**Journal of Algebraic Systems**
**Vol. 1, No. 2, (2013), pp 101-115**

# ZARISKI-LIKE SPACES OF CERTAIN MODULES

H. FAZAELI MOGHIMI* AND F. RASHEDI

ABSTRACT. Let $R$ be a commutative ring with identity and $M$ be a unitary $R$-module. The primary-like spectrum $Spec_L(M)$ is the collection of all primary-like submodules $Q$ such that $M/Q$ is a primeful $R$-module. Here, $M$ is defined to be RSP if $rad(Q)$ is a prime submodule for all $Q \in Spec_L(M)$. This class contains the family of multiplication modules properly. The purpose of this paper is to introduce and investigates a new Zariski space of an RSP module, called a Zariski-like space. In particular, we provide conditions under which the Zariski-like space of a multiplication module has a subtractive basis.

## 1. Introduction

This paper focuses on rings, which all are commutative with an identity and modules are unitary. Let $M$ be an $R$-module and $N$ be a submodule of $M$. The colon ideal of $M$ into $N$ is the ideal $(N : M) = \{r \in R \mid rM \subseteq N\}$ of $R$. A proper submodule $P$ of $M$ is called $p$-prime if for $p = (P : M)$, whenever $rm \in P$, $r \in R$ and $m \in M$, then $m \in P$ or $r \in p$. The collection of all prime submodules of $M$ is denoted by $Spec(M)$. If $N$ is a submodule of $M$, then the radical of $N$, denoted $rad(N)$, is the intersection of all prime submodules of $M$ which contain $N$, unless no such primes exist, in which case $rad(N) = M$.
A proper submodule $Q$ of $M$ is said to be primary-like if $rm \in Q$ implies $r \in (Q : M)$ or $m \in rad(Q)$ [5]. We state that a submodule $N$ of

MSC(2010): Primary: 13C05; Secondary: 13C13, 54B99
Keywords: RSP module, Multiplication module, Zariski-like space, Subtractive subsemimodule, Subtractive basis.
Received: 13 April 2013, Revised: 12 September 2013
*Corresponding author.

an $R$-module $M$ satisfies the primeful property if for each prime ideal $p$ of $R$ with $(N : M) \subseteq p$, there exists a prime submodule $P$ containing $N$ such that $(P : M) = p$. In this case $\sqrt{(N : M)} = (rad(N) : M)$ [10, Proposition 5.3]. For example the zero submodule of the $\mathbb{Z}$-module $M = \prod_{p\in\Omega}(\frac{\mathbb{Z}}{p\mathbb{Z}})$ is not a primary-like submodule of $M$, but it satisfies the primeful property [7, Example 1.1(6)]. On the other hand although $M' = \oplus_{p\in\Omega}(\frac{\mathbb{Z}}{p\mathbb{Z}})$ is a primary-like submodule of $M$, it dose not satisfy the primeful property [10, Example 1(5) and (6)]. In [5, Lemma 2.1] it is shown that, if $Q$ is a primary-like submodule satisfying the primeful property, then $p = \sqrt{(Q : M)}$ is a prime ideal of $R$ and so in this case, is $Q$ called a $p$-primary-like submodule.

The primary-like spectrum $Spec_L(M)$ is defined to be the set of all primary-like submodules of $M$ satisfying the primeful property. For example if $M$ is the $\mathbb{Z}$-module $\mathbb{Q}\bigoplus\mathbb{Z}_p$, where $\mathbb{Q}$ is the abelian group of rational numbers and $\mathbb{Z}_p$ is the cyclic group of order $p$, then $Spec(M) = \{\mathbb{Q}\bigoplus 0, 0\bigoplus\mathbb{Z}_p\}$ by [15, Example 2.6] and $Spec_L(M) = \{\mathbb{Q}\bigoplus 0\}$ by [6, Example 3.1]. In [6, Lemma 2.1], it is shown that if $Spec(M) = \emptyset$, then $Spec_L(M) = \emptyset$. However for the $\mathbb{Z}$-module $\mathbb{Q}$, we have $Spec(\mathbb{Q}) = \{0\}$ and $Spec_L(\mathbb{Q}) = \emptyset$.

