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A NOTE ON BALANCED BIG COHEN–MACAULAY MODULES

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Abstract. Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay complete local ring with the canonical module ω. The aim of this note, is to show that, under mild assumptions, the class of balanced big Cohen–Macaulay modules coincides with the one consisting of those modules admitting a right resolution by modules in $\mathsf{Add}\omega$. This generalizes the well-known result for the class of maximal Cohen–Macaulay modules.

1. Introduction

Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring. Hochster [4] defines a not necessarily finitely generated R-module M is *big Cohen-Macaulay*, if there exists a system of parameters of R which is an M-regular sequence. Sharp [8] called a big Cohen-Macaulay R-module M is (*weak*) *balanced big Cohen-Macaulay,* ((weak) balanced big CM, for short), provided that every system of parameters of R is an (a weak) M-regular sequence. A finitely generated R-module M is maximal Cohen-Macaulay (abbreviated, MCM), if it is either balanced big Cohen-Macaulay or zero. It is known that, over a Cohen–Macaulay complete local ring R with the canonical module ω, the class of maximal Cohen–Macaulay R-modules coincides with the class of thoses modules admitting a right resolution by modules in $\mathsf{add}\omega$; see Remark

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2.5. Here $\mathsf{add}\omega$ denotes the subcategory of finitely generated modules formed by direct summands of finite direct sums of ω. Since balanced big CM modules are generalization of MCM modules, it is conceivable that such a characterization should be true for balanced big CM modules. This note is an effort to give some evidence in this direction. In order to state our main result precisely, let us recall some notions. Let $\mathsf{Add}\omega$ denote the full subcategory of R-modules consisting of all modules isomorphic to direct summands of direct sums of copies of ω. We also use $\mathcal{X}'_\omega$ to denote the subcategory consisting of all modules M admitting a right resolution by modules in $\mathsf{Add}\omega$. A balanced big CM R-module M is said to have an $\mathfrak{m}$-primary cohomological annihilator, if there is an integer $t \geq 1$ such that $\mathfrak{m}^t\mathsf{Ext}^1_R(M,N) = 0$ for all R-modules N. Now our main result can be stated as follows:

Theorem 1.1. *Let $(R,\mathfrak{m})$ be a complete Cohen–Macaulay local ring with the canonical module ω. Let M be a countably generated R-module with an $\mathfrak{m}$-primary cohomological annihilator. Then $M \in \mathcal{X}'_\omega$ if and only if M is balanced big* CM.

Throughout the paper, unless otherwise specified, $(R,\mathfrak{m},k)$ is a d-dimensional commutative Cohen-Macaulay complete local ring with the canonical (or dualizing) module ω. The category of all (finitely generated) R-modules will be denoted by $(\mathsf{mod}R)$ $\mathsf{Mod}R$.

2. Results and proofs

As we have mentioned in the introduction, over a Cohen–Macaulay complete local ring with the canonical module ω, a given module is maximal Cohen–Macaulay if and only if it admits a right resolution by modules in $\mathsf{add}\omega$. The aim of this section is to prove the counerpart of this result for balanced big CM modules. We begin with the followoing definitions.

Definition 2.1. Let M be an R-module. A sequence of elements $\boldsymbol{x} = x_1,\dots,x_n \in \mathfrak{m}$ is called a *weak M-regular sequence*, provided that x_i is a non-zerodivisor on $M/(x_1,\dots,x_{i-1})M$ for any $1 \leq i \leq n$ (for $i=1$, we mean that x_1 is non-zerodivisor on M). If, in addition, $(x_1,\dots,x_n)M \neq M$, then $\boldsymbol{x}$ is said to be an *M-regular sequence.* It is worth remarking that if M is a non-zero finitely generated R-module, then it follows from Nakayama's lemma that any weak M-regular sequence is M-regular sequence, as well. It is also known that over local rings, any permutation of M-regular sequence, is again M-regular sequence.

Definition 2.2. Let M be a not necessarily finitely generated R-module. Following Hochster [4], we say that M is (*weak*) *balanced big Cohen-Macaulay,* ((weak) balanced big CM, for short) provided that every system of parameters of R is an (a weak) M-regular sequence. It follows from the definition that finitely generated balanced big CM modules are MCM. Indeed, M is MCM if it is zero or it is a finitely generated balanced big CM module.

Definition 2.3. We say that a balanced big CM R-module M has an $\mathfrak{m}$-*primary cohomological annihilator*, provided that $\mathfrak{m}^t\mathsf{Ext}^1_R(M,-)=0$ over $\mathsf{Mod}R$, for some non-negative integer t. It should be observed that, by [6, Lemma 2.14], this is equivalent to saying that $\mathfrak{m}^t\mathsf{Ext}^i_R(M,-)=0$ for all $i\geq 1$.

