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GRAPHS WITH TOTAL FORCING NUMBER TWO, REVISITED

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ABSTRACT. A subset of the vertex set of a graph G is called a zero forcing set if by considering them colored and, as far as possible, a colored vertex with exactly one non-colored neighbor forces its non-colored neighbor to get colored, then the whole vertices of G become colored. The total forcing number of a graph G, denoted by $F_t(G)$, is the cardinality of a smallest zero forcing set of G which induces a subgraph with no isolated vertex. The connected forcing number, denoted by $F_c(G)$, is the cardinality of a smallest zero forcing set of G which induces a connected subgraph. In this paper, we first recharacterize the graphs with $F_t(G) = 2$ and, as a corollary, we recharacterize the graphs with $F_c(G) = 2$.

1. INTRODUCTION

Throughout the paper, for each positive number n, the symbol $[n]$ stands for the set $\{1, \ldots, n\}$. All graphs presented in this article are finite, undirected and simple. Let G be a graph. For $S \subseteq V(G)$, the graph $G - S$ denotes the subgraph obtained from G by deleting the vertices in S, the induced subgraph by $V \setminus S$. Also, for an edge $xy \in E(G)$, by $G - xy$ we mean the spanning subgraph of G obtained by removing the edge xy. For a cycle C in G, a *chord of* C is an edge of G which is not an edge of C but connects two vertices of C.

Let G be a graph and $F \subseteq V(G)$ an initial set of colored vertices while the other vertices are non-colored. The *forcing process* on G

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initiating from F is the following process. At first step, F is considered as the set of our initial precolored vertices. Each colored vertex with exactly one non-colored neighbor forces its non-colored neighbor to get colored. If, by iterating this process, all the vertices of G get finally colored, then F is called a *zero forcing set* or simply a *forcing set* of G. The smallest possible cardinality of a zero forcing set of G is called the *zero forcing number* of G, denoted by $F(G)$.

Zero forcing set was first introduced by the AIM-minimum rank group to bound the minimum rank of a graph and consequently its maximum nullity [15]. After that, zero forcing set and parameters related to it have gained much attention in graph theory, physics, logic circuits, coding theory and so on [2, 6, 12, 7].

A *total forcing set* of a graph G is a zero forcing set such that the induced subgraph by it has no isolated vertex. So, it is obvious that total forcing sets are defined for graphs with no isolated vertices. The *total forcing number* of G, denoted by $F_t(G)$, is the cardinality of a smallest total forcing set of G. Any such a total forcing set is also called an $F_t(G)$-set. Total forcing number was first introduced and studied by Davila in [11]. Davila and Henning in [8] studied total forcing sets in trees and in [9] showed that for a connected graph G of order $n \geq 3$ and maximum degree Δ, we have $F_t(G) \leq \left(\frac{\Delta}{\Delta+1}\right)n$.

A *connected forcing set* of a graph G is a zero forcing set which induces a connected subgraph. It is clear that connected forcing sets are defined for connected graphs. The *connected forcing number* of G, denoted by $F_c(G)$, is the cardinality of a smallest connected forcing set of G. Connected forcing number was first introduced and studied by Brimkov and Davila in [3]. Parameters such as maximum nullity, power domination number, leaf number and path cover number of a graph are bounded by the connected forcing number [3, 10]. In general, it is known that $F(G), F_c(G)$, and $F_t(G)$ are difficult to compute. They indeed lie in the class of NP-complete decision problems [1, 5, 9]. So, characterizing and bounding these parameters even for some special cases might be interesting. In this regard, the characterization of graphs with zero forcing number $1, 2$, and $n-1$ are done in [13] and the characterization of graphs with zero forcing number $n-2$ are presented in [15]. Brimkov and Davila in [3] characterized the graphs with connected forcing number 1 and $n-1$ and posed the following open question.

Question 1.1. *Which graphs satisfy* $F_c(G) = 2, F_c(G) = n-2, F_c(G) = 3$*, or* $F_c(G) = n-3$*?*

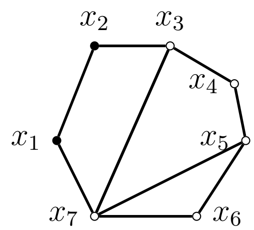

FIGURE 1. The two paths $P_1 : x_5x_6x_7$ and $P_2 : x_7x_1x_2x_3$ are section but the two paths $P' : x_3x_4x_5x_6x_7$ and $P'' : x_3x_4x_5$ are not section. The set $\{x_1, x_2\}$ is a total forcing set of this graph.

