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GEOMETRIC HYPERGROUPS

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Abstract. The aim of this paper is to extend the notion of geometric groups to geometric hypergroups and to investigate the interaction between algebraic and geometric properties of hypergroups. In this regard, we first define a metric structure on hypergroups via word metric and present some examples on it by using generalized Cayley graphs over hypergroups. Then we study a large scale of geometry with respect to the structure of hypergroups and we prove that metric spaces of finitely generated hypergroups coming from different generating sets are quasi-isometric.

1. Introduction

Geometric group theory is the art of studying groups without using algebra. It is about using geometry to help us understand groups. In 1872, Klein proposed group theory as a means of formulating and understanding geometrical constructions. Since that time, the two subjects have been closely linked. The key idea in geometric group theory is to study groups by endowing them with a metric and treating them as geometric objects. This can be done for groups that are finitely generated (e.g. all finite groups, group of integers under addition, ...), i.e. groups that can be reconstructed from a finite subset via multiplication and inversion.

On the other hand, Graph theory is the study of mathematical objects known as graphs which consist of vertices connected by edges. A

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connection between group theory and graph theory was established in 1878 when Cayley [4] introduced Cayley graphs to describe the structure of abstract groups. Cayley graph depicts the elements of groups as vertices connected by edges taking in to consideration the set of generators used. Given a finite generating set S of a group G, one can define a metric on G by constructing a connected graph (the Cayley graph of G) with G serving as the set of vertices and the edges are labeled by elements in S. A Cayley graph $\Gamma(G;S)$ is a connected graph that admits a metric structure where the distance between two points is the length of the shortest path in the graph joining these points. The first obstacle to "geometrizing" groups in this way is the fact that a Cayley graph depends not only on the group but also on a particular choice of the finite generating set. It is known that Cayley graphs associated with different generating sets are not isometric but merely quasi-isometric.

Hypergroup theory was introduced around 80 years ago by Marty [12] as a natural extension of group theory. The law characterizing such a structure is called multi-valued operation, or hyperoperation and the theory of the algebraic structures endowed with at least one multi-valued operation is known as the hyperstructure theory or hypercompositional algebra. Marty's motivation to introduce hypergroups is that the quotient of a group modulo any of its subgroups (not necessarily normal) is a hypergroup. A hypergroup is a non-empty set of elements together with a hyperoperation that combines any two of its elements to form a non-empty set. The set and its hyperoperation must satisfy hypergroup axioms, namely associativity and reproduction axiom. The theory knew an important progress starting with the 70's, when its research area has enlarged and many applications to different fields of Sciences have been found.

A connection between hyperstructure theory and graphs was found in 2019 when Heidari et al. [9, 10] studied the concept of generalized Cayley graphs over polygroups. Moreover, the authors [1] studied generalized Cayley graphs over hypergroups and their graph product. Inspired by the work related to generalized Cayley graphs over polygroups, Arabpur et al. in 2020 [2], studied geometric polygroups by expressing a connection between finitely generated polygroups and geodesic metric spaces.

As a generalization of geometric groups and geometric polygroups, our paper discusses geometric hypergroups and it is constructed as follows: after an Introduction, Section 2 presents the basic definitions about hypergroup theory, graph theory, and metric spaces that are used throughout the paper. Section 3 discusses generalized Cayley graphs over hypergroups and presents examples of these graphs over finite and infinite hypergroups. Finally, Section 4 defines a metric structure on hypergroups and by using the notion of quasi-isometry, we prove that the geometry of finitely generated hypergroups is independent on the choice of generating sets.

2. Preliminaries

In this section, we present some definitions about graph theory [3], hypergroup theory [5, 7], and metric spaces [11] that are used throughout the paper.

A *graph* is an ordered pair $\Gamma = (V, E)$ where V is the vertex set and E is the edge set. There are many types of graphs. For example, a *simple graph* is a graph that has no loops and no multiple edges and a *connected graph* is a graph where there is a path between any pair of its vertices.

Definition 2.1. [3] A *complete graph* is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge.

Let n be a positive integer. Then a complete graph on n vertices is denoted as K_n.

Example 2.2. The complete graph K_7 on 7 vertices is presented in Figure 2.2.

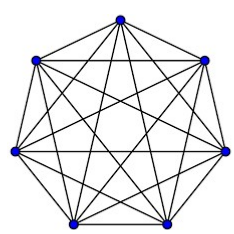

Figure 1. The complete graph K_7

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs. Then Γ_1 and Γ_2 are *isomorphic graphs* ($\Gamma_1 \cong \Gamma_2$) if there is a bijection $\phi : V_1 \to V_2$ such that $\{x, y\} \in E_1$ if and only if $\{\phi(x), \phi(y)\} \in E_2$.

Let H be a non-empty set. Then a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *binary hyperoperation* on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \; x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

Definition 2.3. [7] A hypergroupoid $(H, \circ)$ is called a *hypergroup* if for all $a, b, c \in H$, the following axioms are satisfied:

(1) **Reproduction axiom.** $a \circ H = H \circ a = H$;
(2) **Associative axiom.** $a \circ (b \circ c) = (a \circ b) \circ c$.