There are different module theoretic generalizations of the well-known Zariski topology on the spectrum of a ring $R$ having $\{V(I) \mid I$ is an ideal of $R\}$ as the collection of closed sets, where $V(I) = \{p \in Spec(R) \mid I \subseteq p\}$ (see for example [1, 2, 3, 12]).

We set $\eta^*(M) = \{\nu^*(N) \mid N$ is a submodule of $M\}$, where $\nu^*(N) = \{Q \in Spec_L(M) \mid N \subseteq rad(Q)\}$. This collection of varieties of submodules is not closed under finite unions. An $R$-module $M$ is called top-like if $\eta^*(M)$ satisfies the axioms of a Zariski-like topology $\mathcal{T}^*$ for closed sets [6].

A module $M$ over a ring $R$ is called a multiplication module if each submodule of $M$ is of the form $IM$, where $I$ is an ideal of $R$. In this case, we can take $I = (N : M)$ [4]. Multiplication modules are top-like [7, Theorem 2.2]. Also if $R$ is an Artinian ring, then Bezout $R$-modules and distributive $R$-modules are top-like [6, Proposition 4.1].

From an algebraic point view, some Zariki spaces have been studied related to these topologies [14, 16]. It is easily seen that $\eta^*(M)$ with the binary operation $\nu^*(N)+\nu^*(N') = \nu^*(N+N') = \nu^*(N)\cap\nu^*(N')$ is a semigroup with zero. Moreover $\eta^*(R)$ with the similar addition and multiplication as $\nu^*(I) * \nu^*(J) = \nu^*(IJ) = \nu^*(I \cap J)$ is a semiring.

An $R$-module $M$ is called RSP if the radical of each element of $Spec_L(M)$ is prime. In Section 2, we introduce a Zariski-like space over RSP modules. In fact we show that for an RSP module $M$, the semigroup

$(\eta^*(M), +)$ with the scalar multiplication $\nu^*(I) * \nu^*(N) = \nu^*(IN)$ is an $\eta^*(R)$-semimodule (Theorem 2.4). In this case $(Spec_L(M), \eta^*(M))$ also means an $\eta^*(R)$-space, called the Zariski-like space. In this section we provide some background material and results regarding subtractive subsemimodules of $\eta^*(M)$.
The notion of Z*-radical of a submodule $N$ of $M$, defined in Section 3 and denoted by $\sqrt[Z^*]{N}$, is the intersection of all elements of $\nu^*(N)$, unless $\nu^*(N) = \emptyset$, in which case $\sqrt[Z^*]{N} = M$. It is proved that for submodules $N$ and $N'$ of a multiplication module $M$, $\sqrt[Z^*]{N \cap N'} = \sqrt[Z^*]{N} \cap \sqrt[Z^*]{N'}$. Moreover, if $\mid Spec_L(M) \mid < \infty$, then $\sqrt[Z^*]{\sqrt[Z^*]{N}} = \sqrt[Z^*]{N}$ (Lemma 3.9).
Since these identities are frequently needed to examine the new notion of a subtractive basis for a Zariski-like space, in a main part of Section 3, we restrict ourselves on the class of multiplication modules as a subclass of RSP modules. Such bases provide a means of generating Zariski-like Spaces, which exploits both the algebraic and topological-type properties of these spaces.
It is shown that if $M$ is a Z*-radical Noetherian multiplication $R$-module with $\mid Spec_L(M) \mid < \infty$ such that for every submodule $N$ of $M$ and $Q \in Spec_L(M)$, $N \subseteq \sqrt[Z^*]{N}$ and $rad(Q) \cap N = rad(Q \cap N)$, then $\eta^*(M)$ has a subtractive basis (Corollary 3.14).

## 2. The Zariski-like Space of RSP modules and $\eta^*(R)$-homomorphisms

The saturation of a submodule $N$ of an $R$-module $M$ with respect to a prime ideal $p$ of $R$ is the contraction of $N_p$ in $M$ and designated by $S_p(N)$. It is known that $S_p(N) = \{m \in M \mid rm \in N$ for some $r \in R \backslash p\}$ [11]. Hereafter we will use $\mathcal{X}$ to represent $Spec_L(M)$. Hence for any $Q \in \mathcal{X}$, the ideal $\sqrt{(Q : M)} = (rad(Q) : M)$ is prime and so is $rad(Q) \neq M$.

