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HOOPS WITH QUASI-VALUATION MAPS

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Abstract. Based on subalgebras and filters in hoops, the notions of S-quasi-valuation maps and F-quasi-valuation maps are introduced, and related properties are investigated. Relations between S-quasi-valuation maps and F-quasi-valuation maps are discussed. Using F-quasi-valuation map, a (pseudo) metric space is introduced, and we show that the operations "$\odot$", "$\rightarrow$" and "$\wedge$" in a hoop are uniformly continuous.

1. Introduction

Non-classical logic has become a considerable formal tool for computer science and artificial intelligence to deal with fuzzy information and uncertainty information. Many-valued logic, a great extension and development of classical logic, has always been a crucial direction in non-classical logic. In order to research the many-valued logical system whose propositional value is given in a lattice, Bosbach in [8, 9], proposed the concept of hoops, and discussed their some properties. Hoops are naturally ordered commutative residuated integral monoids. In the last years, hoops theory was enriched with deep structure theorems (See [2, 8, 9]). Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops (See [2, Corollary 2.10]) one obtains an elegant short proof of

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the completeness theorem for propositional basic logic (See [2, Theorem 3.8]), introduced by Hájek in [12]. The algebraic structures corresponding to Hájek's propositional (fuzzy) basic logic, BL-algebras, are particular cases of hoops. The main example of BL-algebras is interval $[0,1]$ endowed with the structure introduced by a t-norm. MV-algebras, product algebras and Gödel algebras are the most known classes of BL-algebras. Recent investigations are concerned with non-commutative generalizations for these structures. During these years, many researchers study on hoops in different way, and got some results on hoops [1, 3, 12, 14, 17, 11]. Song, Roh and Jun, in [18] introduced the notion of quasi-valuation maps based on a subalgebra and an ideal in BCK/BCI-algebras, and then we investigated several properties. They provided relations between a quasi-valuation map based on a subalgebra and a quasi-valuation map based on an ideal. In a BCI-algebra, they gave a condition for a quasi-valuation map based on an ideal to be a quasi-valuation map based on a subalgebra, and found conditions for a real-valued function on a BCK/BCI-algebra to be a quasi-valuation map based on an ideal. Using the notion of a quasi-valuation map based on an ideal, they constructed (pseudo) metric spaces, and shew that the binary operation $\star$ in BCK-algebras is uniformly continuous.

In this paper, we introduce the notion of quasi-valuation maps such as ($S_\odot$, $S_\to$) S-quasi-valuation maps and F-quasi-valuation map based on subhoops and filters and related properties of them are investigated. Also, we study the relation between them and we prove that every F-quasi-valuation map is an S-quasi-valuation map. Finally, by using the notion of F-quasi-valuation map, we induce a (pseudo) metric space and prove that the operations "$\odot$", "$\to$" and "$\wedge$" in a hoop are uniformly continuous.

2. Preliminaries

In this section, we introduced some definitions and results which will be used in this paper.

By a *hoop* we mean an algebra $(H, \odot, \to, 1)$ in which $(H, \odot, 1)$ is a commutative monoid and, for all $x, y, z \in H$, the following assertions are valid.

(H1) $x \to x = 1$.
(H2) $x \odot (x \to y) = y \odot (y \to x)$.
(H3) $x \to (y \to z) = (x \odot y) \to z$.

We define a relation "$\leq$" on a hoop H by

$$(\forall x, y \in H)(x \leq y \;\Leftrightarrow\; x \to y = 1). \tag{2.1}$$

It is easy to see that $(H, \leq)$ is a poset. A hoop H is *bounded* if there is an element $0 \in H$ such that $0 \leq x$ for all $x \in H$. Let $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for any $n \in \mathbb{N}$. If H is a bounded hoop, then we define a negation " $'$ " on H by $x' = x \to 0$ for all $x \in H$. A nonempty subset S of H is called a *subhoop* of H if it satisfies:

$$(\forall x, y \in S)(x \odot y \in S,\ x \to y \in S). \tag{2.2}$$

Note that every subhoop contains the element 1.

Proposition 2.1 ([8, 9]). *Let $(H, \odot, \to, 1)$ be a hoop. For any $x, y, z \in H$, the following conditions hold:*

(a1) *$(H, \leq)$ is a meet-semilattice with $x \wedge y = x \odot (x \to y)$.*
(a2) *$x \odot y \leq z$ if and anly if $x \leq y \to z$.*
(a3) *$x \odot y \leq x, y$ and $x^n \leq x$ for any $n \in \mathbb{N}$.*
(a4) *$x \leq y \to x$.*
(a5) *$1 \to x = x$ and $x \to 1 = 1$.*
(a6) *$x \odot (x \to y) \leq y$ and $x \odot y \leq x \wedge y \leq x \to y$.*
(a7) *$x \to y \leq (y \to z) \to (x \to z)$.*
(a8) *$x \leq y$ implies $x \odot z \leq y \odot z$, $z \to x \leq z \to y$ and $y \to z \leq x \to z$.*
(a9) *$x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$.*

A nonempty subset F of a hoop H is called a *filter* of H (see [8, 9]) if the following assertions are valid.

$$(\forall x, y \in H)(x, y \in F \ \Rightarrow x \odot y \in F), \tag{2.3}$$

$$(\forall x, y \in H)(x \in F,\ x \leq y \ \Rightarrow \ y \in F). \tag{2.4}$$

Note that the conditions (2.3) and (2.4) means that F is closed under the operation $\odot$ and F is upward closed, respectively.

Note. In what follows, let H denote a hoop unless otherwise specified.