Brimkov et al. in [4] characterized the graphs with connected forcing numbers 2 and $n-2$. In this paper, we first characterize the graphs with $F_t(G) = 2$ and consequently, as a corollary, we characterize the graphs with $F_c(G) = 2$. In other words, we provide a partial answer to Question 1.1 and our proof is shorter than the proof presented in [4].

In this section, as the main objective, we characterize the graphs whose total forcing number is 2. To this end, we first define the two following families $\mathcal{F}_1$ and $\mathcal{F}_2$.

The family $\mathcal{F}_1$ consists of Hamiltonian outerplanar graphs which have no cycle going through at least three chords. For any $G \in \mathcal{F}_1$, it is known that G has a unique Hamiltonian cycle which, throughout the paper, is denoted by C_G, see [14]. A *chord* of G is a chord of C_G. Also, throughout the paper, for each graph $G \in \mathcal{F}_1$, we consider a fixed outerplanar drawing of it. A uv-path P in C_G is called a section of G, if uv is a chord and any other vertex on this path is of degree two in G, see Figure 1. Note that, in other words, a Hamiltonian outerplanar graph is in $\mathcal{F}_1$ if and only if it has at most two sections.

If G has no chord, i.e., it is a cycle, then the whole of G is considered as a section. Note that a section consists of at least three vertices. Moreover, in view of the assumption that G has no cycle going through at least three chords, each $G \in \mathcal{F}_1$ with at least one chord has two different sections. Each graph in $\mathcal{F}_2$ is obtained from a graph in $\mathcal{F}_1$ by removing one edge located in a section of it. In other words,

$$\mathcal{F}_2 = \Big\{ H \colon H = G - e \text{ where } G \in \mathcal{F}_1 \text{ and } e \in C_G \text{ is in a section of } G \Big\}.$$

The following observation is trivial due to the fact that every graph in $\mathcal{F}_1$ has no cycle going through at least three chords.

Observation 1.2. *For each $G \in \mathcal{F}_1$, every edge of C_G is in a section if and only if $|E(G)| - |E(C_G)| \leq 1$.*

By the following two lemmas, we derive some properties of the $F_t(G)$-sets of graphs in $\mathcal{F}_1 \cup \mathcal{F}_2$. These two lemmas play key roles in the proof of our main results.

Lemma 1.3. *A set $\{u, v\} \subseteq V(G)$ is an $F_t(G)$-set for a graph $G \in \mathcal{F}_1$ if and only if $uv \in E(C_G)$ and u and v are in the same section of G.*

Proof. If G has no chord, then there is nothing to prove. So, we suppose that G has at least one chord. Let $\{u, v\} \subseteq V(G)$ be an $F_t(G)$-set. Since each end point of a chord has degree at least three, uv is not a chord and hence, $uv \in E(C_G)$. Let x and y be the two first vertices with degree at least three which can be reached from u and v (including u and v themselves) traversing the cycle C_G. It is clear that xy is a chord, i.e., $xy \in E(G) \setminus E(C_G)$. Otherwise, the forcing process would stop at x and y and $\{u, v\}$ is not a zero forcing set, which is a contradiction. So u and v are in the same section.

For the reverse part of the proof, we proceed by induction on $|V(G)|$. The statement is clearly true for the graphs with 3 vertices and also for the graph with no chord. So, we assume that G is a graph with $n \geq 4$ vertices, having at least one chord, and the statement is true for any graph in $\mathcal{F}_1$ with less than n vertices. Now, consider $\{u, v\} \in E(C_G)$ such that u and v belong to the section $P : x_1x_2 \ldots x_t$ of G, where $x_i = u$ and $x_{i+1} = v$ for some $i \in [t-1]$, $deg(x_i) = 2$ for $2 \leq i \leq t-1$ and x_1x_t is a chord of G. Clearly, $\{u, v\}$ is a zero forcing set for G if and only if $\{x_1, x_t\}$ is a zero forcing set for $G' = G - \{x_2, \ldots, x_{t-1}\}$. To complete the proof, it suffices to prove that $\{x_1, x_t\}$ is a zero forcing set for G'. Note that G' is in $\mathcal{F}_1$. We claim that x_1 and x_t are in the same section of G'. To see this, let w_1 and w_2 be the two first vertices with degree at least three which can be reached from x_1 and x_t (including x_1 and x_t themselves) traversing the cycle $C_{G'}$. We need to prove that $w_1w_2 \in E(G')$. For a contradiction, suppose that w_1z_1 and w_2z_2 are two different chords of G'. Now, it is simple to see that G has a cycle going through the three chords x_1x_t, w_1z_1, and w_2z_2, which is impossible. Now, since x_1 and x_t are in the same section of G', using the induction hypothesis, $\{x_1, x_t\}$ is a zero forcing set for it, completing the proof. □

