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FINITENESS PROPERTIES OF FORMAL LOCAL COHOMOLOGY MODULES

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Abstract. Let $\mathfrak{a}$ be an ideal of Noetherian local ring $(R, \mathfrak{m})$, M a finitely generated R-module. In this paper, we prove some results concerning finiteness of formal local cohomology modules. In particular, we investigate some properties of top formal local cohomology module $\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)$. Among other things, we determine $\operatorname{Att}_R(\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M))$, in the case that $\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)$ is an artinian R-module. Also, we show that $\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)$ is an artinian R-module if and only if it is minimax.

1. Introduction

Throughout this paper, $(R, \mathfrak{m})$ is a commutative Noetherian local ring with identity, $\mathfrak{a}$ is an ideal of R and M is a finitely generated R-module. Recall that, the i-th local cohomology module of M with respect to $\mathfrak{a}$ is denoted by $\mathrm{H}_{\mathfrak{a}}^i(M)$. For basic facts about commutative algebra see [4], [6]; for local cohomology refer to [3]. Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. For each $i \geq 0$; $\mathfrak{F}_{\mathfrak{a}}^i(M) := \varprojlim_n \mathrm{H}_m^i(M/\mathfrak{a}^n M)$ is called the i-th formal local cohomology of M with respect to $\mathfrak{a}$.

The basic properties of formal local cohomology modules are found in [1], [2], [5] and [9].

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In this paper, we investigate some artinianness and finiteness properties of formal local cohomology modules. At first, we obtain a relation between attached primes of artinian formal local cohomology modules and attached primes of local cohomology modules, see Theorem 2.3 below. Then by using it, we determine attached primes of top formal local cohomology module $\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)$ and we show that

$$\operatorname{Att}_R(\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)) = \operatorname{Assh}(M/\mathfrak{a}M) = \operatorname{Min} \operatorname{V}(\operatorname{Ann}_R(\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)).$$

Note that, by [9, Theorem 4.5], $l := \dim M/\mathfrak{a}M$ is the largest integer i such that $\mathfrak{F}_{\mathfrak{a}}^{i}(M) \neq 0$.

In the second main result, we investigate a relation between coassociated primes of finitely generated formal local cohomology modules and attached primes of local cohomology modules, see Theorem 2.16 below. By applying this result, we show that for any ideal $\mathfrak{b}$, if $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M) \neq 0$ then $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)$ is not finitely generated. Also, we prove that $\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)$ is artinian if and ony if it is minimax. Recall that an R-module M is called minimax, if there is a finite submodule N of M such that M/N is Artinian (see [12]).

2. Main results

In this section, we obtain some results about artinianness and finiteness properties of formal local cohomology modules. We show that for any ideal $\mathfrak{b}$ and any integer i, if $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is artinian then the set $\operatorname{Coass}_R \mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is a subset of $\cup_{k\in\mathbb{N}} \operatorname{Att}_R(\operatorname{H}_{\mathfrak{m}}^{i}(M/\mathfrak{a}^k M))$. By using this result, we show that if $\mathfrak{F}_{\mathfrak{a}}^{0}(M)$ is artinian, then it is finitely generated. Also, we determine attached primes of top formal local cohomology module $\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)$. For two ideals $\mathfrak{a}$ and $\mathfrak{b}$ and any integer i, we obtain some properties of $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{i}(M)$. For example, we prove that if $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M) \neq 0$ then it is not finitely generated. Finally, it is shown that $\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)$ is minimax if and only if it is artinian.

We first recall the concept of coassociated primes, cosupport and attached primes of an R-module M. A module is called cocyclic if it is a submodule of $E(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ of R. A prime ideal $\mathfrak{p}$ is called coassociated to a non-zero R-module M if there is a cocyclic homomorphic image T of M with $\mathfrak{p} = \operatorname{Ann}_R T$ [10]. The set of coassociated primes of M is denoted by $\operatorname{Coass}_R(M)$. Also, Yassemi [10] defined the cosupport of an R-module M, denoted by $\operatorname{Cosupp}_R(M)$, to be the set of primes $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\operatorname{Ann}_R(L) \subseteq \mathfrak{p}$. In [10] we can see that $\operatorname{Coass}_R(M) \subseteq \operatorname{Cosupp}_R(M)$ and every minimal element of the set $\operatorname{Cosupp}_R(M)$ belongs to $\operatorname{Coass}_R(M)$. Recall that, a prime ideal $\mathfrak{p}$ of R is said to be an attached prime of M if $\mathfrak{p} = \mathrm{Ann}_R(M/N)$ for some submodule N of M. If M has a secondary representation, this definition agrees with the usual definition of attached primes and $\mathrm{Coass}_R(M) = \mathrm{Att}_R(M)$, ([10, Theorem 1.14]).