Example 2.4. Let H be an non-empty set and define "$\star$" on H as follows:

$$x \star y = \{x, y\} \text{ for all } x, y \in H.$$

Then $(H, \star)$ is a hypergroup. Such hypergroups are called **Biset hypergroups**.

Example 2.5. Let H be an non-empty set and define "$\cdot$" on H as follows:

$$x \cdot y = H \text{ for all } x, y \in H.$$

Then $(H, \cdot)$ is a hypergroup. Such hypergroups are called **Total hypergroups**.

Definition 2.6. [7] Let $(H, \star)$ and $(K, \star')$ be two hypergroups. Then $f : H \to K$ is said to be *hypergroup homomorphism* if $f(x \star y) = f(x) \star' f(y)$ for all $x, y \in H$. $(H, \star)$ and $(K, \star')$ are called *isomorphic hypergroups*, and written as $H \cong K$, if there exists a bijective function $f : H \to K$ that is also a homomorphism.

Definition 2.7. [5] Let $(H, \circ)$ be a hypergroup and S be a finite non-empty subset of H. Then S is called a *finitely generating set* of H, denoted as, $H =< S >$ if for every $h \in H$, there exists $n \in \mathbb{N}$ such that $h \in \underbrace{S \circ S \circ \ldots \circ S}_{n}$.

Definition 2.8. [11] A metric space is a pair (X, d) consisting of a set X and a map $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfying the following conditions: For all $x, y, z \in X$,

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 2.9. [11] Let $f : X \to Y$ be a map between the metric spaces (X, d_X) and (Y, d_Y). Then

- f is *isometric embedding* if for all $x, x' \in X$,

$$d_X(x, x') = d_Y(f(x), f(x')).$$

- f is *isometry* if it is isometric embedding and there is an isometric embedding $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$.
- Two metric spaces are *isometric* if there exists an isometry betweeen them.

Remark 2.10. An isometric embedding is isometry if and only if it is bijective.

Definition 2.11. [11] Let $f : X \to Y$ be a map between the metric spaces (X, d_X) and (Y, d_Y). Then

- f is *bilipschitz embedding* if there is a constant $c \in \mathbb{R}_{>0}$ such that for all $x, x' \in X$,
$$\frac{d_X(x, x')}{c} \leq d_Y(f(x), f(x')) \leq c d_X(x, x').$$
- f is *bilipschitz equivalence* if it is bilipschitz embedding and there is a bilipschitz embedding $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$.
- Two metric spaces are *bilipschitz equivalent* if there exists a bilipschitz equivalence betweeen them.

Remark 2.12. A bilipschitz embedding is a bilipschitz equivalence if and only if it is bijective.

Definition 2.13. [11] Let $f : X \to Y$ be a map between the metric spaces (X, d_X) and (Y, d_Y). Then

- f is *quasi-isometric embedding* if there are constants $c \in \mathbb{R}_{>0}, b \in \mathbb{R}_{\geq 0}$ such that for all $x, x' \in X$,
$$\frac{d_X(x, x')}{c} - b \leq d_Y(f(x), f(x')) \leq c d_X(x, x') + b.$$
- A map $g : X \to Y$ has finite distance from f if there exists $c \in \mathbb{R}_{\geq 0}$ such that $d_Y(f(x), g(x)) \leq c$ for all $x \in X$.
- f is *quasi-isometry* if it is quasi-isometric embedding and there is a quasi-isometric embedding $g : Y \to X$ such that $f \circ g$ has finite distance from id_Y and $g \circ f$ has finite distance from id_X.
- Two metric spaces are *quasi-isometric* if there exists a quasi-isometry betweeen them.

Remark 2.14. Every isometry is bilipschitz equivalence and every bilipschitz equivalence is quasi-isometry.

Definition 2.15. [11] Let (X, d) be a metric space. The diameter of (X, d) is defined as:
$$diam(X) = sup_{x,y \in X} d(x, y).$$

If $diam(X) < \infty$, we say that the metric space has finite diameter. Otherwise, it has infinite diameter.

3. Generalized Cayley graphs over hypergroups

In [1], the authors defined generalized Cayley graphs over finite hypergroups and investigated their properties. In this section, we extend their definition to cover infinite hypergroups and we present some examples.

Definition 3.1. [1] Let $(H, \circ)$ be any hypergroup, $S \subseteq H$ be a connection set with the property that $x \in S \circ y \Longleftrightarrow y \in S \circ x$. Then we define the *generalized Cayley graph* $GCH(H; S)$ to be the simple graph with vertex set H and edge set E given as follows:

$$E = \{\{x, y\} : x \neq y \text{ and } x \in S \circ y\}.$$

A graph Λ is called a GCH-graph if there exist a hypergroup H and a connection set S such that $\Lambda \cong GCH(H; S)$.

Notation 3.2. *Let x, y be two vertices in a graph G. We say that $x \sim y$ if there is an edge between x and y and $x \nsim y$ otherwise.*

In what follows, we provide some examples of generalized Cayley graphs over finite and infinite hypergroups. The examples on the finite case are found in [1].