3. Quasi-valuation maps based on subhoops and filters

In this section, we introduce the notion of quasi-valuation maps such as ($S_{\odot}$, $S_{\to}$) S-quasi-valuation maps and F-quasi-valuation map and related properties of them are investigated. Also, we study the relation between them and we prove that every F-quasi-valuation map is an S-quasi-valuation map.

Definition 3.1. A real valued function λ of H is called

- an *$S_{\odot}$-quasi-valuation map* of H if

$$(\forall x, y \in H)(\lambda(x \odot y) \geq \lambda(x) + \lambda(y)), \tag{3.1}$$

- an $S_{\to}$-*quasi-valuation map* of H if
$$(\forall x, y \in H)(\lambda(x \to y) \geq \lambda(x) + \lambda(y)). \tag{3.2}$$
- an S-*quasi-valuation map* of H if it is an $S_{\odot}$-quasi-valuation map and an $S_{\to}$-quasi-valuation map of H.

Example 3.2. Let $H = \{0, a, b, 1\}$ be a set with Cayley tables (Tables 1 and 2). Then $(H, \odot, \to, 1)$ is a hoop (see [5]).

TABLE 1. Cayley table for the binary operation "$\odot$"

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

TABLE 2. Cayley table for the binary operation "$\to$"

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

(1) Define a map λ on H as follows:
$$\lambda : H \to \mathbb{R},\ x \mapsto \begin{cases} -2 & \text{if } x = 0, \\ -4 & \text{if } x = a, \\ -6 & \text{if } x = b, \\ -7 & \text{if } x = 1. \end{cases} \tag{3.3}$$
Then λ is an $S_{\odot}$-quasi-valuation map of H. But it is not an $S_{\to}$-quasi-valuation map of H since
$$\lambda(0 \to 0) = \lambda(1) = -7 < -4 = \lambda(0) + \lambda(0).$$

(2) Define a map λ on H as follows:
$$\lambda : H \to \mathbb{R},\ x \mapsto \begin{cases} -45 & \text{if } x = 0, \\ -20 & \text{if } x = a, \\ -20 & \text{if } x = b, \\ -25 & \text{if } x = 1. \end{cases} \tag{3.4}$$
Then λ is an S-quasi-valuation map of H.

Example 3.3. Let $H = \{0, a, b, 1\}$ be a set with Cayley tables (Tables 3 and 4). Then $(H, \odot, \rightarrow, 1)$ is a hoop (see [5]). Define a map λ on

Table 3. Cayley table for the binary operation "$\odot$"

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Table 4. Cayley table for the binary operation "$\rightarrow$"

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

H as follows:

$$\lambda : H \to \mathbb{R},\ x \mapsto \begin{cases} -5 & \text{if } x = 0, \\ -2 & \text{if } x = a, \\ -2 & \text{if } x = b, \\ -1 & \text{if } x = 1. \end{cases} \tag{3.5}$$

Then λ is an $S_{\rightarrow}$-quasi-valuation map of H. But it is not an $S_{\odot}$-quasi-valuation map of H since

$$\lambda(a \odot a) = \lambda(0) = -5 < -4 = \lambda(a) + \lambda(a).$$

We know that any $S_{\odot}$-quasi-valuation map (resp., $S_{\rightarrow}$-quasi-valuation map and S-quasi-valuation map) is not order preserving in Example 3.2.

Proposition 3.4. *Every $S_{\odot}$-quasi-valuation map (resp., $S_{\rightarrow}$-quasi-valuation map) λ of H safisfies:*

$$(\forall x \in H)(\lambda(x) \leq 0). \tag{3.6}$$

Proof. Let λ be an $S_{\rightarrow}$-quasi-valuation map of H. For any $x \in H$, we have $\lambda(1) = \lambda(x \rightarrow 1) \geq \lambda(x) + \lambda(1)$, and so $\lambda(x) \leq 0$. If λ is an $S_{\odot}$-quasi-valuation map of H, then $\lambda(0) = \lambda(x \odot 0) \geq \lambda(x) + \lambda(0)$ and so $\lambda(x) \leq 0$ for all $x \in H$. $\square$

Theorem 3.5. *Let S be a subhoop of H. For any negative real numbers k_1 and k_2 with $k_1 > k_2$, let λ_S be a real valued function on H defined by*

$$\lambda_S : H \to \mathbb{R},\ x \mapsto \begin{cases} k_1 & \text{if } x \in S, \\ k_2 & \text{otherwise.} \end{cases} \tag{3.7}$$

Then λ_S is an S-quasi-valuation map of H.

Proof. Straightforward. □

Theorem 3.6. *If $\lambda : H \to \mathbb{R}$ is an S-quasi-valuation map of H, then the set*

$$S_\lambda := \{x \in H \mid \lambda(x) = 0\} \tag{3.8}$$

is a subhoop of H.

Proof. Let $x, y \in S_\lambda$. Then $\lambda(x) = 0$ and $\lambda(y) = 0$. Thus $\lambda(x \odot y) \geq \lambda(x)+\lambda(y) = 0$ and $\lambda(x \to y) \geq \lambda(x)+\lambda(y) = 0$. Since $\lambda(x\odot y) \leq 0$ and $\lambda(x \to y) \leq 0$ by Proposition 3.4, we have $x \odot y \in S_\lambda$ and $x \to y \in S_\lambda$. Therefore S_λ is a subhoop of H. □

The converse of Theorem 3.6 is not true in general as seen in the following example.