In what follows, we always assume that $(R, \mathfrak{m})$ is a Noetherian local ring.

The following lemma is used in the sequel.

Lemma 2.1. *Let $\mathfrak{a}$ be an ideal of R and M an R-module. If $\mathfrak{a}^k M = 0$ for some $k \in \mathbb{N}$, then $\mathfrak{F}^i_{\mathfrak{a}}(M) \cong \mathrm{H}^i_{\mathfrak{m}}(M)$ for all $i \geq 0$. Therefore $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is artinian for all $i \geq 0$.*

Proof. It is clear that $\mathfrak{F}^i_{\mathfrak{a}}(M) \cong \varprojlim_n \mathrm{H}^i_m(M/\mathfrak{a}^n M) \cong \mathrm{H}^i_m(M)$. But by [3, Theorem 7.1.3] $\mathrm{H}^i_{\mathfrak{m}}(M)$ is artinian for all $i \geq 0$ and so the proof is complete. □

Next Lemma plays a significant role in our proofs. It is well known that, if $\{M_i\}_{i \in I}$ is a direct system of R-modules then $\mathrm{Ass}_R(\varinjlim_i M_i) \subseteq \cup \mathrm{Ass}_R M_i$. In the following, by using the Matlis Duality functor, we prove the following duality result. Recall that, for an R-module M, $\mathrm{E}(R/\mathfrak{m})$ denotes the injective envelope of $R/\mathfrak{m}$ and $\mathrm{D}(.)$ denotes the Matlis duality functor $\mathrm{Hom}_R(., \mathrm{E}(R/\mathfrak{m}))$. It is well known that, $\mathrm{Ass}_R \mathrm{D}(M) = \mathrm{Coass}_R M$ and if A is an artinian R-module then $A \simeq \mathrm{D}\,\mathrm{D}(A)$, (see [10]).

Lemma 2.2. *Let $\{M_i\}_{i \in I}$ be an inverse system of artinian R-modules and $\mathfrak{a}$ be an ideal of R. Then*

i) If $\mathfrak{a}\varprojlim_i M_i$ is artinian, then $\mathrm{Coass}_R(\mathfrak{a}\varprojlim_i M_i) \subseteq \cup \mathrm{Coass}_R M_i$,

ii) If $\varprojlim_i M_i$ is artinian, then $\mathrm{Coass}_R(\varprojlim_i M_i) \subseteq \cup \mathrm{Coass}_R M_i$.

Proof. i) Since M_i is artinian $M_i \simeq \mathrm{D}\,\mathrm{D}(M_i)$ and we have

$$\mathfrak{a}\varprojlim_i M_i \simeq \mathfrak{a}\varprojlim_i \mathrm{D}\,\mathrm{D}(M_i) \simeq \mathrm{D}(\mathfrak{a}\varinjlim_i \mathrm{D}(M_i)).$$

By assumption $\mathfrak{a}\varprojlim_i M_i$ is artinian, and so $\mathfrak{a}\varinjlim_i \mathrm{D}(M_i)$ is finitely generated. Now, by [10, Theorem 1.18] we have

$$\mathrm{Coass}_R(\mathfrak{a}\varprojlim_i M_i) = \mathrm{Coass}_R(\mathrm{D}(\mathfrak{a}\varinjlim_i \mathrm{D}(M_i)) = \mathrm{Ass}_R(\mathfrak{a}\varinjlim_i \mathrm{D}(M_i).$$

But

$$\mathrm{Ass}_R(\mathfrak{a}\varinjlim_i \mathrm{D}(M_i) \subseteq \mathrm{Ass}_R(\varinjlim_i \mathrm{D}(M_i)) \subseteq \cup \mathrm{Ass}_R \mathrm{D}(M_i) = \cup \mathrm{Coass}_R M_i,$$

and so we get the result.

ii) By putting $\mathfrak{a} = R$ in the part (i). $\square$

Theorem 2.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R and M a finitely generated R-module. Let i be a natural number.*

i) If $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is artinian, then

$$\operatorname{Att}_R \mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{i}(M) \subseteq \cup_{k\in\mathbb{N}} \operatorname{Att}_R(\mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{a}^k M)),$$

ii) If $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is artinian, then $\operatorname{Att}_R \mathfrak{F}_{\mathfrak{a}}^{i}(M) \subseteq \cup_{k\in\mathbb{N}} \operatorname{Att}_R(\mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{a}^k M))$.