Example 3.3. [1] Let $H = \{1, 2\}$ and define the hypergroup $(H, \circ)$ by the following table:

$\circ$	1	2
1	1	2
2	2	H

Then $GCH(H; \{1\})$ and $GCH(H; \{2\})$ are shown in Figure 2 and Figure 3 respectively.

Figure 2. $GCH(\{1, 2\}; \{1\})$

Remark 3.4. Different connection sets may lead to different GCH-graphs. This is easily seen in Example 3.3.

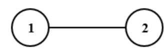

FIGURE 3. $GCH(\{1,2\};\{2\})$

Proposition 3.5. *Let H be any non-empty set and $(H,\circ)$ be the Biset hypergroup. Then the only connection set satisfying Definition 3.1 is H. Moreover, if $|H|=n$ then $GCH(H;S)\cong K_n$.*

Proof. The proof is straightforward. □

Proposition 3.6. *Let H be any non-empty set and $(H,\circ)$ be the total hypergroup. Then every non-empty subset S of H is a connection set. Moreover, if $|H|=n$ then $GCH(H;S)\cong K_n$.*

Proof. The proof is straightforward. □

Example 3.7. [1] Let $H=\{1,2,3\}$, $S=\{1\}$ be a connection set for H and $(H,\circ)$ be the hypergroup defined by the following table:

$\circ$	1	2	3
1	H	$\{1,2\}$	$\{1,3\}$
2	$\{1,2\}$	H	$\{2,3\}$
3	$\{1,3\}$	$\{2,3\}$	H

Then the generalized Cayley graph of H is shown in Figure 4.

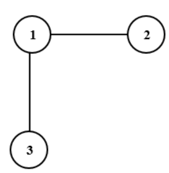

FIGURE 4. $GCH(\{1,2,3\};\{1\})$

Example 3.8. Let $(\mathbb{Z},\star)$ be the hypergroup on the set of integers $\mathbb{Z}$ where "$\star$" is defined as follows:

$$m\star n=\{m+n-1,m+n,m+n+1\}.$$

One can easily see that $S=\{-1,1\}$ is a generating set for $(\mathbb{Z},\star)$ and $GCH(H;S)$ is shown in Figure 5. Here, a,b,c,d are any consecutive integers.

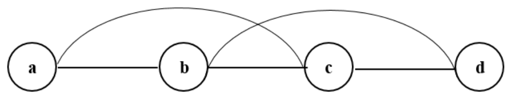

FIGURE 5. $GCH(\mathbb{Z}; \{-1, 1\})$

Example 3.9. Let $(\mathbb{Z}, \star)$ be the hypergroup on the set of integers where "$\star$" is defined as follows:

$$m \star n = m + n + 2\mathbb{Z}.$$

One can easily see that $S = \{1\}$ is a generating set for $(\mathbb{Z}, \star)$. Having

$$1 \star m = \begin{cases} 2\mathbb{Z} & \text{if } m \text{ is odd;} \\ 2\mathbb{Z} + 1 & \text{if } m \text{ is even} \end{cases}$$

implies that an even integer is connected to all odd integers and an odd integer is connected to all even integers. Thus, $GCH(H; S)$ is shown in Figure 6. Here, a, b, c are even integers and d, e, f are odd integers.

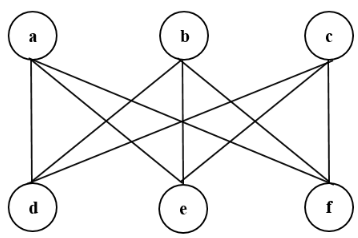

FIGURE 6. $GCH(\mathbb{Z}; \{1\})$

The necessary and sufficient condition for a GCH-graph over an infinite finitely generated hypergroup to be connected is the same as that for GCH-graph over a finite hypergroup to be connected. We illustrate this by Theorem 3.10 which has a similar proof to that for the finite case (see [1]).

Theorem 3.10. *Let $(H, \circ)$ be a hypergroup, $S \subseteq H$ be a finite connection set for H satisfying Definition 3.1. Then S is a generating set of H if and only if $GCH(H; S)$ is connected.*

Proof. Let $S = \{s_1, s_2, \ldots, s_k\}$ be a finite generating set for H and let $x, y \in H$. We need to show that there exists a path from x to y. Having

$H = < S >$ implies that $H = s_{i_1} \circ \ldots \circ s_{i_m}$ with $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, k\}$. Having $x, y \in H$ implies that $x, y \in s_{i_1} \circ \ldots \circ s_{i_m}$ which in turns implies that there exists $x_1 \in s_{i_2} \circ \ldots \circ s_{i_m}$ such that $x \in s_{i_1} \circ x_1$. Thus, $x \sim x_1$. Having $x_1 \in s_{i_2} \circ \ldots s_{i_m}$ implies that there exists $x_2 \in s_{i_3} \circ \ldots \circ s_{i_m}$ such that $x_2 \in s_{i_2} \circ x_2$. Thus, $x_1 \sim x_2$. Continuing on this pattern, we get that $x_{m-2} \in s_{i_{m-1}} \circ s_{i_m}$. Thus $x_{m-2} \sim s_{i_m}$. We get now:

$$x \sim x_1 \sim x_2 \sim \ldots \sim x_{m-2} \sim s_{i_m}.$$

Similarly, we can obtain the same thing for y, i.e.,

$$y \sim y_1 \sim y_2 \sim \ldots \sim y_{m-2} \sim s_{i_m}.$$

Our path from x to y is given as follows:

$$x \sim x_1 \sim x_2 \sim \ldots \sim x_{m-2} \sim s_{i_m} \sim y_{m-2} \sim \ldots \sim y_1 \sim y.$$