**Lemma 2.1.** *Let $M$ be an $R$-module and $Q$ be a primary-like submodule of $M$. Then $S_p(Q) \subseteq rad(Q)$ for every $p \in V(Q : M)$. In particular, if $S_p(Q)$ is a prime submodule of $M$ for some $p \in V(Q : M)$, then $S_p(Q) = rad(Q)$.*

*Proof.* Straightforward. □

**Lemma 2.2.** *Let $M$ be an $R$-module and $Q$ be a submodule of $M$. Consider the following statements.*

(1) *$rad(Q)$ is a $p$-prime submodule of $M$.*
(2) *$rad(Q)$ is a $p$-primary-like submodule of $M$.*
(3) *$Q$ is a $p$-primary-like submodule of $M$*

*Then* (1) $\Leftrightarrow$ (2)*. Furthermore, if* $Q \in \mathcal{X}$ *and* $(Q : M)$ *is a radical ideal of* $R$*, then* (1) $-$ (3) *are equivalent.*

*Proof.* (1) $\Leftrightarrow$ (2) is clear since $rad(rad(Q)) = rad(Q)$.
(1) $\Rightarrow$ (3) Clear. (3) $\Rightarrow$ (1) Since $S_p(Q) \subseteq rad(Q)$, then $S_p(Q) \neq M$. Thus by [11, Proposition 2.4] and Lemma 2.1 $rad(Q)$ is prime. The verification of the other implications is straightforward. □

Recall that an $R$-module $M$ is called RSP if the radical of each element of $\mathcal{X}$ is prime. In the following we list some conditions under which an $R$-module $M$ is RSP.

**Theorem 2.3.** *Let* $M$ *be an* $R$*-module. Then* $M$ *is RSP in each of the following cases.*

(1) $R$ *is a zero-dimensional ring.*
(2) *For each* $Q \in \mathcal{X}$ *and* $p = \sqrt{(Q : M)}$, $(S_p(Q) : M)$ *is a radical ideal.*
(3) *For each* $Q \in \mathcal{X}$ *and* $p = (Q : M)$, $S_p(Q) \neq M$.
(4) $M$ *is a multiplication module.*
(5) $R$ *is a Noetherian domain and* $Q \in \mathcal{X}$ *is contained in only finitely many prime submodules of* $M$.

*Proof.* (1) Suppose $Q \in \mathcal{X}$. Since $\sqrt{(Q : M)} = (rad(Q) : M)$ is prime and hence maximal, $\sqrt{(Q : M)} = (P : M)$ for all prime submodules $P$ containing $Q$. Now if $rm \in rad(Q)$ and $m \notin rad(Q)$, there is a prime submodule $P$ containing $Q$ such that $rm \in P$ and $m \notin P$ and so $r \in (P : M) = \sqrt{(Q : M)} = (rad(Q) : M)$. Thus $rad(Q)$ is prime.
(2) $p = \sqrt{(Q : M)} \subseteq \sqrt{(S_p(Q) : M)} \subseteq (rad(Q) : M) = \sqrt{(Q : M)} = p$. It follows that $\sqrt{(S_p(Q) : M)} = p$. Now since $(S_p(Q) : M)$ is a radical ideal, we have $(S_p(Q) : M) = p$. It follows from [11, Theorem 2.3] and Lemma 2.1, $rad(Q)$ is a prime submodule of $M$.
(3) Suppose $S_p(Q) \neq M$. By [11, Proposition 2.4 ], $S_p(Q)$ is a prime submodule of $M$. It follows from Lemma 2.1 $rad(Q)$ is a prime submodule of $M$.
(4) Since $(rad(Q) : M)$ is a prime ideal of $R$ for every $Q \in \mathcal{X}$, $rad(Q)$ is a prime submodule of $M$ by [4, Corollary 2.11].
(5) By Lemma 2.2 we may assume that $(Q : M) \neq 0$. If $P$ is a prime submodule containing $Q$, then $0 \subset \sqrt{(Q : M)} \subseteq (P : M)$ is a chain of prime ideals of $R$. If $\sqrt{(Q : M)} \subseteq (P : M)$ is a proper containment, then by [9, P.144] there are infinitely many prime ideals $p$ with $(Q : M) \subset p \subset (P : M)$ and so we have infinitely prime submodules $P$ containing $Q$, a contradiction. Hence we have $\sqrt{(Q : M)} = (P : M)$, for all prime submodules $P$ containing $N$. Now if $rm \in rad(Q)$ and

$m \notin rad(Q)$, there is a prime submodule $P$ containing $Q$ such that $rm \in P$ and $m \notin P$ and so that $r \in (P : M) = \sqrt{(Q : M)} = (rad(Q) : M)$. □

For the remainder of this section, we assume that $M$ and $M'$ are RSP $R$-modules.