Lemma 1.4. *Let H be a graph in $\mathcal{F}_1$ and $S = z_1 \ldots z_t$ a section of it. Set $G = H - xy \in \mathcal{F}_2$, where xy is an edge in the section S.*

- **I)** *H is a cycle, i.e., it has no chord, if and only if any edge of G is a total forcing set for it.*
- **II)** *When H has some chord, $uv \in E(C_H)$ is a total forcing set for G if u and v are in a section of H different from the section S.*

Proof. First assume that H is a cycle with no chord, then it is obvious that any edge of G is a total forcing set for it. Now, for a contradiction, suppose that any edge of G is a total forcing set for it and H is a cycle with some chord. It is clear that G has a cycle and a vertex of degree one, say a. The vertex a and its neighbor must be a zero forcing set, which is impossible since G has some vertex of degree 3.

For the proof of second part, we use induction on $|V(H)|$. Since H has some chord, we have $|V(H)| \geq 4$. The statement is clearly true when $|V(H)| = 4$. Let H be a graph with $n \geq 5$ vertices and suppose that the statement is true for graphs H with less than n vertices. We know that $uv \in E(C_H)$ and the vertices u and v are in a section in H and this section is different form the section of x and y. Suppose that u and v belong to the section $P : x_1x_2 \dots x_t$ of H, where $x_i = u$ and $x_{i+1} = v$ for some $i \in [t-1]$, $\deg(x_i) = 2$ for $2 \leq i \leq t-1$ and x_1x_t is a chord of H. Clearly, $\{u, v\}$ is a forcing set for G if and only if $\{x_1, x_t\}$ is a forcing set for $G' = G - \{x_2, \dots, x_{t-1}\}$. Therefore, to complete the proof, it suffices to prove that $\{x_1, x_t\}$ is a forcing set for G'. Set $H' = H - \{x_2, \dots, x_{t-1}\}$. First note $H' \in \mathcal{F}_1$ since otherwise, H' should have a cycle going through at least 3 chords. This cycle is a cycle of H as well, which contradicts the fact that H is in F_1. When H' is a cycle $\{x_1, x_t\}$ is a forcing set for $G' = H' - xy$ as desired. So, we may assume that H' has some chords. Since $G' = H' - xy$, to use the induction hypothesis, we need to prove that x and y are in a section of H' and also, x_1 and x_t are in a section of H' which is different from the section of x and y. Since $\{x, y\} \cap \{x_2, \dots, x_{t-1}\} = \varnothing$, the section of H which has x and y (considered in the statement of the lemma as S) is still a section of H' which implies that x and y are in S as a section of H'. This also implies $G' = H' - xy \in \mathcal{F}_2$. Note that since H' is not a cycle, x_1 and x_t are not in S. So, what is left is to prove that x_1 and x_t are in the same section of H'. To see this, let w_1 and w_2 be the two first vertices with degree at least three which can be reached from x_1 and x_t (including x_1 and x_t themselves) traversing the cycle $C_{H'}$. We should prove that $w_1w_2 \in E(H')$. For a contradiction, suppose that w_1z_1 and w_2z_2 are two different chords of H'. Now, it is simple to see that H has a cycle containing three chords x_1x_t, w_1z_1 and w_2z_2, which is impossible. Now, since x_1 and x_t are in the same section of H', using the induction hypothesis, $\{x_1, x_t\}$ is a forcing set for it, completing the proof.

□

We now are in a position to prove the main result of this paper.