Conversely, let $GCH(H; S)$ be a connected graph, $S = \{s_1, s_2, \ldots, s_k\}$ be a connection set for H and $x \in H$. Since $GCH(H; S)$ is a connected graph, it follows that there exists a path from x to s_k, say

$$x \sim x_1 \sim x_2 \ldots \sim x_r \sim s_k.$$

Having $x_r \sim s_k$ implies that there exist $s_{j_r} \in S$ such that $x_r \in s_{j_r} \circ s_k$. And having $x_{r-1} \sim x_r$ implies that there exist $s_{j_{r-1}} \in S$ such that $x_{r-1} \in s_{j_{r-1}} \circ x_r$. The latter and using the associativity of "$\circ$" imply that $x_{r-1} \in s_{j_{r-1}} \circ s_{j_r} \circ s_k$. Continuing on this pattern, we get that $x \in s_{j_1} \circ \ldots \circ s_{j_r} \circ s_k \subseteq < S >$. Thus, S is a generating set for H. □

4. Quasi-isometry types of hypergroups

In this section, we define the word metric on hypergroups, provide some examples, and prove some properties related to isometry, bilipschitz equivalence, and quasi-isometry.

Let $(H, \circ)$ be a hypergroup with a finite generating set S and define $d_S : H \times H \to \mathbb{R}_{\geq 0}$ as follows:

$$d_S(x, y) = \begin{cases} 0 & \text{if } x = y; \\ \min\{n \in \mathbb{N} : x \in s_1 \circ s_2 \circ \ldots \circ s_n \circ y\} & \text{otherwise.} \end{cases}$$

It is clear that $d_S(x, y) \geq 0$.

Proposition 4.1. *Let $(H, \circ)$ be a hypergroup with a finite generating set S and $x, y \in H$. Then $d_S(x, y) = 0$ if and only if $x = y$.*

Proof. If $x = y$ then $d_S(x, y) = 0$ by definition of d_S. Let $x \neq y$. Since S is a generating set of $(H, \circ)$, it follows by the connectivity of $GCH(H; S)$ (Theorem 3.10) that there is a path from x to y. Thus, $d_S(x, y) \geq 1$. □

Proposition 4.2. *Let $(H, \circ)$ be a hypergroup with a finite generating set S and $x, y, z \in H$. Then $d_S(x, z) \leq d_S(x, y) + d_S(y, z)$.*

Proof. If $x = y$ or $y = z$ then the conclusion is clear. We suppose that $x \neq y$ and $y \neq z$. Let $x, y, z \in H$ with $d_S(x, y) = k$ and $d_S(y, z) = l$. Then there exist

$$s_1, s_2, \ldots, s_k, s_{k+1}, s_{k+2}, \ldots, s_{k+l} \in S$$

such that $x \in s_1 \circ s_2 \circ \ldots \circ s_k \circ y$ and $y \in s_{k+1} \circ s_{k+2} \circ \ldots \circ s_{k+l} \circ z$. We get that $x \in s_1 \circ s_2 \circ \ldots \circ s_k \circ s_{k+1} \circ s_{k+2} \circ \ldots \circ s_{k+l} \circ z$. The latter implies that $d_S(x, z) \leq k + l$. Therefore, $d_S(x, z) \leq d_S(x, y) + d_S(y, z)$. □

Proposition 4.3. *Let $(H, \circ)$ be a hypergroup with a finite generating set S and $x, y \in H$ with $d_S(x, y) = n$. If $\{s_i\}_{i=1}^n$ is the shortest chain such that $x \in s_1 \circ \ldots s_n \circ y$ and $y' \in s_n \circ y$ then $d_S(x, y') = n - 1$.*

Proof. It is clear that $d_S(y, y') = 1$. If not, then $y = y'$ and $d_S(x, y) = d_S(x, y') \leq n - 1$. Suppose, to get contradiction, that $d_S(x, y') = k < n - 1$. Proposition 4.2 asserts that $d_S(x, y) \leq d_S(x, y') + d_S(y', y) \leq k + 1 < n$. □

Proposition 4.4. *Let $(H, \circ)$ be a hypergroup with a finite generating set S and $x, y \in H$. Then $d_S(x, y) = d_S(y, x)$.*