Let $(X, \Omega)$ be a topological space, and let $\Gamma$ be a collection of subsets of a set $Y$ such that $Y \in \Gamma$ and $\Gamma$ is closed with respect to finite intersections. Further suppose that there exists a mapping $* : \Omega * \Gamma \to \Gamma$ such that $(\Gamma, \cap)$ is an $\Omega$-semimodule. That is to say, for all $\tau, \tau' \in \Omega$ and for all $\gamma, \gamma' \in \Gamma$, the following properties hold.

(1) $\tau * (\gamma \cap \gamma') = (\tau * \gamma) \cap (\tau * \gamma')$;
(2) $(\tau \cap \tau') * \gamma = (\tau * \gamma) \cap (\tau' * \gamma)$;
(3) $(\tau \cup \tau') * \gamma = \tau * (\tau' * \gamma)$;
(4) $\emptyset * \gamma = \gamma$;
(5) $\tau * Y = Y = X * \gamma$.

Then $(Y, \Gamma)$ is called an $\Omega$-space [14].

**Theorem 2.4.** *Let $M$ be an $R$-module and let the $\eta^*(R)$-action on $\eta^*(M)$ be given by $\nu^*(I) * \nu^*(N) = \nu^*(IN)$, where $I$ is an ideal of $R$ and $N$ is a submodule of $M$. Then $(\mathcal{X}, \eta^*(M))$ is an $\eta^*(R)$-space.*

*Proof.* It is easy to see that $(\eta^*(M), \cap)$ is a commutative monoid with identity $\mathcal{X} = \nu^*(0)$. Now assume that $\nu^*(I) = \nu^*(J)$ and $\nu^*(N) = \nu^*(N')$, where $I, J$ are ideals of $R$ and $N, N'$ are submodules of $M$. We must show that $\nu^*(IN) = \nu^*(JN')$. Suppose $Q \in \nu^*(IN)$. Therefore $IN \subseteq rad(Q)$. Since $rad(Q)$ is prime, $N \subseteq rad(Q)$ or $I \subseteq (rad(Q) : M)$ by [15, Lemma 1.1]. Hence $JN' \subseteq rad(Q)$ or $JN' \subseteq (rad(Q) : M)N' \subseteq rad(Q)$. By symmetry we have $\nu^*(IN) = \nu^*(JN')$. Hence the operation $(*)$ is well-defined. Now we check the condition (3) of the above definition. $\nu^*(I) * (\nu^*(J) * \nu^*(N)) = \nu^*(I) * \nu^*(JN) = \nu^*(I(JN)) = \nu^*(IJ) * \nu^*(N) = (\nu^*(I) \cup \nu^*(J)) * \nu^*(N)$. The other properties follow similarly. □

The $\eta^*(R)$-space $(\mathcal{X}, \eta^*(M))$ is called a Zariski-like space. As mentioned in the introduction, from another point view, $(\eta^*(M), +)$ may be considered as an semimodule over a semiring $\eta^*(R)$ with addition and multiplication defined as:

$$\nu^*(N) + \nu^*(N') = \nu^*(N + N') = \nu^*(N) \cap \nu^*(N'),$$
$$\nu^*(I) * \nu^*(N) = \nu^*(IN) = \nu^*(IM) \cup \nu^*(N).$$

Let $\mathcal{R}$ be a semiring. By a $\mathcal{R}$-semimodule homomorphism, we mean a map $f : \mathcal{M} \to \mathcal{M}'$ of $\mathcal{R}$-semimodules $\mathcal{M}$ and $\mathcal{M}'$ which is $\mathcal{R}$-linear. Also subsemimodules and subspaces are defined naturally (For further reading about semirings, semimodules, and Zariski spaces, see for example [8, 14, 13]).