Theorem 1.5. *For a graph G, $F_t(G) = 2$ if and only if $G \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{P_2\}$.*

Proof. For the first part, we proceed the proof by induction on $|E(G)|$. For $|E(G)| \in \{1, 2\}$, the assertion trivially holds. Assume that the assertion is true for any graph with less than $m \geq 3$ edges. Let G be a graph with m edges, $F_t(G) = 2$, and $F = \{u, v\}$ be an $F_t(G)$-set. It is clear that $uv \in E(G)$ and at least one of the vertices u and v has degree two, say u, otherwise $\{u, v\}$ is not a zero forcing set. In the first step of the forcing process on F, the vertex u forces its unique non-colored neighbor w. Obtain the graph G' of G by removing the vertex u and adding the edge vw (if they are not already adjacent in G) that gives us $|E(G')| \leq |E(G)| - 1$. Since F is an $F_t(G)$-set, $\{w, v\}$ must be an $F_t(G')$-set. By induction, $G' \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{P_2\}$ which implies that $G \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{P_2\}$. Conversely, let $G \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{P_2\}$. When $G \in \mathcal{F}_1$, choose a section and an edge uv in that section. In view of Lemma 1.3, $\{u, v\}$ is an $F_t(G)$-set. When $G \in \mathcal{F}_2 \cup \{P_2\}$, either $G = P_n$ for $n \geq 2$ or G contains a cycle. In the first case, we clearly have the assertion. In the second case, G is in $\mathcal{F}_2$ and, in view of the definition of $\mathcal{F}_2$, there is a graph $H \in \mathcal{F}_1$ such that $G = H - xy$ as in the definition of $\mathcal{F}_2$. Choose a section different from the section containing xy (note that such a section exists since G is not a path) and an edge uv in that section. In view of Lemma 1.4, the set $\{u, v\}$ is an $F_t(G)$-set, as desired. □

In the following, as a corollary of Theorem 1.5, we characterize the graphs with connected forcing number 2.

Corollary 1.6. *For a graph G, $F_c(G) = 2$ if and only if $G \in (\mathcal{F}_1 \cup \mathcal{F}_2) \setminus \{P_n \colon n \geq 2\}$.*

Proof. Let G be a graph with $F_c(G) = 2$. It is clear that G is not a path since the connected forcing number of a path is one. Note that any connected forcing set with cardinality at least two is also a total forcing set. Therefore, $F_t(G) = 2$ and, in view of Theorem 1.5, the graph G is thus in $(\mathcal{F}_1 \cup \mathcal{F}_2) \setminus \{P_n \colon n \geq 2\}$.

Now, suppose that $G \in (\mathcal{F}_1 \cup \mathcal{F}_2) \setminus \{P_n \colon n \geq 2\}$. Since G is not a path, $F_c(G) \geq 2$. On the other hand, Theorem 1.5 implies $F_t(G) = 2$. Since any total forcing set of size 2 is also a connected forcing set, we conclude that $F_c(G) \leq 2$, which completes the proof. □

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زیر مجموعه ای از مجموعه رئوس گراف G را یک مجموعه تحمیلی صفر گوییم هرگاه با رنگی در نظر گرفتن این رئوس و اجرای فرایندی که در آن تا جای ممکن هر رأس رنگی با دقیقا یک همسایه غیر رنگی، آن همسایه غیر رنگی را تحمیل به رنگی شدن می کند، کل رئوس گراف رنگی شوند. عدد تحمیلی کلی گراف G که با $F_t(G)$ نمایش داده می‌شود اندازه کوچکترین مجموعه تحمیلی صفر است که زیر گراف القایی روی آن فاقد راس تنها باشد. عدد تحمیلی همبند که با $F_c(G)$ نمایش داده می‌شود، اندازه کوچکترین مجموعه تحمیلی صفر است که زیر گراف القایی روی آن همبند باشد. در این مقاله ما به رده‌بندی گراف‌هایی با ویژگی $F_t(G) = ۲$ می‌پردازیم و به عنوان یک نتیجه از آن، رده‌بندی گرافهایی با ویژگی $F_c(G) = ۲$ را نیز ارائه می‌دهیم.

کلمات کلیدی: مجموعه تحمیلی صفر ،عدد تحمیلی کلی، عدد تحمیلی همبند.