Proof. If $d_S(x, y) = 0$ then $d_S(y, x) = d_S(x, y) = 0$. And if $d_S(x, y) = 1$ then there exist $s \in S$ such that $x \in s \circ y$. The latter implies that there exist $s' \in S$ such that $y \in s' \circ x$. Since $y \neq x$, it follows that $d_S(y, x) = 1$. Assume that $d_S(x, y) = m$ and $d_S(y, x) = n$ where m and n are natural numbers greater than one. We prove that $n \leq m$ and the proof of $m \leq n$ can be done in a similar way. Let $\{s_i\}_{i=1}^m$ be the shortest chain such that $x \in s_1 \circ \ldots \circ s_m \circ y$. Then there exists $y_1 \in s_2 \circ \ldots \circ s_m \circ y$ such that $x \in s_1 \circ y_1$. The latter implies that there exist $s'_1 \in S$ with $y_1 \in s'_1 \circ x$. Thus, $d_S(y_1, x) \leq 1$. Having $y_1 \in s_2 \circ \ldots \circ s_m \circ y$ implies that there exist $y_2 \in s_3 \circ \ldots \circ s_m \circ y$ with $y_1 \in s_2 \circ y_2$. The latter implies that $s'_2 \in S$ with $y_2 \in s'_2 \circ y_1$. Thus, $d_S(y_2, y_1) \leq 1$. Continuing on this pattern, we get that there exist $y_{m-1} \in s_m \circ y$ with $y_{m-2} \in s_{m-1} \circ y_{m-1}$. The latter implies that there exist $s'_{m-1} \in S$ with $y_{m-1} \in s'_{m-1} \circ y_{m-2}$. Thus, $d_S(y_{m-1}, y_{m-2}) \leq 1$. Proposition 4.2 asserts that $d_S(y, x) \leq d_S(y, y_{m-1}) + d_S(y_{m-1}, y_{m-2}) + \ldots + d_S(y_2, y_1) + d_S(y_1, x) \leq m \leq d_S(x, y)$. By interchanging the roles of x and y, we get that $d_S(x, y) \leq d_S(y, x)$. Therefore, $d_S(x, y) = d_S(y, x)$. □

Theorem 4.5. *Let $(H, \circ)$ be a hypergroup with a finite generating set S. Then (H, d_S) is a metric space.*

Proof. The proof follows from Propositions 4.1, 4.2 and 4.4. □

Let $(H, \circ)$ be a hypergroup and S be a generating set of it. We define d on $GCH(H; S)$ as follows: For vertices v_g, v_h corresponding to the elements $g, h \in H$ respectively,

$$d(v_g, v_h) = \text{length of the shortest path from } v_g \text{ to } v_h.$$

Proposition 4.6. *Let $(H, \circ)$ be a hypergroup and S be a generating set of it. Then $d_S(g, h) = d(v_g, v_h)$ for all $g, h \in H$.*

Proof. Let $d_S(g, h) = n$ and $d(v_g, v_h) = k$. Having $d(v_g, v_h) = k$ implies that we can find a path of length k from v_g to v_h, say $v_g \sim v_{x_1} \sim v_{x_2} \sim \ldots \sim v_{x_{k-1}} \sim v_h$. Having $v_g \sim v_{x_1}$ implies that there exist $s_1 \in S$ such that $g \in s_1 \circ x_1$ and having $v_{x_{i-1}} \sim v_{x_i}$ for $i = 1, \ldots, k-2$ implies that there exist $s_i \in S$ such that $x_{i-1} \in s_i \circ x_i$. Moreover, $v_{x_{k-1}} \sim v_h$ implies that there exist $s_k \in S$ such that $x_{k-1} \in s_k \circ h$. We get that $g \in s_1 \circ s_2 \circ \ldots \circ s_k \circ h$. Thus, $d_S(g, h) \leq k$.

Having $d_S(g, h) = n$ implies that there exist $s_1, \ldots, s_n \in S$ such that $g \in s_1 \circ \ldots \circ s_n \circ h$. The latter implies that there exist $g_1 \in s_2 \circ \ldots s_n \circ h$ such that $g \in s_1 \circ g_1$. Having $g_1 \in s_2 \circ \ldots s_n \circ h$ implies that there exist $g_2 \in s_3 \circ \ldots s_n \circ h$ such that $g_1 \in s_2 \circ g_3$. Continuing on this pattern, we get $v_g \sim v_{g_1} \sim \ldots v_{g_{n-1}} \sim v_h$. Thus, $d(v_g, v_h) \leq n$. ☐

Example 4.7. Let $(H, \circ)$ be the hypergroup in Example 3.7. Then (H, d_S) is a metric space where $d_S(1, 2) = d_S(1, 3) = 1$ and $d_S(2, 3) = 2$.

Example 4.8. Let $(H, \circ)$ be the total hypergroup or Biset hypergroup. Then (H, d_S) is the discrete metric space. i.e.

$$d_S(x, y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{otherwise.} \end{cases}$$

Example 4.9. Let $(\mathbb{Z}, \star)$ be the hypergroup defined in Example 3.9. Then $(\mathbb{Z}, d_S)$ is a metric space. Here, d_S is defined as follows:

$$d_S(x, y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{if } x \text{ and } y \text{ have different parities;} \\ 2 & \text{if } x \neq y \text{ have same parity.} \end{cases}$$

Example 4.10. Let $(H, \circ)$ be the hypergroup defined in Example 3.3 and $S = \{2\}$ be its connection set. Having $d_S(1, 2) = 0$ implies that (H, d_S) is not a metric space.