Proof. i) By definition $\mathfrak{F}_{\mathfrak{a}}^{i}(M) = \varprojlim_k \mathrm{H}_{m}^{i}(M/\mathfrak{a}^k M)$. Since $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is artinian we have $\operatorname{Att}_R \mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{i}(M) = \operatorname{Coass}_R \mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ by [10, Theorem 1.14] also by [3, Theorem 7.1.3] $\mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{a}^k M)$ is artinian for any integer k and so $\operatorname{Coass}_R(\mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{a}^k M)) = \operatorname{Att}_R(\mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{a}^k M))$. Now by lemma 2.2 (i), we have

$$\operatorname{Att}_R \mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{i}(M) \subseteq \cup_{k\in\mathbb{N}} \operatorname{Coass}_R(\mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{a}^k M)) = \cup_{k\in\mathbb{N}} \operatorname{Att}_R(\mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{a}^k M)).$$

ii) It follows by (i) with $\mathfrak{b} = R$. $\square$

Theorem 2.4. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R and M a finitely generated R-module. If $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{0}(M)$ is artinian, then $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{0}(M)$ is finitely generated. In particular, if $\mathfrak{F}_{\mathfrak{a}}^{0}(M)$ is artinian, then $\mathfrak{F}_{\mathfrak{a}}^{0}(M)$ is finitely generated.*

Proof. It is well known that $\mathrm{H}_{\mathfrak{m}}^{0}(M/\mathfrak{a}^k M)$ is of finite length and so $\operatorname{Att}_R(\mathrm{H}_{\mathfrak{m}}^{0}(M/\mathfrak{a}^k M)) \subseteq \{\mathfrak{m}\}$ for any integer k, by [3, Corollary 7.2.12]. By Lemma 2.3(i) we have

$$\operatorname{Att}_R(\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{0}(M)) \subseteq \cup_{k\in\mathbb{N}} \operatorname{Att}_R(\mathrm{H}_{\mathfrak{m}}^{0}(M/\mathfrak{a}^k M)) \subseteq \{\mathfrak{m}\}.$$

By assumption, $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{0}(M)$ is artinian and so by [3, Corollary 7.2.12] we conclude that $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{0}(M)$ is finitely generated. $\square$

In [2, Theorem 3.1], we proved that $\operatorname{Att}_R \mathfrak{F}_{\mathfrak{a}}^{\dim M}(M) = \operatorname{Assh}_R(M) \cap \mathrm{V}(\mathfrak{a})$, where $\operatorname{Assh}_R(M) := \{\mathfrak{p} \in \operatorname{Ass}_R M \,|\, \dim(R/\mathfrak{p}) = \dim M\}$. In the following main result, we determine the set $\operatorname{Att}_R \mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)$.

Theorem 2.5. *Let $\mathfrak{a}$ be an ideal of R and M a finitely generated R-module. If $\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)$ is artinian, then*

$$\operatorname{Att}_R \mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M) = \operatorname{Att}_R(\mathrm{H}_{\mathfrak{m}}^{\dim M/\mathfrak{a}M}(M/\mathfrak{a}M)) = \operatorname{Assh}_R(M/\mathfrak{a}M).$$

Proof. Let $l := \dim M/\mathfrak{a}M$. By Theorem 2.3(ii)

$$\operatorname{Att}_R \mathfrak{F}_{\mathfrak{a}}^{l}(M) \subseteq \cup_{k\in\mathbb{N}} \operatorname{Att}_R(\mathrm{H}_{\mathfrak{m}}^{l}(M/\mathfrak{a}^k M)).$$

But, by [3, Theorem 7.3.2]

$$\mathrm{Att}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM)) = \mathrm{Assh}_R(M/\mathfrak{a}^kM) = \mathrm{Assh}_R(M/\mathfrak{a}M)$$

for any integer k. It follows that

$$\mathrm{Att}_R\,\mathfrak{F}^l_{\mathfrak{a}}(M) \subseteq \mathrm{Att}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M)) = \mathrm{Assh}_R(M/\mathfrak{a}M).$$