Definition 2.4. (1) By $\mathsf{Add}\omega$ (resp. $\mathsf{add}\omega$) we mean the full subcategory of $\mathsf{Mod}R$ (resp. $\mathsf{mod}R$) consisting of all modules isomorphic to direct summands of direct sums (resp. finite direct sums) of copies of ω.
(2) We let $\mathcal{X}_\omega$ (resp. $\mathcal{X}'_\omega$) denote the subcategory of $\mathsf{mod}R$ (resp. $\mathsf{Mod}R$) consisting of all R-modules M admitting a right resolution by modules in $\mathsf{add}\omega$ (resp. $\mathsf{Add}\omega$), that is, an exact sequence of R-modules;

$$0\longrightarrow M\longrightarrow w_0\xrightarrow{d_0}w_1\xrightarrow{d_1}\cdots\xrightarrow{d_{i-1}}w_i\xrightarrow{d_i}\cdots,$$

with $w_i\in\mathsf{add}\omega$ (resp. $w_i\in\mathsf{Add}\omega$).
(3) It is worth remarking that since ω is a selforthogonal R-module of finite injective dimension, $\mathsf{Ext}^{i>0}_R(W,W')=0$ for all modules $W,W'\in\mathsf{Add}\omega$. Indeed, this follows from the isomorphisms $\mathsf{Ext}^i_R(\oplus_{j\in J}\omega,W')\cong\prod_{j\in J}\mathsf{Ext}^i_R(\omega,W')$ and $\mathsf{Ext}^i_R(\omega,\oplus_{j\in J'})\cong\oplus_{j\in J'}\mathsf{Ext}^i_R(\omega,\omega)$. One should observe that the latter isomorphism holds, because ω is a finitely generated R-module. So it is easily seen that a given R-module M belongs to $\mathcal{X}'_\omega$ if and only if it is ω-Gorenstein projective in the sense of Holm and Jrgensen [5]. This, in particular, yeilds that $\mathcal{X}'_\omega$ is closed under direct summands. Recall that an R-module M is *selforthogonal,* if $\mathsf{Ext}^i_R(M,M)=0$ for all $i\geq 1$.

Remark 2.5. It is easy to see that $\mathsf{MCM}=\mathcal{X}_\omega$. To see this, according to [2, Theorem 3.3.10], a given module $M\in\mathsf{mod}R$ is MCM if and only if $\mathsf{Ext}^i_R(M,\omega)=0=\mathsf{Ext}^i_R(M^*,\omega)$ for all $i\geq 1$ and the natural homomorphism $\delta:M\longrightarrow M^{**}$ is an isomorphism, where $M^*=\mathsf{Hom}_R(M,\omega)$. Now assume that M is an arbitrary MCM module and $\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow M^*\longrightarrow 0$ is an exact sequence in $\mathsf{mod}R$ such that each P_i is projective. So, applying the functor $\mathsf{Hom}_R(-,\omega)$, implies that M admits a right resolution by modules in $\mathsf{add}\omega$, giving the containment $\mathsf{MCM} \subseteq \mathcal{X}_\omega$. For the opposite containment, take an R-module M in $\mathcal{X}_\omega$. Since ω has finite injective dimension and $\mathsf{Ext}^i_R(\omega, \omega) = 0$ for all $i \geq 1$, one may deduce that $\mathsf{Ext}^i_R(M, \omega) = 0$ for all $i \geq 1$. This, in turn, implies that $M^* \in \mathcal{X}_\omega$ and so $\mathsf{Ext}^i_R(M^*, \omega) = 0$ for all $i \geq 1$. Now, one may use the fact that $\delta_\omega : \omega \longrightarrow \omega^{**}$ is an isomorphism, and conclude that the same is true for $\delta_M : M \longrightarrow M^{**}$. Hence M will be a MCM module.

2.6. For a given R-module M and any non-negative integer n, $\Omega^n M$, (or $\Omega^n_R M$ when there is some fear of confusion), stands for the n-th syzygy of a projective resolution of M, and so it is uniquely determined, up to projective direct summands.

The following result which is needed in the next theorem, has been also appeared in the proof of [9, Theorem 2.4]. We include its proof only for the reader's convenience.