The following theorem finds a necessary and sufficient condition for (H, d_S) to be a metric space.

Theorem 4.11. *Let $(H, \circ)$ be a hypergroup, $S \subseteq H$ be a finite connection set for H satisfying Definition 3.1. Then S is a generating set of H if and only if (H, d_S) is a metric space.*

Proof. The proof is straightforward by means of Theorem 3.10 and Proposition 4.6. □

Definition 4.12. Let $(H, \circ)$ be a hypergroup with finite generating set S and $x \in S$. Then

$$|x|_S = \begin{cases} \min\{n \in \mathbb{N} : x \in s_1 \circ \ldots \circ s_n\} & \text{if } x \notin S; \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.13. Such an n exists as $|S| < \infty$.

Example 4.14. Let $(H, \circ)$ be the hypergroup in Example 3.7. Then $|1|_S = 0$ and $|2|_S = |3|_S = 2$.

Example 4.15. If $(H, \circ)$ is the total hypergroup then

$$|x|_S = \begin{cases} 2 & \text{if } x \notin S; \\ 0 & \text{otherwise.} \end{cases}$$

And if $(H, \circ)$ is the Biset hypergroup then $|x|_S = 0$ for all $x \in H$.

Example 4.16. Let $(\mathbb{Z}, \star)$ be the hypergroup defined in Example 3.9. Then

$$|m|_S = \begin{cases} 0 & \text{if } m = 1; \\ 1 & \text{if } m \text{ is even;} \\ 2 & \text{if } m \neq 1 \text{ is odd.} \end{cases}$$

Remark 4.17. Different generating sets lead to different metric spaces. By considering the hypergroup in Example 3.7, $d_S(2, 3) = 2 \neq d_H(2, 3) = 1$.

Example 4.18. Let $(H, \circ)$ be the hypergroup defined in Example 3.7. One can easily see that $(H, d_{\{2\}})$ and (H, d_H) are not isometric metric spaces. This is clear as (H, d_H) is the discrete metric space and $d_{\{2\}}(2, 3) = 2$.

Proposition 4.19. *Let $(H_1, \circ_1)$ and $(H_2, \circ_2)$ be isomorphic finitely generated hypergroups with $f : H_1 \to H_2$ an isomorphism and S be a generating set for $(H_1, \circ_1)$. Then $f(S)$ is a generating set for $(H_2, \circ_2)$.*

Proof. Let $S = \{s_1, \ldots, s_n\}$ be a generating set for $(H_1, \circ_1)$ and let $y \in H_2$. Since f is bijective, it follows that there exists $x \in H_1 =< S >$ such that $y = f(x)$. The latter implies that there exist $s_{mi} \in S$ with$1 \leq i \leq l$, $1 \leq mi \leq k$ such that $x \in s_{m1} \circ_1 \ldots \circ_1 s_{ml}$. And having f a homomorphism implies that

$$y = f(x) \in f(s_{m1} \circ_1 \ldots \circ_1 s_{ml}) = f(s_{m1}) \circ_2 \ldots \circ_2 f(s_{ml}).$$

Thus, $f(S)$ is a generating set for $(H_2, \circ_2)$. □

Proposition 4.20. *Let* $(H_1, \circ_1)$ *and* $(H_2, \circ_2)$ *be isomorphic finitely generated hypergroups with* $f : H_1 \to H_2$ *an isomorphism and* S *be a generating set for* $(H_1, \circ_1)$. *Then* (H_1, d_S) *and* $(H_2, d_{f(S)})$ *are isometric metric spaces.*

Proof. Since f is a bijective function and we need to prove that f is isometry, it suffices to show that $d_S(x, y) = d_{f(S)}(f(x), f(y))$ for all $x, y \in H_1$. For all $a, b \in H_2$ there exist $x, y \in H_1$ with $f(x) = a, f(y) = b$. Let $d_S(x, y) = m$ and $d_{f(S)}(a, b) = n$. Then there exist $s_1, \ldots, s_m \in S$ with $x \in s_1 \circ_1 \ldots \circ_1 s_m \circ_1 y$. Having f a homomorphism implies that $a = f(x) \in f(s_1 \circ_1 \ldots \circ_1 s_m \circ_1 y) = f(s_1) \circ_2 \ldots \circ_2 f(s_m) \circ_2 b$. The latter implies that $n \le m$. Having $d_{f(S)}(a, b) = n$ implies that there exist $s_1, \ldots, s_n \in S$ with $a \in f(s_1) \circ_2 \ldots \circ_2 f(s_n) \circ_2 b$. The latter and having f a homomorphism implies that $f(x) \in f(s_1 \circ_1 \ldots \circ_1 s_n \circ_1 y)$. Thus, $x \in s_1 \circ_1 \ldots \circ_1 s_n \circ_1 y$ and hence $m \le n$. □

Remark 4.21. Isomorphic hypergroups may not be isometric. This is clear from Example 4.18.