On the other hand, the exact sequence

$$0 \longrightarrow \mathfrak{a}M \longrightarrow M \longrightarrow M/\mathfrak{a}M \longrightarrow 0$$

induces a long exact sequence

$$\cdots \to \mathfrak{F}^l_{\mathfrak{a}}(M) \to \mathfrak{F}^l_{\mathfrak{a}}(M/\mathfrak{a}M) \to \mathfrak{F}^{l+1}_{\mathfrak{a}}(\mathfrak{a}M) \to \cdots .$$

Since

$$\sup\{i \in \mathbb{N}_0 : \mathfrak{F}^i_{\mathfrak{a}}(\mathfrak{a}M) \neq 0\} = \dim(\mathfrak{a}M/\mathfrak{a}^2M) \leq \dim(M/\mathfrak{a}M) = l$$

it follows that $\mathfrak{F}^{l+1}_{\mathfrak{a}}(\mathfrak{a}M) = 0$. On the other hand, since $M/\mathfrak{a}M$ is an $\mathfrak{a}$-torsion R-module by Lemma 2.1 $\mathfrak{F}^l_{\mathfrak{a}}(M/\mathfrak{a}M) \simeq \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M)$. Thus, from the above long exact sequence we obtain the exact sequence $\mathfrak{F}^l_{\mathfrak{a}}(M) \to \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M) \to 0$. Therefore

$$\mathrm{Att}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M)) \subseteq \mathrm{Att}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)),$$

as required. □

Corollary 2.6. *Let $\mathfrak{a}$ be an ideal of R and M a finitely generated R-module. Let $l := \dim M/\mathfrak{a}M$. If $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is artinian, then it follows that $\mathrm{Att}_R\,\mathfrak{F}^l_{\mathfrak{a}}(M) = \mathrm{Min}\,\mathrm{V}(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)))$.*

Proof. Since $\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ for all $i > l$, [8, Corollary 2.14] implies that $\mathfrak{F}^l_{\mathfrak{a}}(M) \simeq \mathfrak{F}^l_{\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))}(M)$.

At first, we show that $\dim(M/(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)))M) = l$. By [2, Corollary 2.10] it follows that $\mathrm{V}(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \mathrm{V}(\mathfrak{a})$. Therefore

$$\mathrm{V}(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))) \cap \mathrm{V}(\mathrm{Ann}_R M) \subseteq \mathrm{V}(\mathfrak{a}) \cap \mathrm{V}(\mathrm{Ann}_R M)$$

and so

$$\mathrm{Supp}_R(M/(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)))M) \subseteq \mathrm{Supp}_R(M/\mathfrak{a}M).$$

Thus $\dim(M/(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)))M) \leq l$. If $\dim(M/(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)))M) \lneqq l$ then it follows that $\mathfrak{F}^l_{\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))}(M) = 0$ and by the above isomorphism we get $\mathfrak{F}^l_{\mathfrak{a}}(M) = 0$ which is a contradiction by [9, Theorem 4.5]. Thus, it follows that $\dim(M/(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)))M) = l$. Now, by Theorem 2.5 we conclude that

$$\mathrm{Att}_R\,\mathfrak{F}^l_{\mathfrak{a}}(M) = \mathrm{Att}_R(\mathfrak{F}^l_{\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))}(M)) = \mathrm{Assh}_R(M/(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)))M).$$

But, it is easy to see that

$$\mathrm{Assh}_R(M/(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)))M) \subseteq \mathrm{Min}\,\mathrm{Supp}_R(M/(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)))M).$$

On the other hand, since $\mathrm{Ann}_R M \subseteq \mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))$ it follows that

$$\mathrm{V}(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))) \cap \mathrm{V}(\mathrm{Ann}_R(M)) = \mathrm{V}(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))).$$

Thus $\mathrm{Supp}_R(M/(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)))M) = \mathrm{V}(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)))$ and so we have $\mathrm{Min}\,\mathrm{Supp}_R(M/(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)))M) = \mathrm{Min}\,\mathrm{V}(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)))$. Therefore we conclude that $\mathrm{Att}_R\,\mathfrak{F}^l_{\mathfrak{a}}(M) \subseteq \mathrm{Min}\,\mathrm{V}(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)))$.