Proposition 2.7. *Let M be a balanced big CM R-module with an $\mathfrak{m}$-primary cohomological annihilator. Then M is a direct summand of $\Omega^d_R(M/\boldsymbol{x}M)$, for some R-sequence $\boldsymbol{x} = x_1, x_2, \ldots, x_d$.*

Proof. By the hypothesis, there exists an integer $t > 0$ such that $\mathfrak{m}^t \mathsf{Ext}^1_R(M, -) = 0$ over $\mathsf{Mod}R$. Take an R- and also M-sequence $\boldsymbol{x} = x_1, \ldots, x_d$ in $\mathfrak{m}^t$. Applying the same argument given in the proof of [3, Lemma 2.2] (see also [9, Lemma 2.1]) would imply that M is a direct summand of $\Omega_R(M/x_1M)$. Moreover, for any $i \geq 0$, [7, Lemma 2(ii), page 140] gives rise to the isomorphism $\mathsf{Ext}^i_R(M, \Omega_{R/x_1R}(M/x_1M)) \cong \mathsf{Ext}^i_{R/x_1R}(M/x_1M, \Omega_{R/x_1R}(M/x_1M))$. This, in particular, yields that $x_2 \mathsf{Ext}^1_{R/x_1R}(M/x_1M, \Omega_{R/x_1R}(M/x_1M)) = 0$. Now one may apply [3, Lemma 2.2] again and conclude that M/x_1M is an R/x_1R-direct summand (and so R-direct summand) of $\Omega_{R/x_1R}(M/(x_1, x_2)M)$ and hence M will be an R-direct summand of $\Omega_R \Omega_{R/x_1R}(M/(x_1, x_2)M)$. Take the following commutative diagram;

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & R^{\oplus s} & = & R^{\oplus s} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega_R(M/(x_1,x_2)M) & \longrightarrow & R^{\oplus s} & \longrightarrow & M/(x_1,x_2)M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega_{R/x_1R}(M/(x_1,x_2)M) & \longrightarrow & (R/x_1R)^{\oplus s} & \longrightarrow & M/(x_1,x_2)M & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Now applying the first syzygy functor over R on the left column of this diagram, gives rise to an R-isomorphism $\Omega^2_R(M/(x_1,x_2)M) \cong \Omega_R\Omega_{R/x_1R}(M/(x_1,x_2)M)$, up to projective summands, implying that M is an R-direct summand of $\Omega^2_R(M/(x_1,x_2)M)$. Repeating this way, one may infer that M is indeed a direct summand of $\Omega^d_R(M/\boldsymbol{x}M)$, as desired. □

Proof of Theorem 1.1. First we prove the 'only if' part. To this end, assume that $\boldsymbol{x}$ is an arbitrary system of parameters of R. As $M \in \mathcal{X}'_\omega$ and $\boldsymbol{x}$ is a W-regular sequence for any $W \in \mathsf{Add}\omega$, it is fairly easy to see that $\boldsymbol{x}$ is a weak M-regular sequence. Since M has an $\mathfrak{m}$-primary cohomological annihilator, one may find an integer $t > 0$ such that $\boldsymbol{x}^t\mathsf{Ext}^1_R(M,-) = 0$. If $M/\boldsymbol{x}M = 0$, by [1, Lemma 4.2] $M/\boldsymbol{x}^tM = 0$ then by making use of [1, Lemma 4.1], we infer that M is projective over R. This would imply that $M/\boldsymbol{x}M \neq 0$, because R is Cohen-Macaulay. So we obtain a contradiction. For the reverse implication, assume that $t > 0$ is an integer such that $\mathfrak{m}^t\mathsf{Ext}^1_R(M,-) = 0$ over $\mathsf{Mod}R$. In particular, taking a system of parameters $\boldsymbol{x} \in \mathfrak{m}^t$, we have $\boldsymbol{x}^t\mathsf{Ext}^1_R(M,-) = 0$. So by Proposition 2.7, M is a direct summand of $\Omega^d_R(M/\boldsymbol{x}M)$. Consequently, the result will be obtained if we show that $\Omega^d_R(M/\boldsymbol{x}M) \in \mathcal{X}'_\omega$, because $\mathcal{X}'_\omega$ is closed under direct summands. Suppose that $M/\boldsymbol{x}M = \varinjlim_{i\in\mathbb{N}} S_i$ where all S_i's are finitely generated submodules of $M/\boldsymbol{x}M$ and for any $i \leq j$, $\alpha^i_j : S_i \longrightarrow S_j$ is a monomorphism. Therefore, for any $i < j$ one may find an R-monomorphism $h^i_j : \Omega^d_R S_i \longrightarrow \Omega^d_R S_j$ with $\mathrm{Coker} h^i_j$ is MCM. To see this, take exact sequences of R-modules, $0 \longrightarrow \Omega^d_R S_i \longrightarrow P_{d-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow S_i \longrightarrow 0$ and $0 \longrightarrow \Omega^d_R \alpha^i_j \longrightarrow Q_{d-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow \alpha^i_j \longrightarrow 0$, where for $0 \leq s \leq d-1$, P_s, Q_s are projective. Hence we will obtain the following commutative diagram of R-modules;