**Lemma 2.5.** *Let $M$, $M'$ be $R$-modules and $f : \eta^*(M) \to \eta^*(M')$ be an $\eta^*(R)$-homomorphism. If $N$, $N'$ are submodules of $M$ such that $\nu^*(N) \subseteq \nu^*(N')$, then $f(\nu^*(N)) \subseteq f(\nu^*(N'))$.*

*Proof.* Since $\nu^*(N) \subseteq \nu^*(N')$, we have $\nu^*(N) = \nu^*(N) \cap \nu^*(N') = \nu^*(N) + \nu^*(N')$. Hence $f(\nu^*(N)) = f(\nu^*(N) + \nu^*(N')) = f(\nu^*(N)) + f(\nu^*(N')) = f(\nu^*(N)) \cap f(\nu^*(N')) \subseteq f(\nu^*(N'))$. □

**Lemma 2.6.** *Let $M$, $M'$ be $R$-modules and $f : \eta^*(M) \to \eta^*(M')$ be an $\eta^*(R)$-surjective homomorphism. Then $f(\nu^*(M)) = \nu^*(M')$.*

*Proof.* Since $f$ is surjective, there exists a submodule $N$ of $M$ such that $f(\nu^*(N)) = \nu^*(M')$. Hence $f(\nu^*(M)) = f(\nu^*(M + N)) = f(\nu^*(M) + \nu^*(N)) = f(\nu^*(M)) + f(\nu^*(N)) = f(\nu^*(M)) + \nu^*(M') = \nu^*(M')$. □

**Lemma 2.7.** *Let $M$, $M'$ be $R$-modules and $f : \eta^*(M) \to \eta^*(M')$ be an $\eta^*(R)$-injective homomorphism. If $N$, $N'$ are submodule of $M$ such that $f(\nu^*(N)) \subseteq f(\nu^*(N'))$, then $\nu^*(N)) \subseteq \nu^*(N')$.*

*Proof.* Since $f(\nu^*(N)) \subseteq f(\nu^*(N'))$, we have $f(\nu^*(N)) = f(\nu^*(N)) \cap f(\nu^*(N')) = f(\nu^*(N) \cap \nu^*(N'))$. Hence $\nu^*(N) = \nu^*(N) \cap \nu^*(N')$ because $f$ is injective. Thus $\nu^*(N)) \subseteq \nu^*(N')$. □

A subsemimodule $\Delta$ is a subtractive subsemimodule of $\eta^*(M)$ if for submodules $N$, $N'$ of $M$ the conditions $\nu^*(N) \in \Delta$ and $\nu^*(N) + \nu^*(N') \in \Delta$ implies that $\nu^*(N') \in \Delta$. In this paper, we use Bourne factor semimodule of a semimodule $\Gamma$ over a semiring $\Omega$ (that is, the elements of $\frac{\Gamma}{\Delta}$ are the equivalency classes $[\gamma]$ $(\gamma \in \Gamma)$ of the congruence $\gamma \sim \gamma' \Leftrightarrow \exists \delta, \delta' \in \Delta$: $\gamma + \delta = \gamma' + \delta'$. Also addition and scalar multiplication is defined naturally; $[\gamma] + [\gamma'] = [\gamma + \gamma']$ and $\omega * [\gamma] = [\omega * \gamma]$).

**Lemma 2.8.** *Let $f : \eta^*(M) \to \eta^*(M')$ be an $\eta^*(R)$-homomorphism. Then $Kerf$ is a subtractive subsemimodule of $\eta^*(M)$. Conversely, if $\Delta$ is a subtractive subsemimodule of $\eta^*(M)$ , then $\pi : \eta^*(M) \to \frac{\eta^*(M)}{\Delta}$ which is defined by $\pi(\nu^*(N)) = [\nu^*(N)]$ is an $\eta^*(R)$-surjective homomorphism with $Ker\pi = [0]$.*

*Proof.* It is clear that $Kerf$ is a subtractive subsemimodule of $\eta^*(M)$ by [8]. Conversely, it is easy to see that $\pi$ is a surjective homomorphism. Now we have $\pi(\nu^*(N) + \nu^*(N')) = [\nu^*(N) + \nu^*(N')] = [\nu^*(N)] + [\nu^*(N'] = \pi(\nu^*(N)) + \pi(\nu^*(N'))$ and $\pi(\nu^*(I) * \nu^*(N)) = [\nu^*(I) * \nu^*(N)] = \nu^*(I) * [\nu^*(N)] = \nu^*(I) * \pi(\nu^*(N)))$. Thus $\pi$ is an $\eta^*(R)$-homomorphism. Also $Ker\pi = \{\nu^*(N) \in \eta^*(M) \mid [\nu^*(N)] = [0]\} = \{\nu^*(N) \in \eta^*(M) \mid \nu^*(N) \in [0]\} = [0]$. □