On the other hand, it is well known that for an artinian R-module A the set of all minimal prime ideals containing $\mathrm{Ann}_R A$ is exactly the set of all minimal elements of $\mathrm{Att}_R A$. Thus

$$\mathrm{Min}\,\mathrm{V}(\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))) = \mathrm{Min}\,\mathrm{Att}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \mathrm{Att}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)),$$

as required. □

Corollary 2.7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R and M a finitely generated R-module such that $l := \dim(M/\mathfrak{a}M) = \dim(M/\mathfrak{b}M)$. Let $\mathfrak{F}^l_{\mathfrak{a}}(M)$ and $\mathfrak{F}^l_{\mathfrak{b}}(M)$ be artinian and $\mathrm{Att}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Att}_R(\mathfrak{F}^l_{\mathfrak{b}}(M))$. Then*

i) $\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M) \simeq \mathfrak{F}^l_{\mathfrak{a}}(M/\mathfrak{a}M) \simeq \mathfrak{F}^l_{\mathfrak{b}}(M/\mathfrak{a}M)$,

ii) $\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{b}M) \simeq \mathfrak{F}^l_{\mathfrak{a}}(M/\mathfrak{b}M) \simeq \mathfrak{F}^l_{\mathfrak{b}}(M/\mathfrak{b}M)$.

Proof. i) By Theorem 2.5, $\mathrm{Att}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Assh}_R(M/\mathfrak{a}M)$ and $\mathrm{Att}_R(\mathfrak{F}^l_{\mathfrak{b}}(M)) = \mathrm{Assh}_R(M/\mathfrak{b}M)$. By assumption we get

$$\mathrm{Assh}_R(M/\mathfrak{a}M) = \mathrm{Assh}_R(M/\mathfrak{b}M).$$

On the other hand, by [2, Theorem 3.1] we have

$$\mathrm{Att}_R\,\mathfrak{F}^l_{\mathfrak{a}}(M/\mathfrak{a}M) = \mathrm{Assh}_R(M/\mathfrak{a}M) \cap \mathrm{V}(\mathfrak{a}) = \mathrm{Assh}_R(M/\mathfrak{a}M),$$

also

$$\begin{aligned}\mathrm{Att}_R\,\mathfrak{F}^l_{\mathfrak{b}}(M/\mathfrak{a}M) &= \mathrm{Assh}_R(M/\mathfrak{a}M) \cap \mathrm{V}(\mathfrak{b})\\ &= \mathrm{Assh}_R(M/\mathfrak{b}M) \cap \mathrm{V}(\mathfrak{b})\\ &= \mathrm{Assh}_R(M/\mathfrak{b}M).\end{aligned}$$

Since $\mathrm{Assh}_R(M/\mathfrak{a}M) = \mathrm{Assh}_R(M/\mathfrak{b}M)$ we conclude that $\mathrm{Att}_R\,\mathfrak{F}^l_{\mathfrak{a}}(M/\mathfrak{a}M) = \mathrm{Att}_R\,\mathfrak{F}^l_{\mathfrak{b}}(M/\mathfrak{a}M)$. Now by [2, Theorem 3.4] it follows that $\mathfrak{F}^l_{\mathfrak{a}}(M/\mathfrak{a}M) \simeq \mathfrak{F}^l_{\mathfrak{b}}(M/\mathfrak{a}M)$. But $M/\mathfrak{a}M$ is an $\mathfrak{a}$-torsion R-module and so by Lemma 2.1, $\mathfrak{F}^l_{\mathfrak{a}}(M/\mathfrak{a}M) \simeq \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M)$ and the proof of (i) is complete.

ii) The proof is similar to the proof of (i). □

Theorem 2.8. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R and M a finitely generated R-module. Let i be a natural number. If $\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M)$ is artinian, then $\mathfrak{a} \subseteq \sqrt{(0 : \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M))}$.*

Proof. By Theorem 2.3 (i),

$$\mathrm{Att}_R\, \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M) \subseteq \cup_{k\in\mathbb{N}} \mathrm{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^k M)).$$

But, $\mathfrak{a}^k \subseteq \mathrm{Ann}_R(\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^k M)))$ and so $\mathrm{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^k M))) \subseteq \mathrm{V}(\mathfrak{a})$ by [3, 7.2.11], for any integer $k \geq 0$. Thus $\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M))) \subseteq \mathrm{V}(\mathfrak{a})$, as required. □

Theorem 2.9. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{a} \subseteq \mathfrak{b}$ and M a finitely generated R-module, and let $n \in \mathbb{N}$. Then the following statements are equivalent:*
i) $\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M)$ is artinian for all $i < n$,
ii) $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is artinian for all $i < n$.

