از R باشد. در این مقاله نشان می‌دهیم که به ازای هر زیر مجموعه‌ی T از $\mathrm{Assh}_R M$، ایده‌آل $\mathfrak{a}$ از R موجود است به‌طوری که $\mathrm{Att}_R \mathrm{H}^n_{\mathfrak{a}}(M) = T$. با استفاده از این مطلب برخی ارتباطات بین بالاترین مدول‌های کوهمولوژی موضعی و کوهمولوژی موضعی صوری بیان می‌شود. مطالب بیان شده، نتایج اصلی به‌دست آمده توسط دیبایی و جعفری در مرجع [۳] و رضایی در مرجع [۸] را تعمیم می‌دهد.

کلمات کلیدی: ایده‌آل‌های اول چسبیده، مدول‌های کوهمولوژی موضعی، مدول‌های کوهمولوژی موضعی صوری، حلقه‌های نوتری موضعی.