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Omega^d_R S_i & \longrightarrow & P_{d-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & S_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h^i_i} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \alpha^i_j} & & \\ 0 & \longrightarrow & \Omega^d_R S_j & \longrightarrow & P_{d_1} \oplus Q_{d-1} & \longrightarrow & \cdots & \longrightarrow & P_0 \oplus Q_0 & \longrightarrow & S_j & \longrightarrow & 0, \end{array}$$

where all middle columns are split monomorphism. This, in conjunction with [10, Proposition 1.4], gives rise to the short exact sequence of MCM modules, $0 \longrightarrow \Omega^d_R S_i \xrightarrow{h^i_j} \Omega^d_R S_j \longrightarrow \Omega^d_R \alpha^i_j \longrightarrow 0$. Moreover, it can be easily seen that $\{\Omega^d_R S_i, h^i_j\}_{i,j\in\mathbb{N}}$ forms a direct system. In particular, there exist free R-modules P, Q such that $\varinjlim \Omega^d_R S_i \oplus P = \Omega^d_R(M/\boldsymbol{x}M) \oplus Q$. Now one may invoke [1, Lemma 6.2] and deduce that $\varinjlim \Omega^d_R S_i \in \mathcal{X}'_\omega$. Consequently, $\Omega^d_R(M/\boldsymbol{x}M) \in \mathcal{X}'_\omega$, as well. So the proof is finished.

Remark 2.8. Assume that R is a Gorenstein local ring. Then R is Cohen–Macaulay with $\omega = R$, and in particular, $\mathcal{X}'_R$ is exactly the class of all Gorenstein projective R-modules. Indeed, since R is Gorenstein, every projective R-module has finite injective dimension and then, for any $M \in \mathcal{X}'_R$, $\mathsf{Ext}^{i\geq 1}_R(M, P) = 0$, for all projective R-modules P. Consequently, M is Gorenstein projective.

As a direct consequence of the above theorem, we include the result below.

Corollary 2.9. *Let R be a Gorenstein local ring and let M be a countably generated R-module with an $\mathfrak{m}$-primary cohomological annihilator. Then M is Gorenstein projective if and only if it is a balanced big* CM *module.*

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References

1. A. Bahlekeh, F. Fotouhi and Sh. Salarian, Representation-theoretic properties of balanced big Cohen-Macaulay modules, *Math. Z.*, **293** (2019), 1673–1709.
2. W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Stud. Adv. Math., Vol. 39, Cambridge Univ. Press, (1993).
3. J. Herzog and D. Popescu, Thom-Sebastiani problems for maximal Cohen-Macaulay modules, *Math. Ann.*, **309** (1997), 677–700.

4. M. Hochster, Big and small Cohen-Macaulay modules, Module theory (Proc. Special Session, Amer. Math. Soc., Univ. Washington, Seattle, Wash., 1977), *Lecture Notes in Math.*, Springer, Berlin, (1979), 119–142.
5. H. Holm and P. Jrgensen, Semi-dualizing modules and related Gorenstein homological dimension, *J. Pure Appl. Algebra*, **205** (2006), 423–445.
6. S. B. Iyengar and R. Takahashi, Annihilation of cohomology and strong generation of module categories, *Int. Math. Res. Not. IMRN*, **2** (2016), 499–535.
7. H. Matsumura, *Commutative Ring Theory*, Cambridge Univerdity Press, (1989).
8. R. Y. Sharp, Cohen-Macaulay properties for balanced big Cohen-Macaulay modules, *In mathematical proceeding of the Cambridge Philosophical Society*, Vol. 90, Cambridge Univ. Press, (1981), 229–238.
9. R. Takahashi, Classifying thick subcategories of the stable category of Cohen-Macaulay modules, *Adv. Math.*, **225** (2010), 2076–2116.
10. Y. Yoshino, *Cohen-Macaulay Modules over Cohen-Macaulay Rings*, Lond. Math. Soc. Lecture Notes Series, 146, Cambridge Univ. Press, Cambridge, (1990).

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A NOTE ON BALANCED COHEN–MACAULAY MODULES

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مطالعه مدول‌های کوهن–مکالی بزرگ متعادل

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فرض کنید (R, m) یک حلقه‌ی جابجایی کوهن–مکالی کامل موضعی با مدول کانونی ω باشد. در این مقاله نشان می‌دهیم که تحت برخی شرایط ضعیف، رده مدول‌های کوهن–مکالی بزرگ متعادل با رده مدول‌هایی که دارای تحلیل راست از مدول‌ها در $\mathsf{Add}\omega$ بوده، برابر می‌باشد. این مطلب تعمیم یک قضیه شناخته شده برای مدول‌های کوهن–مکالی ماکزیمال می‌باشد.

کلمات کلیدی: مدول‌های کوهن–مکالی ماکزیمال، مدول‌های کوهن–مکالی بزرگ متعادل، مدول کانونی.