Proof. i)⇒ ii): Let $i < n$ be an integer. By Theorem 2.8 there exists an integer t such that $\mathfrak{a}^t\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$. But $\mathfrak{a} \subseteq \mathfrak{b}$ implies that $\mathfrak{a}^{t+1}\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$. Thus $\mathfrak{a} \subseteq \sqrt{(0 : \mathfrak{F}^i_{\mathfrak{a}}(M))}$ for all $i < n$. Now by [2, Theorem 2.6] we conclude that $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is artinian for all $i < n$.
ii) ⇒ i): It is clear. □

The formal grade of M with respect to $\mathfrak{a}$ is defined to be the least integer i such that $\mathfrak{F}^i_{\mathfrak{a}}(M) \neq 0$ and it is denoted by $\mathrm{fgrade}(\mathfrak{a}, M)$.

In the next result, there is a characterization of the artinianness of $\mathfrak{F}^{\mathrm{fgrade}(\mathfrak{a},M)}_{\mathfrak{a}}(M)$.

Corollary 2.10. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{a} \subseteq \mathfrak{b}$ and M a finitely generated R-module, and let $g := \mathrm{fgrade}(\mathfrak{a}, M)$. Then $\mathfrak{b}\mathfrak{F}^g_{\mathfrak{a}}(M)$ is artinian if and only if $\mathfrak{F}^g_{\mathfrak{a}}(M)$ is artinian.*

Proof. Since $\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ for all $i < g$, the result follows by Theorem 2.9. □

Theorem 2.11. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{a} \subseteq \mathfrak{b}$ and M a finitely generated R-module, and let $n \in \mathbb{N}$. Then the following statements are equivalent:*
i) $\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M)$ is artinian for all $i > n$,
ii) $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is artinian for all $i > n$.

Proof. i)⇒ ii): Let $i > n$ be an integer. By Theorem 2.8 there exists an integer t such that $\mathfrak{a}^t\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$. Since $\mathfrak{a} \subseteq \mathfrak{b}$ we conclude that $\mathfrak{a}^{t+1}\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$. Thus $\mathfrak{a} \subseteq \sqrt{(0 : \mathfrak{F}^i_{\mathfrak{a}}(M))}$ for all $i > n$. Now, the result follows by [2, Theorem 2.9].
ii) ⇒ i): It is clear. □

Corollary 2.12. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{a} \subseteq \mathfrak{b}$ and M a finitely generated R-module . Let $l := \dim(M/\mathfrak{a}M)$. Then $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ is artinian, if and only if $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is artinian.*

Proof. Since $\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ for all $i > l$, the result follows by Theorem 2.11. □

Corollary 2.13. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and M a finitely generated R-module, and let $i \in \mathbb{N}$. Then the following statements are equivalent:*
i) there exists an integer k such that $\mathfrak{b}^k\mathfrak{F}^i_{\mathfrak{a}}(M)$ is artinian,
ii) there exists an integer k such that $\mathfrak{b}^k\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$.

Proof. i)⇒ ii): By Theorem 2.8 there exists an integer t such that $\mathfrak{a}^t\mathfrak{b}^k\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$. Since $\mathfrak{b} \subseteq \mathfrak{a}$ it follows that $\mathfrak{b}^{t+k}\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$.
ii) ⇒ i): It is clear. □

Theorem 2.14. *Suppose that $(R, \mathfrak{m})$ is a local ring which is a homomorphic image of a Gorenstein local ring. Let $\mathfrak{a}$ be an ideal of R and M be a finitely generated R-module. Let j be an integer. If $\mathfrak{F}^j_{\mathfrak{a}}(M)$ is artinian, then $\operatorname{Att}_R(\mathfrak{F}^j_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{p} \in \operatorname{spec}(R) : \dim(R/\mathfrak{p}) \leq j\}$.*