Example 3.7. Let $H = \{0, a, b, 1\}$ be the hoop as in Example 3.3. If we define a map λ on H by

$$\lambda : H \to \mathbb{R},\ x \mapsto \begin{cases} -33 & \text{if } x = 0, \\ -13 & \text{if } x = a, \\ 0 & \text{if } x \in \{b, 1\}, \end{cases} \tag{3.9}$$

then $S_\lambda = \{b, 1\}$ is a subhoop of H. But λ is not an S-quasi-valuation map of H since

$$\lambda(a \odot a) = \lambda(0) = -33 < -26 = \lambda(a) + \lambda(a).$$

Definition 3.8. A real valued function λ of H is called an *F-quasi-valuation map of H* if

$$\lambda(1) = 0, \tag{3.10}$$

$$(\forall x, y \in H)(\lambda(y) \geq \lambda(x) + \lambda(x \to y)). \tag{3.11}$$

Example 3.9. Let $H = \{0, a, b, 1\}$ be the hoop as in Example 3.3. If we define a map λ on H by

$$\lambda : H \to \mathbb{R},\ x \mapsto \begin{cases} -30 & \text{if } x = 0, \\ -25 & \text{if } x = a, \\ -20 & \text{if } x = b, \\ 0 & \text{if } x = 1, \end{cases} \tag{3.12}$$

It is routine to verify that λ is an F-quasi-valuation map of H.

Theorem 3.10. *Every F-quasi-valuation map of H is an S-quasi-valuation map of H.*

Proof. Let λ be an F-quasi-valuation map of H. Then

$$\begin{aligned} 0 = \lambda(1) &= \lambda(x \to (x \to 1)) = \lambda(x \to (x \to (y \to y))) \\ &= \lambda(x \to (y \to (x \to y))) \leq \lambda(y \to (x \to y)) - \lambda(x) \\ &\leq \lambda(x \to y) - \lambda(y) - \lambda(x), \end{aligned}$$

and so $\lambda(x \to y) \geq \lambda(x) + \lambda(y)$ for all $x, y \in H$. Hence λ is an $S_{\to}$-quasi-valuation map of H. Also, we have

$$\begin{aligned} \lambda(a \odot b) &\geq \lambda(b) + \lambda(b \to (a \odot b)) \\ &\geq \lambda(b) + \lambda(a) + \lambda(a \to (b \to (a \odot b))) \\ &= \lambda(b) + \lambda(a) + \lambda((a \odot b) \to (a \odot b)) \\ &= \lambda(b) + \lambda(a) + \lambda(1) = \lambda(b) + \lambda(a) \end{aligned}$$

for all $a, b \in H$. Hence λ is an $S_{\odot}$-quasi-valuation map of H, and therefore λ is an S-quasi-valuation map of H. $\square$

In general, any S-quasi-valuation map of H is not an F-quasi-valuation map as seen in Example 3.2(2). We provide conditions for an S-quasi-valuation map to be an F-quasi-valuation map.

Theorem 3.11. *Let λ be an $S_{\odot}$-quasi-valuation map which satisfies the condition (3.10). If λ is order preserving, then it is an F-quasi-valuation map.*

Proof. Let λ be an order preserving $S_{\odot}$-quasi-valuation map of H which satisfies the condition (3.10). Since $x \odot (x \to y) \leq y$ for all $x, y \in H$, we have

$$\lambda(y) \geq \lambda(x \odot (x \to y)) \geq \lambda(x) + \lambda(x \to y)$$

for all $x, y \in H$. Hence λ is an F-quasi-valuation map of H. $\square$

Corollary 3.12. *Every order preserving S-quasi-valuation map satisfying the condition (3.10) is an F-quasi-valuation map.*

The following example shows that any order preserving $S_{\to}$-quasi-valuation map satisfying the condition (3.10) is not an F-quasi-valuation map.

Example 3.13. Consider the hoop H as in Example 3.3 and a map

$$\lambda : H \to \mathbb{R},\ x \mapsto \begin{cases} -40 & \text{if } x = 0, \\ -15 & \text{if } x = a, \\ -10 & \text{if } x = b, \\ 0 & \text{if } x = 1, \end{cases}$$

It is routine to verify that λ is an order preserving $S_{\to}$-quasi-valuation map satisfying the condition (3.10). But it is not an F-quasi-valuation map of H since

$$\lambda(0) = -40 < -30 = \lambda(a \to 0) + \lambda(a).$$

Proposition 3.14. *For any F-quasi-valuation map λ of H, we have the following assertions.*

(1) *λ is order preserving.*
(2) $(\forall x \in H)(\lambda(x) \leq 0)$.
(3) $(\forall x, y, z \in H)\ (\lambda(x \to z) \geq \lambda(x \to y) + \lambda(y \to z))$.
(4) $(\forall x, y, x \in H)\ (\lambda(x \to (y \to z)) \geq \lambda((x \to y) \to z))$.

Proof. (1) Let $x, y \in H$ such that $x \leq y$. Then $x \to y = 1$, which implies from (3.11) that

$$\lambda(y) \geq \lambda(x) + \lambda(x \to y) = \lambda(x) + \lambda(1) = \lambda(x) + 0 = \lambda(x).$$

(2) It is by Theorem 3.10 and Proposition 3.4.
(3) For any $x, y, z \in H$, we have

$$\lambda(x \to y) \leq \lambda((y \to z) \to (x \to z)) \leq \lambda(x \to z) - \lambda(y \to z)$$

by $(a7)$, (1) and (3.11). Hence $\lambda(x \to z) \geq \lambda(x \to y) + \lambda(y \to z)$ for all $x, y, z \in H$.