**Lemma 2.9.** *Let $\Delta$ be a subspace of $\eta^*(M)$. Then the following are equivalent.*

1. *$\Delta$ is a subtractive subsemimodule of $\eta^*(M)$;*
2. *For submodules $N$, $N'$ of $M$ the conditions $\nu^*(N) \in \Delta$ and $\nu^*(N) \subseteq \nu^*(N')$ implies that $\nu^*(N') \in \Delta$.*

*Proof.* (1)⇒(2) By Lemma 2.8, $\Delta$ is the kernel of the $\eta^*(R)$-surjective homomorphism $\pi : \eta^*(M) \to \frac{\eta^*(M)}{\Delta}$. Suppose $N$, $N'$ are submodules of $M$. Assume $\nu^*(N) \in \Delta$ and $\nu^*(N) \subseteq \nu^*(N')$. Hence $\pi(\nu^*(N)) + \pi(\nu^*(N')) = \nu^*(0)$. Thus $\pi(\nu^*(N')) = \nu^*(0)$ and so $\nu^*(N') \in \Delta$.
(2)⇒(1) Assume $N$, $N'$ are submodules of $M$. Suppose $\nu^*(N) \in \Delta$ and $\nu^*(N) \cap \nu^*(N') \in \Delta$. Since $\nu^*(N) \cap \nu^*(N') \subseteq \nu^*(N')$, then $\nu^*(N') \in \Delta$. Thus $\Delta$ is a subtractive subsemimodule of $\eta^*(M)$. □

**Proposition 2.10.** *Every proper subtractive subspace of $\eta^*(M)$ is contained in a maximal subtractive subspace.*

*Proof.* Suppose $\Delta$ is a proper subtractive subspace of $\eta^*(M)$. Put $\mathcal{A} = \{\Phi \mid \Delta \subseteq \Phi\}$. Assume $\mathcal{C} = \{\Phi_i \mid i \in I\}$ is a chain of elements of $\mathcal{A}$. It is easy to see that $\Delta \in \mathcal{A}$ and $\cup_{i\in I}\Phi_i \in \mathcal{A}$. Thus the assertion holds by Zorn's lemma. □

**Proposition 2.11.** *Let $M$, $M'$ be $R$-modules and $f : \eta^*(M) \to \eta^*(M')$ be an $\eta^*(R)$-homomorphism. If $\Delta$ is a subtractive subspace of $\eta^*(M')$, then the following hold.*

1. *$f^{-1}(\Delta)$ is a subtractive subspace of $\eta^*(M)$ containing $Kerf$.*
2. *$f$ induces an $\eta^*(R)$-homomorphism $\phi : \frac{\eta^*(M)}{f^{-1}(\Delta)} \to \frac{\eta^*(M')}{\Delta}$ having kernel $f^{-1}(\Delta)$.*

*Proof.* (1) Suppose $N$, $N'$ are submodules of $M$. Assume $\nu^*(N) \in f^{-1}(\Delta)$ and $\nu^*(N) \cap \nu^*(N') \in f^{-1}(\Delta)$. Hence $f(\nu^*(N)) \in \Delta$ and $f(\nu^*(N)) \cap f(\nu^*(N')) \in \Delta$. Since $\Delta$ is a subtractive subspace of $\eta^*(M')$, then $f(\nu^*(N')) \in \Delta$. Thus $\nu^*(N') \in f^{-1}(\Delta)$ and so $f^{-1}(\Delta)$ is a subtractive subspace of $\eta^*(M)$. It is easy to see that $Kerf \subseteq f^{-1}(\Delta)$.
(2) Use [8, Corollary 13.48]. □

It is common that if $\{\Delta_\lambda\}_{\lambda\in\Lambda}$ be a family of subtractive subspaces of $\eta^*(M)$, then $\cap_{\lambda\in\Lambda}\Delta_\lambda$ is subtractive. Let $\Upsilon$ be a subset of $\eta^*(M)$. The subtractive closure of $\Upsilon$, denoted $\gamma(\Upsilon)$, is the smallest subtractive subspace of $\eta^*(M)$ which contains $\Upsilon$. It is clear that if $\Upsilon \subseteq \Upsilon'$ be subsemimodules of $\eta^*(M)$, then $\gamma(\Upsilon) \subseteq \gamma(\Upsilon')$.