Proof. i) By Theorem 2.3(ii), $\operatorname{Att}_R \mathfrak{F}^j_{\mathfrak{a}}(M) \subseteq \cup_{k\in\mathbb{N}} \operatorname{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^kM))$. But by [3, Corollary 11.3.5] we have

$$\operatorname{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^kM)) \subseteq \{\mathfrak{p} \in \operatorname{spec}(R) : \dim(R/\mathfrak{p}) \leq j\}$$

for any integer k and so the proof is complete. □

Lemma 2.15. *Let $\{M_i\}_{i\in I}$ be an inverse system of artinian R-modules and $\mathfrak{a}$ be an ideal of R. Then*
i) If $\mathfrak{a}\varprojlim_i M_i$ is a finitely generated R-module, then

$$\operatorname{Coass}_R(\mathfrak{a}\varprojlim_i M_i) \subseteq \cup \operatorname{Coass}_R M_i.$$

ii) If $\varprojlim_i M_i$ is a finitely generated R-module, then

$$\operatorname{Coass}_R(\varprojlim_i M_i) \subseteq \cup \operatorname{Coass}_R M_i.$$

Proof. i) Since M_i is artinian $M_i \simeq \mathrm{D}\,\mathrm{D}(M_i)$ for any $i \in I$ and so

$$\mathfrak{a}\varprojlim_i(M_i) \simeq \mathfrak{a}\varprojlim_i \mathrm{D}\,\mathrm{D}(M_i) \simeq \mathfrak{a}\,\mathrm{D}(\varinjlim_i \mathrm{D}(M_i)) \simeq \mathrm{D}(\mathfrak{a}\varinjlim_i \mathrm{D}(M_i)).$$

By assumption $\mathfrak{a}\varprojlim_i M_i$ is finitely generated and so $\mathfrak{a}\varinjlim_i \mathrm{D}(M_i)$ is artinian R-module and we have $\mathrm{D}\,\mathrm{D}(\mathfrak{a}\varinjlim_i \mathrm{D}(M_i)) \simeq \mathfrak{a}\varinjlim_i \mathrm{D}(M_i)$. Thus we conclude that

$$\begin{aligned}\operatorname{Coass}_R(\mathfrak{a}\varprojlim_i M_i) &= \operatorname{Coass}_R(\mathrm{D}(\mathfrak{a}\varinjlim_i \mathrm{D}(M_i))) \\ &= \operatorname{Ass}_R(\mathrm{D}\,\mathrm{D}(\mathfrak{a}\varinjlim_i \mathrm{D}(M_i)))\end{aligned}$$

$$= \mathrm{Ass}_R(\mathfrak{a}\varinjlim_i \mathrm{D}(M_i))$$
$$\subseteq \mathrm{Ass}_R(\varinjlim_i \mathrm{D}(M_i))$$
$$\subseteq \cup \mathrm{Ass}_R \mathrm{D}(M_i) = \cup \mathrm{Coass}_R M_i.$$

ii) By using part (i) with $\mathfrak{a} = R$. □

Theorem 2.16. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R and M a finitely generated R-module. Let i be a natural number.*

i) If $\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M)$ is finitely generated, then

$$\mathrm{Coass}_R \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M) \subseteq \cup_{k\in\mathbb{N}} \mathrm{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^k M)),$$

ii) If $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is finitely generated, then

$$\mathrm{Coass}_R \mathfrak{F}^i_{\mathfrak{a}}(M) \subseteq \cup_{k\in\mathbb{N}} \mathrm{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^k M)).$$

Proof. i) Since $\mathfrak{F}^i_{\mathfrak{a}}(M) = \varprojlim_k \mathrm{H}^i_m(M/\mathfrak{a}^k M)$ and $\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M)$ is finitely generated, by lemma 2.15 (i), we have

$$\mathrm{Coass}_R \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M) \subseteq \cup_{k\in\mathbb{N}} \mathrm{Coass}_R(\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^k M))$$
$$= \cup_{k\in\mathbb{N}} \mathrm{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^k M)).$$

ii) It follows by (i) with $\mathfrak{b} = R$. □

The following Theorem is a generalization of [1, Theorem 2.6(ii)].