(4) Since $x \odot y \leq x \to y$ for all $x, y \in H$, it follows from $(a8)$ and $(a9)$ that

$$(x \to y) \to z \leq (x \odot y) \to z = x \to (y \to z)$$

for all $x, y, z \in H$. Therefore $\lambda(x \to (y \to z)) \geq \lambda((x \to y) \to z)$ for all $x, y, z \in H$ since λ is order preserving. □

Theorem 3.15. *If $\lambda : H \to \mathbb{R}$ is an F-quasi-valuation map of H, then the set*

$$F_\lambda := \{x \in H \mid \lambda(x) = 0\} \tag{3.13}$$

is a filter of H.

Proof. Obviously $1 \in F_\lambda$ by (3.10). Let $x, y \in H$ such that $x \in F_\lambda$ and $x \to y \in F_\lambda$. Then $\lambda(x) = 0$ and $\lambda(x \to y) = 0$. It follows from (3.11) that $\lambda(y) \geq \lambda(x) + \lambda(x \to y) = 0$. By Theorem 3.10 and Proposition

3.4, we get that $\lambda(y) \leq 0$, and so $\lambda(y) = 0$. Hence, $y \in F_\lambda$. Therefore F_λ is a filter of H. $\square$

The following example shows that the converse of Theorem 3.15 is not true in general.

Example 3.16. Let $H = \{0, a, b, c, 1\}$ be a set with the Hasse diagram (Figure 1) and Cayley tables (Tables 5 and 6).

Figure 1: Hasse diagram of $(H, \leq)$

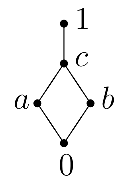

TABLE 5. Cayley table for the binary operation "$\odot$"

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

TABLE 6. Cayley table for the binary operation "$\to$"

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $(H, \odot, \to, 1)$ is a hoop (see [17]). Define a map λ on H as follows:

$$\lambda : H \to \mathbb{R},\ x \mapsto \begin{cases} -5 & \text{if } x = 0, \\ -3 & \text{if } x = a, \\ -1 & \text{if } x = b, \\ 0 & \text{if } x \in \{c, 1\}. \end{cases} \tag{3.14}$$

Then $F_\lambda = \{c, 1\}$ is a filter of H. Since $\lambda(0) = -5 < -4 = \lambda(a) + \lambda(a \to 0)$, λ is not an F-quasi-valuation map of H.

Theorem 3.17. *Let F be a filter of H. For any negative real number k, let λ_F be a real valued function on H defined by*

$$\lambda_F : H \to \mathbb{R},\ x \mapsto \begin{cases} 0 & \text{if } x \in F, \\ k & \text{otherwise.} \end{cases} \tag{3.15}$$

Then λ_F is an F-quasi-valuation map of H and $F_{\lambda_F} = F$.

Proof. It is clear that $F_{\lambda_F} = F$ and $\lambda_F(1) = 0$. Let $x, y \in H$. If $y \in F$, then

$$\lambda_F(y) = 0 \geq \lambda_F(x) + \lambda_F(x \to y).$$

Assume that $y \notin F$. Then $x \notin F$ or $x \to y \notin F$. If $x \in F$ and $x \to y \notin F$ (or $x \notin F$ and $x \to y \in F$), then $\lambda_F(x) = 0$ and $\lambda_F(x \to y) = k$ (or $\lambda_F(x) = k$ and $\lambda_F(x \to y) = 0$). Hence

$$\lambda_F(y) = k = \lambda_F(x) + \lambda_F(x \to y).$$

If $x \notin F$ and $x \to y \notin F$, then

$$\lambda_F(y) = k \geq 2k = \lambda_F(x) + \lambda_F(x \to y).$$

Therefore λ_F is an F-quasi-valuation map of H. □

Corollary 3.18. *Let F be a filter of H. For any negative real number k, let λ_F be a real valued function on H in (3.15). Then λ_F is an S-quasi-valuation map of H.*

Proposition 3.19. *Every F-quasi-valuation map λ of H satisfies the following assertion.*

$$(\forall x, y, z \in H)(z \leq y \to x \ \Rightarrow\ \lambda(x) \geq \lambda(y) + \lambda(z)). \tag{3.16}$$

Proof. Let $x, y, z \in H$ be such that $z \leq y \to x$. Then $z \to (y \to x) = 1$. Using (3.10) and (3.11), we have

$$\lambda(y \to x) \geq \lambda(z \to (y \to x)) + \lambda(z) = \lambda(1) + \lambda(z) = \lambda(z)$$

and so $\lambda(x) \geq \lambda(y \to x) + \lambda(y) \geq \lambda(y) + \lambda(z)$. □

Corollary 3.20. *Every F-quasi-valuation map λ of H satisfies the following assertion.*

$$(\forall x, y, z \in H)(z \odot y \leq x \ \Rightarrow\ \lambda(x) \geq \lambda(y) + \lambda(z)). \tag{3.17}$$

Proposition 3.21. *Every F-quasi-valuation map λ of H satisfies the following assertion.*

$$(\forall x, y \in H)(\lambda(x) + \lambda(y) \leq \lambda(x \odot y) \leq \min\{\lambda(x) - \lambda(y), \lambda(y) - \lambda(x)\}), \tag{3.18}$$

$$(\forall x, y \in H)(\lambda(x) + \lambda(y) \leq \lambda(x \wedge y) \leq \min\{\lambda(x) - \lambda(y), \lambda(y) - \lambda(x)\}), \tag{3.19}$$

$$(\forall a, b, x, y \in H)(\lambda(x \to y) + \lambda(a \to b) \leq \lambda((x \wedge a) \to (y \wedge b)), \tag{3.20}$$

$$(\forall a, b, x, y \in H)(\lambda(x \to y) + \lambda(a \to b) \leq \lambda((y \to a) \to (x \to b)). \tag{3.21}$$