Theorem 4.22. *Let* $(H_1, \circ_1)$ *and* $(H_2, \circ_2)$ *be finitely generated hypergroups with generating sets* S, T *respectively. Then* (H_1, d_S) *and* (H_2, d_T) *are isometric if and only if* $GCH(H_1; S)$ *and* $GCH(H_2; T)$ *are isomorphic graphs.*

Proof. Let $GCH(H_1; S)$ and $GCH(H_2; T)$ be isomorphic graphs. Then there exist a bijection $\phi : V(H_1) \to V(H_2)$ with the property that if $v_x \sim v_y$ then $\phi(v_x) \sim \phi(v_y)$ for all vertices v_x, v_y representing $x, y \in H_1$. Let $f : H_1 \to H_2$ be defined as follows:

$$f(a) = b \text{ whenever } \phi(v_a) = v_b.$$

It is clear that f is a bijection. Proposition 4.6 asserts that $d_S(a, x) = d(v_a, v_x)$. And having $d(v_a, v_x) = d(\phi(v_a), \phi(v_x)) = d_T(f(a), f(x))$ completes the proof.

Conversely, let (H_1, d_S) and (H_2, d_T) be isometric. Then there exists a bijective isometric embedding $f : H_1 \to H_2$. Let $\phi : V(H_1) \to V(H_2)$ be defined as follows:

$$\phi(v_a) = v_b \text{ whenever } f(a) = b.$$

Proposition 4.6 asserts that $d(v_a, v_x) = d_S(a, x)$. The latter implies that if $v_a \sim v_x$ then $d_S(a, x) = 1$. And having f an isometry implies that $d_T(f(a), f(x)) = 1$. Thus, $\phi(v_a) \sim \phi(v_x)$ and hence, $GCH(H_1; S)$ and $GCH(H_2; T)$ are isomorphic graphs. □

Example 4.23. The Biset hypergroup on n elements and the total hypergroup on n elements are isometric. This is clear since their generalized Cayley graph is K_n.

Remark 4.24. Finite hypergroups of different cardinalities are not isometric.

From Proposition 4.20, Remark 4.21, Theorem 4.22 and Remark 4.24, we can see that the notion of isometry is too rigid. We want a notion of similarity between metric spaces. In general, the word metric on a given hypergroup depends on the chosen set of generators. However, the difference is negligible when looking at the hypergroup from far away, i.e. by using a large scale geometry. In other words, the geometry of finitely generated hypergroups is independent on the choice of the generating sets. This idea is illustrated in Theorem 4.25.

Theorem 4.25. *Let $(H, \circ)$ be a hypergroup and S and T be finite generating sets for it. Then (H, d_S) and (H, d_T) are bilipschitz equivalent.*

Proof. Let $f : H \to H$ be the identity map and $x, y \in H$ with $d_S(x, y) = k$. If $k = 0$ then $d_S(x, y) = d_T(x, y)$. We assume that $k \neq 0$. This implies that there exist $s_1, \ldots, s_k \in S$ such that $x \in s_1 \circ \ldots \circ s_k \circ y$. Since $(H, \circ)$ is finitely generated by T and $s_i \in S$ for $i = 1, 2, \ldots, k$, it follows that there exist $t_{ij} \in T$, $1 \leq i \leq k$, $1 \leq j \leq n_i$ such that $s_i \in t_{i1} \circ \ldots \circ t_{in_i}$. Thus, $x \in t_{11} \circ \ldots \circ t_{1n_1} \circ \ldots \circ t_{k1} \circ \ldots \circ t_{kn_k} \circ y$. The latter implies that $d_T(x, y) \leq |s_1|_T + \ldots + |s_k|_T \leq kM$ where $M = \max\{|s_i|_T : 1 \leq i \leq k\}$. We get now that

$$\frac{d_T(x, y)}{M} \leq d_S(x, y). \qquad (1)$$

In a similar manner, we get

$$d_S(x, y) \leq N d_T(x, y) \qquad (2)$$

where $N = \max\{|t_i|_S : 1 \leq i \leq l\}$. By setting $M^\star = \max\{M, N\}$ and using (1) and (2), we get

$$\frac{d_T(x, y)}{M^\star} \leq d_S(x, y) \leq M^\star d_T(x, y).$$

We get now that f is bilipschitz embedding. And having f a bijective function implies that the two metric spaces are bilipschitz equivalent. □

Corollary 4.26. *Let $(H, \circ)$ be a hypergroup and S and T be finite generating sets for it. Then (H, d_S) and (H, d_T) are quasi-isometric.*

Proof. The proof follows from Theorem 4.25 and the fact that every bilipschitz equivalence is quasi-isometry. □

Corollary 4.27. *Let $(H, \circ)$ be a hypergroup and S and T be finite generating sets for it. Then every metric space that is bilipschitz equivalent (or quasi-isometric) to (H, d_S) is also bilipschitz equivalent (or quasi-isometric) to (H, d_T).*