Theorem 2.17. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R and M a finitely generated R-module. Let $l := \dim(M/\mathfrak{a}M) > 0$. If $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M) \neq 0$ then $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ is not finitely generated.*

Proof. Assume that $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ is finitely generated. Thus by [10, Theorem 2.10] $\phi \neq \mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{m}\}$ and so $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) = \{\mathfrak{m}\}$. On the other hand, by Theorem 2.16(i) we have

$$\mathrm{Coass}_R \mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M) \subseteq \cup_{k\in\mathbb{N}} \mathrm{Att}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M))$$

and by [3, Theorem 7.3.2] for any integer k we have

$$\mathrm{Att}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)) = \{\mathfrak{p} \in \mathrm{Ass}_R(M/\mathfrak{a}M) | \dim R/\mathfrak{p} = l > 0\}.$$

Thus $\mathfrak{m} \in \{\mathfrak{p} \in \mathrm{Ass}_R(M/\mathfrak{a}M) | \dim R/\mathfrak{p} = l > 0\}$ which is a contradiction. □

Theorem 2.18. *Let $\mathfrak{a}$ be an ideal of a R and M a finitely generated R-module. Let $l := \dim(M/\mathfrak{a}M) > 0$. Then $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is minimax if and only if $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is artinian.*

Proof. Any artinian R-module is minimax. Now, assume that $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is minimax. By [7, Theorem 2.2] we have $\mathrm{Cosupp}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \mathrm{V}(\mathfrak{a})$. Thus $\mathrm{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \mathrm{V}(\mathfrak{a})$ and so there exists an integer k such that $\mathfrak{a}^k\mathfrak{F}^l_{\mathfrak{a}}(M)$ is finitely generated by [11, Satz 2.4] and by Theorem 2.17 we have $\mathfrak{a}^k\mathfrak{F}^l_{\mathfrak{a}}(M) = 0$. Now, by using [2, Corollary 2.10] we conclude that $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is artinian, as required. □

Theorem 2.19. *Let $\mathfrak{a}$ be an ideal of R and M a finitely generated R-module. Let $g := fgrade(\mathfrak{a}, M)$. Let $\mathfrak{m} \notin \mathrm{Att}_R(\mathrm{H}^g_{\mathfrak{m}}(M/\mathfrak{a}^kM))$ for any integer k. If $\mathfrak{F}^g_{\mathfrak{a}}(M)$ is minimax then $\mathfrak{F}^g_{\mathfrak{a}}(M)$ is artinian.*

Proof. Since $\mathfrak{F}^g_{\mathfrak{a}}(M)$ is minimax,

$$\mathrm{Coass}_R(\mathfrak{F}^g_{\mathfrak{a}}(M)) \subseteq \mathrm{Cosupp}_R(\mathfrak{F}^g_{\mathfrak{a}}(M)) \subseteq \mathrm{V}(\mathfrak{a}),$$

by [7, Theorem 2.2]. Thus there exists an integer k such that $\mathfrak{a}^k\mathfrak{F}^g_{\mathfrak{a}}(M)$ is finitely generated by [11, Satz 2.4]. Assume that $\mathfrak{a}^k\mathfrak{F}^g_{\mathfrak{a}}(M) \neq 0$. Then, we have $\mathrm{Coass}_R(\mathfrak{a}^k\mathfrak{F}^g_{\mathfrak{a}}(M)) = \{\mathfrak{m}\}$ and by Theorem 2.16(i), $\mathfrak{m} \in \cup_{k\in\mathbb{N}}\mathrm{Att}_R(\mathrm{H}^g_{\mathfrak{m}}(M/\mathfrak{a}^kM))$ which is a contradiction. Thus $\mathfrak{a}^k\mathfrak{F}^g_{\mathfrak{a}}(M) = 0$ and [2, Corollary 2.7] implies that $\mathfrak{F}^g_{\mathfrak{a}}(M)$ is artinian, as required. □

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FINITENESS PROPERTIES OF FORMAL LOCAL COHOMOLOGY MODULES

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خصوصیات متناهی بودن مدول‌های کوهمولوژی موضعی صوری

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فرض کنید $\mathfrak{a}$ یک ایده‌آل از حلقه نوتری موضعی $(R, \mathfrak{m})$ و M یک R−مدول متناهی‌مولد باشد. در این مقاله چندین نتیجه درباره متناهی‌مولد بودن و آرتینی بودن مدول‌های کوهمولوژی موضعی صوری به‌دست می‌آوریم. به‌ویژه آخرین مدول ناصفر کوهمولوژی موضعی صوری $\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)$ را مورد بررسی قرار می‌دهیم و مجموعه ایده‌آل‌های اول چسبیده آن را در حالتی که آرتینی باشد به‌دست می‌آوریم. در یکی دیگر از نتایج به‌دست آمده نشان می‌دهیم که $\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)$ آرتینی است اگر و تنها اگر مینیماکس باشد.

کلمات کلیدی: کوهمولوژی موضعی صوری، کوهمولوژی موضعی، آرتینی.