Proof. Let λ be an F-quasi-valuation map of H. Then λ is an S-quasi-valuation map of H (see Theorem 3.10), and so $\lambda(x)+\lambda(y) \leq \lambda(x \odot y)$ for all $x, y \in H$. Since $x \odot (x \odot y) \leq y$ for all $x, y \in H$, it follows from Proposition 3.14(1) that $\lambda(x) + \lambda(x \odot y) \leq \lambda(y)$. Thus $\lambda(x \odot y) \leq \lambda(y) - \lambda(x)$. By the similar way, $\lambda(x \odot y) \leq \lambda(x) - \lambda(y)$. Hence $\lambda(x \odot y) \leq \min\{\lambda(y) - \lambda(x), \lambda(x) - \lambda(y)\}$, that is, (3.18) is valid. Since $x \odot y \leq x \wedge y$ for all $x, y \in H$, we have

$$\lambda(x) + \lambda(y) \leq \lambda(x \odot y) \leq \lambda(x \wedge y).$$

Also, since $x \odot (x \wedge y) \leq x \odot y \leq y$, it follows that $\lambda(x)+\lambda(x \wedge y) \leq \lambda(y)$. Thus $\lambda(x \wedge y) \leq \lambda(y) - \lambda(x)$. By the similar way, $\lambda(y)+\lambda(x \wedge y) \leq \lambda(x)$ and so $\lambda(x \wedge y) \leq \lambda(x) - \lambda(y)$. Hence

$$\lambda(x \wedge y) \leq \min\{\lambda(y) - \lambda(x), \lambda(x) - \lambda(y)\}.$$

Therefore (3.19) is true. Note that $x \to y \leq (x \wedge a) \to (y \wedge a)$ and $a \to b \leq (y \wedge a) \to (y \wedge b)$ for all $x, y, a, b \in H$. Using (1) and (3) in Proposition 3.14, we get

$$\begin{aligned}\lambda(x \to y) + \lambda(a \to b) &\leq \lambda((x \wedge a) \to (y \wedge a)) + \lambda((y \wedge a) \to (y \wedge b)) \\ &\leq \lambda((x \wedge a) \to (y \wedge b)).\end{aligned}$$

Using (1) and (3) in Proposition 3.14 and (a7), we have

$$\begin{aligned}\lambda((y \to a) \to (x \to b)) &\geq \lambda((y \to a) \to (x \to a)) + \lambda((x \to a) \to (y \to b)) \\ &\geq \lambda(x \to y) + \lambda(a \to b)\end{aligned}$$

for all $x, y, a, b \in H$. □

We provide conditions for a real valued function on H to be an F-quasi-valuation map of H.

Theorem 3.22. *Let λ be a real valued function on H satisfying the condition (3.10). If λ satisfies the condition (3.16), then it is an F-quasi-valuation map of H.*

Proof. Since $x \leq (x \to y) \to y$ for all $x, y \in H$, it follows from (3.16) that $\lambda(y) \geq \lambda(x \to y) + \lambda(x)$ for all $x, y \in H$. Hence λ is an F-quasi-valuation map of H. □

Corollary 3.23. *Let λ be a real valued function on H satisfying the condition (3.10). If λ satisfies the condition (3.16), then it is an S-quasi-valuation map of H.*

Corollary 3.24. *Let λ be a real valued function on H satisfying the condition (3.10). If λ satisfies the condition (3.17), then it is an F-quasi-valuation map of H.*

Proof. Since $x \odot (x \to y) \leq y$ for all $x, y \in H$, it follows from (3.17) that $\lambda(y) \geq \lambda(x \to y) + \lambda(x)$ for all $x, y \in H$. Hence λ is an F-quasi-valuation map of H. □

Corollary 3.25. *Let λ be a real valued function on H satisfying the condition (3.10). If λ satisfies the condition (3.17), then it is an S-quasi-valuation map of H.*

Theorem 3.26. *Let λ be a real valued function on H satisfying the condition (3.10). If λ satisfies the condition*

$$(\forall x, y, z \in H)(\lambda(x \to y) \geq \lambda(z \to (x \to (x \to y))) + \lambda(z)), \tag{3.22}$$

then λ is an F-quasi-valuation map of H.

Proof. If we take $x = 1$ and $z = x$ in (3.22) and use ($a5$), then

$$\lambda(y) = \lambda(1 \to y) \geq \lambda(x \to (1 \to (1 \to y))) + \lambda(x) = \lambda(x \to y) + \lambda(x).$$

Therefore λ is an F-quasi-valuation map of H. □

Corollary 3.27. *Let λ be a real valued function on H satisfying the condition (3.10). If λ satisfies the condition (3.22), then λ is an S-quasi-valuation map of H.*

Proposition 3.28. *Every F-quasi-valuation map λ of H satisfies the condition*

$$(\forall x, y, z \in H)(\lambda(x \to (x \to z)) \geq \lambda(x \to y) + \lambda(x \to (y \to z))). \tag{3.23}$$

Proof. Let λ be an F-quasi-valuation map of H that satisfies (3.23). Then λ is an S-quasi-valuation map of H (see Theorem 3.10). Since

$$(x \to y) \odot (x \to (y \to z)) = (x \to y) \odot (y \to (x \to z)) \leq x \to (x \to z)$$

for all $x, y, z \in H$, it follows that

$$\begin{aligned} \lambda(x \to (x \to z)) &\geq \lambda((x \to y) \odot (x \to (y \to z))) \\ &\geq \lambda(x \to y) + \lambda(x \to (y \to z)) \end{aligned}$$

for all $x, y, z \in H$. $\square$

Theorem 3.29. *If a real valued function λ on H satisfies the conditions (3.10) and (3.23), then λ is an F-quasi-valuation map of H.*