Proof. The proof follows from Theorem 4.25, Corollary 4.26, and the fact that composition of bilipschitz equivalences (quasi-isometries) is bilipschitz equivalence (quasi-isometry). □

Theorem 4.28. *Let $(H_1, \circ_1)$ and $(H_2, \circ_2)$ be hypergroups with finite generating sets S and T respectively. If (H_1, d_S) and (H_2, d_T) are metric spaces with finite diameters then (H_1, d_S) and (H_2, d_T) are quasi-isometric.*

Proof. Let $f : H_1 \to H_2$, $g : H_2 \to H_1$ be any functions, $k_1 = diam(H_1, d_S)$, $k_2 = diam(H_2, T)$, and $k = \max\{k_1, k_2\}$. It is clear that

$$d_S(x, y) - k \leq d_T(f(x), f(y)) \leq d_S(x, y) + k$$

and

$$d_T(x, y) - k \leq d_S(g(x), g(y)) \leq d_T(x, y) + k.$$

Thus, f and g are quasi-isometric embedding. Since $d_S(g \circ f(x), x) \leq k_1$ and $d_T(f \circ g(x), x) \leq k_2$, it follows that H_1 and H_2 are quasi-isometric. □

Corollary 4.29. *Finite hypergroups are quasi-isometric.*

Proof. Since finite hypergroups have finite diameters, it follows by Theorem 4.28 that finite hypergroups are quasi-isometric. □

Corollary 4.30. *Every finite hypergroup is quasi-isometric to the trivial hypergroup.*

Proof. The proof follows from Corollary 4.29. □

Remark 4.31. It is known that a finitely generated group is quasi-isometric to a finite group if and only if it is finite. This result in geometric group theory does not hold in geometric hypergroup theory.

We illustrate Remark 4.31 via the following example.

Example 4.32. Let $(G, \star)$ be the trivial hypergroup i.e. $G = \{0\}$, $(H, \circ)$ be the total hypergroup on the infinite set $\{1, 2, \ldots\}$ and $T = \{0\}$, $S = \{1\}$ be their generating sets respectively. Let $f : H \to G$ and $g : G \to H$ be the functions defined by $f(x) = 0$ for all $x \in H$ and $g(0) = 1$. Since $d_S(x, y) - 1 \leq d_T(0, 0) \leq d_S(x, y) + 1$ and $d_T(0, 0) - 1 \leq d_S(x, y) \leq d_T(0, 0) + 1$, it follows that f and g are quasi-isometric embedding. Having $g \circ f$ and $f \circ g$ finite distances from the identity implies that G and H are quasi-isometric.

In geometric group theory, finite groups are bilipschitz equivalent if and only if they have the same number of elements. The same result holds in geometric hypergroup theory.

Theorem 4.33. *Finite hypergroups are bilipschitz if and only if they have the same number of elements.*

Proof. Let $H_1 = \{a_1, \ldots, a_n\}$, $H_2 = \{b_1, \ldots, b_n\}$, and $f : H_1 \to H_2$ be a function defined as $f(a_i) = b_i$ for $i = 1, \ldots, n$. Since H_1 and H_2 are finite then they have finite diameters k_1 and k_2 respectively. By setting $k = \max\{k_1, k_2\}$, we get:

$$\frac{d_S(a_i, a_j)}{k} \leq d_T(b_i, b_j) \leq k d_S(a_i, a_j).$$

And having f a bijective function implies that H_1 and H_2 are bilipschitz equivalent.

Conversely, let H_1 and H_2 be bilipschitz equivalent. Then there is a bijective function from H_1 to H_2. Since H_1 and H_2 are finite sets, it follows that they have the same number of elements. □

Example 4.34. The Biset hypergroup on 3 elements, the total hypergroup on 3 elements, and the hypergroup in Example 3.7 are bilipschitz equivalent.

5. Conclusion

One of the objectives for geometric hypergroup theory is to view hypergroups as geometric objects so that one can study hypergroups without studying hyperstructures. This paper studied a connection between hyperstructures and geometry by introducing geometric hypergroups. Many known results in geometric group theory were proved to be valid for geometric hypergroup theory. More precisely, it was shown that metric spaces of finitely generated hypergroups coming from different generating sets are quasi-isometric.

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GEOMETRIC HYPERGROUPS

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ابرگروه های هندسی

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هدف این مقاله تعمیم دادن گروه‌های هندسی به ابرگروه‌های هندسی و بررسی اثر متقابل بین خاصیت‌های جبری و هندسی ابرگروه‌هاست. در این راستا، ابتدا یک ساختار متریک روی ابرگروه‌ها تعریف کرده و با استفاده از گراف‌های کیلی تعمیم‌یافته روی ابرگروه‌ها، مثال‌هایی را ارائه می‌دهیم. سپس مفاهیمی از هندسه را با توجه به ساختار ابرگروه‌ها مطالعه کرده و ثابت می‌کنیم که فضاهای متریک ابرگروه‌های متناهیاً تولید شده که از مجموعه‌های مختلف تولید می‌شوند، شبه ایزومتریک هستند.

کلمات کلیدی: گراف کیلی، ابرگروه، ابرگروه هندسی.