Proof. If we take $x = 1$ in (3.23) and use $(a5)$, then

$$\lambda(z) = \lambda(1 \to (1 \to z)) \geq \lambda(1 \to y) + \lambda(1 \to (y \to z)) = \lambda(y) + \lambda(y \to z)$$

for all $y, z \in H$. Hence λ is an F-quasi-valuation map of H. $\square$

Corollary 3.30. *If a real valued function λ on H satisfies the conditions (3.10) and (3.23), then λ is an S-quasi-valuation map of H.*

Theorem 3.31. *If a real valued function λ on H satisfies the conditions (3.10) and*

$$(\forall x, y, z \in H)(\lambda(x) \geq \lambda(z \to ((x \to y) \to x)) + \lambda(z)), \tag{3.24}$$

then λ is an F-quasi-valuation map of H.

Proof. If we take $y = 1$ in (3.24) and use $(a5)$, then

$$\lambda(x) \geq \lambda(z \to ((x \to 1) \to x)) + \lambda(z) = \lambda(z \to x) + \lambda(z)$$

for all $x, z \in H$. Therefore λ is an F-quasi-valuation map of H. $\square$

Corollary 3.32. *If a real valued function λ on H satisfies the conditions (3.10) and (3.24), then λ is an S-quasi-valuation map of H.*

The following example shows that there is an F-quasi-valuation map λ of H which does not satisfy the condition (3.24).

Example 3.33. Let λ be the F-quasi-valuation map of H as in Example 3.9. Then we have

$$0 = \lambda(1)+\lambda(1 \to ((b \to a) \to b)) = \lambda((b \to a) \to b)) = \lambda(a \to b) \not\leq \lambda(b) = -20$$

Given a real valued function λ on H, consider the following mapping

$$d_\lambda : H \times H \to \mathbb{R},\ (x, y) \mapsto -(\lambda(x \to y) + \lambda(y \to x)). \tag{3.25}$$

Lemma 3.34. *If a real-valued function λ on H is an F-quasi-valuation map of H, then d_λ is a pseudo-metric [1] on H, and so (H, d_λ) is a pseudo-metric space.*

We say that d_λ is the pseudo-metric introduced by an F-quasi-valuation map λ of H.

[1]By a *pseudo-metric* on H, we mean a real-valued function $d : H \times H \to \mathbb{R}$ satisfying the following properties: $d(x, y) \geq 0$, $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in H$.

Proof. If λ is an F-quasi-valuation map of H, then λ is an S-quasi-valuation map of H and so $\lambda(x) \leq 0$ for all $x \in H$ by Proposition 3.4. Thus $d_\lambda(x, y) \geq 0$ for all $x, y \in H$. It is clear that $d_\lambda(x, x) = 0$ and $d_\lambda(x, y) = d_\lambda(y, x)$ for all $x, y \in H$. Using (3) in Proposition 3.14, we get

$$\begin{aligned} d_\lambda(x, y) + d_\lambda(y, z) &= -(\lambda(x \to y) + \lambda(y \to x)) - (\lambda(y \to z) + \lambda(z \to y)) \\ &= -(\lambda(x \to y) + \lambda(y \to z)) - (\lambda(z \to y) + \lambda(y \to x)) \\ &\geq -(\lambda(x \to z) + \lambda(z \to x)) = d_\lambda(x, z). \end{aligned}$$

Hence (H, d_λ) is a pseudo-metric space. $\square$

Theorem 3.35. *If an F-quasi-valuation map λ of H satisfies the following condition*

$$(\forall x \in H)(\lambda(x) = 0 \;\Rightarrow\; x = 1), \tag{3.26}$$

then (H, d_λ) is a metric space.

Proof. Let λ be an F-quasi-valuation map of H satisfying (3.26). Then (H, d_λ) is a pseudo-metric space (see Lemma 3.34). Suppose that $d_\lambda(x, y) = 0$ for all $x, y \in H$. Then $0 = d_\lambda(x, y) = -(\lambda(x \to y) + \lambda(y \to x))$, and so $\lambda(x \to y) = 0 = \lambda(y \to x)$. It follows from (3.26) that $x \to y = 1 = y \to x$. Hence $x = y$, and therefore (H, d_λ) is a metric space. $\square$

Proposition 3.36. *If λ is an F-quasi-valuation map of H, then, for all $a, b, x, y \in H$, the pseudo-metric d_λ introduced by λ satisfies the following assertions.*

(1) $d_\lambda(x, y) \geq d_\lambda(a \to x, a \to y)$.
(2) $d_\lambda(x, y) \geq d_\lambda(x \to a, y \to a)$.
(3) $d_\lambda(x \to y, a \to b) \leq d_\lambda(x \to y, a \to y) + d_\lambda(a \to y, a \to b)$.
(4) $d_\lambda(x, y) \geq d_\lambda(a \odot x, a \odot y)$.
(5) $d_\lambda(x \odot y, a \odot b) \leq d_\lambda(x \odot y, a \odot y) + d_\lambda(a \odot y, a \odot b)$.

Proof. Let λ be an F-quasi-valuation map of H and $a, b, x, y \in H$. Since

$$(x \to y) \leq (a \to x) \to (a \to y) \text{ and } (y \to x) \leq (a \to y) \to (a \to x),$$

it follows from Proposition 3.14(1) that

$$\lambda(x \to y) \leq \lambda((a \to x) \to (a \to y)) \text{ and } \lambda(y \to x) \leq \lambda((a \to y) \to (a \to x)).$$

Hence

$$\begin{aligned} d_\lambda(x, y) &= -(\lambda(x \to y) + \lambda(y \to x)) \\ &\geq -(\lambda((a \to x) \to (a \to y)) + \lambda((a \to y) \to (a \to x))) \\ &= d_\lambda(a \to x, a \to y), \end{aligned}$$

which proves (1). Similarly, we can prove the second condition. Using Proposition 3.14(3), we have

$$\lambda((x \to y) \to (a \to b)) \geq \lambda((x \to y) \to (a \to y)) + \lambda((a \to y) \to (a \to b)),$$
$$\lambda((a \to b) \to (x \to y)) \geq \lambda((a \to b) \to (a \to y)) + \lambda((a \to y) \to (x \to y))$$

for all $a, b, x, y \in H$. Hence

$$\begin{aligned} d_\lambda(x \to y, a \to b) &= -(\lambda((x \to y) \to (a \to b)) + \lambda((a \to b) \to (x \to y))) \\ &\leq -((\lambda((x \to y) \to (a \to y)) + \lambda((a \to y) \to (a \to b))) \\ &+ \lambda((a \to b) \to (a \to y)) + \lambda((a \to y) \to (x \to y))) \\ &= d_\lambda(x \to y, a \to y) + d_\lambda(a \to y, a \to b) \end{aligned}$$

for all $a, b, x, y \in H$, which proves (3). Since $x \to y \leq (a \odot x) \to (a \odot y)$ and $y \to x \leq (a \odot y) \to (a \odot x)$ for all $a, x, y \in H$, it follows that

$$\lambda(x \to y) \leq \lambda((a \odot x) \to (a \odot y)) \text{ and } \lambda(y \to x) \leq \lambda((a \odot y) \to (a \odot x)).$$

Thus

$$\begin{aligned} d_\lambda(x, y) &= -(\lambda(x \to y) + \lambda(y \to x)) \\ &\geq -(\lambda((a \odot x) \to (a \odot y)) + \lambda((a \odot y) \to (a \odot x))) \\ &= d_\lambda(a \odot x, a \odot y) \end{aligned}$$

for all $a, x, y \in H$. This proves (4). Note that

$$((x \odot y) \to (a \odot y)) \odot ((a \odot y) \to (a \odot b)) \leq (x \odot y) \to (a \odot b),$$

for all $a, b, x, y \in H$. Thus

$$\begin{aligned} \lambda((x \odot y) \to (a \odot b)) &\geq \lambda(((x \odot y) \to (a \odot y)) \odot ((a \odot y) \to (a \odot b))) \\ &\geq \lambda((x \odot y) \to (a \odot y)) + \lambda((a \odot y) \to (a \odot b)). \end{aligned}$$

Similarly, $\lambda((a \odot b) \to (x \odot y)) \geq \lambda((a \odot b) \to (a \odot y)) + \lambda((a \odot y) \to (x \odot y))$. It follows that

$$\begin{aligned} d_\lambda(x \odot y, a \odot b) &= -(\lambda((x \odot y) \to (a \odot b)) + \lambda((a \odot b) \to (x \odot y))) \\ &\leq -(\lambda((x \odot y) \to (a \odot y)) + \lambda((a \odot y) \to (a \odot b)) \\ &+ \lambda((a \odot b) \to (a \odot y)) + \lambda((a \odot y) \to (x \odot y))) \\ &= d_\lambda(x \odot y, a \odot y) + d_\lambda(a \odot y, a \odot b) \end{aligned}$$

which proves (5). □

Let $(H_1, \odot_1, \to_1, 1_1)$ and $(H_2, \odot_2, \to_2, 1_2)$ be hoops. Define binary operations $\odot$ and $\to$ on $H_1 \times H_2$ by

$$(\forall (x, y), (a, b) \in H_1 \times H_2) \left(\begin{array}{l} (x, y) \odot (a, b) = (x \odot_1 a,\ y \odot_2 b) \\ (x, y) \to (a, b) = (x \to_1 a,\ y \to_2 b) \end{array} \right). \quad (3.27)$$

Then $(H_1 \times H_2, \odot, \to, (1_1, 1_2))$ is a hoop (see [10]).

Lemma 3.37. *Let d_λ be the pseudo-metric introduced by an F-quasi-valuation map λ of H. Then $(\mathcal{H}, d_\lambda^*)$ is a pseudo-metric space, where $\mathcal{H} := H \times H$ and*

$$d_\lambda^* : \mathcal{H} \times \mathcal{H} \to \mathbb{R}, \ ((x, y), (a, b)) \mapsto \max\{d_\lambda(x, a), d_\lambda(y, b)\}. \tag{3.28}$$

Proof. Let $(x, y), (a, b), (u, v) \in H \times H$. It is clear that $d_\lambda^*((x, y), (a, b)) \geq 0$, and we get

$$d_\lambda^*((x, y), (x, y)) = \max\{d_\lambda(x, x), d_\lambda(y, y)\} = 0,$$

$$\begin{aligned} d_\lambda^*((x, y), (a, b)) &= \max\{d_\lambda(x, a), d_\lambda(y, b)\} \\ &= \max\{d_\lambda(a, x), d_\lambda(b, y)\} \\ &= d_\lambda^*((a, b), (x, y)). \end{aligned}$$

and

$$\begin{aligned} &d_\lambda^*((x, y), (u, v)) + d_\lambda^*((u, v), (a, b)) \\ &= \max\{d_\lambda(x, u), d_\lambda(y, v)\} + \max\{d_\lambda(u, a), d_\lambda(v, b)\} \\ &\geq \max\{d_\lambda(x, u) + d_\lambda(u, a), \ d_\lambda(y, v) + d_\lambda(v, b)\} \\ &\geq \max\{d_\lambda(x, a) + d_\lambda(y, b)\} = d_\lambda^*((x, y), (a, b)). \end{aligned}$$

Therefore $(\mathcal{H}, d_\lambda^*)$ is a pseudo-metric space. □

Theorem 3.38. *Let λ be an F-quasi-valuation map of H. If λ satisfies the condition (3.26), then $(\mathcal{H}, d_\lambda^*)$ is a metric space.*

Proof. Let λ be an F-quasi-valuation map of H satisfying the condition (3.26). Then d_λ is a pseudo-metric on H (see Lemma 3.34), and so $(\mathcal{H}, d_\lambda^*)$ is a pseudo-metric space (see Lemma 3.37). Assume that $d_\lambda^*((x, y), (a, b)) = 0$ for all $(x, y), (a, b) \in H \times H$. Then

$$0 = d_\lambda^*((x, y), (a, b)) = \max\{d_\lambda(x, a), d_\lambda(y, b)\},$$

and thus $0 = d_\lambda(x, a) = -(\lambda(x \to a) + \lambda(a \to x))$ and $0 = d_\lambda(y, b) = -(\lambda(y \to b) + \lambda(b \to y))$. Hence $\lambda(x \to a) = 0 = \lambda(a \to x)$ and $\lambda(y \to b) = 0 = \lambda(b \to y)$. It follows from (3.26) that $x \to a = 1$, $a \to x = 1$, $y \to b = 1$ and $b \to y = 1$. Thus $(x, y) = (a, b)$, and therefore $(\mathcal{H}, d_\lambda^*)$ is a metric space. □

Theorem 3.39. *If an F-quasi-valuation map λ of H satisfies the condition (3.26), then the operations "$\to$", "$\odot$" and "$\wedge$" in H are uniformly continuous.*

Proof. For any $a, b, x, y \in H$ and $\varepsilon > 0$, let $d_\lambda^*((x, y), (a, b)) < \frac{\varepsilon}{2}$. Then $d_\lambda(x, a) < \frac{\varepsilon}{2}$ and $d_\lambda(y, b) < \frac{\varepsilon}{2}$. It follows from Proposition 3.36 that

$$\begin{aligned} d_\lambda(x \to y, a \to b) &\leq d_\lambda(x \to y, a \to y) + d_\lambda(a \to y, a \to b) \\ &\leq d_\lambda(x, a) + d_\lambda(y, b) \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore the operation "$\rightarrow$" in H is uniformly continuous. By Proposition 3.36, the proof of other cases is similar. $\square$

Theorem 3.39 is illustrated in the following example.

Example 3.40. The F-quasi-valuation map λ of H in Example 3.9 satisfies the condition (3.26), and so (H, d_λ) is a metric space by Theorem 3.35 where d_λ is obtained by (3.25) and it is given by Table 7.

TABLE 7. Tabular representation of "d_λ"

$H \times H$	$(0,0)$	$(0,a)$	$(0,b)$	$(0,1)$	(a,a)	(a,b)	$(a,1)$	(b,b)	$(b,1)$	$(1,1)$
$d_\lambda(x,y)$	0	25	30	30	0	25	25	0	20	0

Also, $(\mathcal{H}, d_\lambda^*)$ is a metric space by Theorem 3.38 where d_λ^* is obtained by (3.28), for example,

$$d_\lambda^*((a,b),(1,a)) = \max\{d_\lambda(a,1), d_\lambda(b,a)\} = \max\{25,25\} = 25,$$
$$d_\lambda^*((b,1),(0,b)) = \max\{d_\lambda(b,0), d_\lambda(1,b)\} = \max\{30,20\} = 30,$$

and so on. It is routine to check that the operations "$\rightarrow$", "$\odot$" and "$\wedge$" in H are uniformly continuous.

4. Conclusions

In this paper, we have introduced the notion of quasi-valuation maps such as $S_\odot$-, $S_\rightarrow$-, S- and F-quasi-valuation map based on sub-hoop and filter. We have investigated several properties, and we have discussed relations between S-quasi-valuation map and F-quasi-valuation map. Using the notion of F-quasi-valuation map, we have introduced a (pseudo) metric space and have shown that the operations "$\rightarrow$", "$\odot$" and "$\wedge$" in a hoop are uniformly continuous.

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HOOPS WITH QUASI-VALUATION MAPS

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هوپ‌ها با نگاشت‌های شبه-ارزشی

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با استفاده از مفهوم زیرجبرها و فیلترها روی هوپ‌ها، مفهوم نگاشت‌های S -شبه ارزشی و F -شبه ارزشی معرفی شده و ویژگی‌های مربوط به آن‌ها مورد بررسی قرار گرفته‌اند. همچنین، رابطه‌ی بین نگاشت‌های S -شبه ارزشی و F -شبه ارزشی مورد بررسی قرار گرفته‌است. با استفاده از مفهوم نگاشت‌های F -شبه ارزشی یک فضای متریک روی هوپ‌ها معرفی شده و ما نشان دادیم که عمل‌های $\odot$، $\to$ و $\wedge$ به صورت یکنواخت پیوسته هستند.

کلمات کلیدی: هوپ، نگاشت شبه-ارزشی، فیلتر، فضای متریک.