**Lemma 2.12.** *Let $N$ be a submodule of an $R$-module $M$ and $\Delta$ be a subsemimodule of $\eta^*(M)$. Then the following hold.*

1. $\gamma(\Delta) = \{\nu^*(N') \mid \nu^*(N'') \subseteq \nu^*(N')$ *for some* $\nu^*(N'') \in \Delta\}$.

(2) $\gamma(\eta^*(R) * \nu^*(N)) = \{\nu^*(N') \mid \nu^*(N) \subseteq \nu^*(N')\}$.

*Proof.* (1) Suppose $A = \{\nu^*(N') \mid \nu^*(N'') \subseteq \nu^*(N')$ for some $\nu^*(N'') \in \Delta\}$ and $\nu^*(N') \in A$. Therefore there exists $\nu^*(N'') \in \Delta$ such that $\nu^*(N'') \subseteq \nu^*(N')$. Since $\gamma(\Delta)$ is the smallest subtractive subspace of $\eta^*(M)$ which contains $\Delta$, then $\nu^*(N') \cap \nu^*(N'') = \nu^*(N'') \in \Delta \subseteq \gamma(\Delta)$. Thus $\nu^*(N') \in \gamma(\Delta)$ and so $A \subseteq \gamma(\Delta)$. For the reverse inclusion we show that $A$ is a subtractive subspace of $\eta^*(M)$ which contains $\Delta$. It is clear that $\Delta \subseteq A$. Now assume $\nu^*(N_1)$, $\nu^*(N_2) \in A$ and $\nu^*(N_1'')$, $\nu^*(N_2'') \in \Delta$ such that $\nu^*(N_1'') \subseteq \nu^*(N_1)$ and $\nu^*(N_2'') \subseteq \nu^*(N_2)$. Hence $\nu^*(N_1'') \cap \nu^*(N_2'') \subseteq \nu^*(N_1) \cap \nu^*(N_2)$. Thus $\nu^*(N_1) \cap \nu^*(N_2) \in A$. Suppose $\nu^*(I) \in \eta^*(R)$. So $\nu^*(I) * \nu^*(N_1'') \subseteq \nu^*(I) * \nu^*(N_1)$. Hence $\nu^*(I) * \nu^*(N_1) \in A$. Thus $A$ is a subspace of $\eta^*(M)$. Now suppose $\nu^*(N) \in \eta^*(M)$ and $\nu^*(N')$, $\nu^*(N) \cap \nu^*(N') \in A$. Then there exists $\nu^*(N'') \in \Delta$ such that $\nu^*(N'') \subseteq \nu^*(N) \cap \nu^*(N')$. Thus $\nu^*(N'') \subseteq \nu^*(N)$ and so $\nu^*(N) \in A$. Therefore $A$ is a subtractive subspace of $\eta^*(M)$ containing $\Delta$. Thus $A = \gamma(\Delta)$.

(2) We have $\eta^*(R) * \nu^*(N) = \{\nu^*(IN) \mid I$ is an ideal of $R\}$. Therefore $\eta^*(R) * \nu^*(N)$ is a subspace of $\eta^*(M)$. Hence $\gamma(\eta^*(R)\nu^*(N)) = \{\nu^*(N') \mid \nu^*(N'') \subseteq \nu^*(N')$ for some $\nu^*(N'') \in \eta^*(R) * \nu^*(N)\} = \{\nu^*(N') \mid \nu^*(IN) \subseteq \nu^*(N')$ for some ideal $I$ of $R\} \subseteq \{\nu^*(N') \mid \nu^*(N) \subseteq \nu^*(N')\}$ by (1). By the similar argument we have $\{\nu^*(N') \mid \nu^*(N) \subseteq \nu^*(N')\} \subseteq \gamma(\eta^*(R) * \nu^*(N))$. □

**Proposition 2.13.** *Let $N$, $N'$ be submodules of an $R$-module $M$ and $\nu^*(N') \in \gamma(\nu^*(N))$. Then $\nu^*(N) \subseteq \nu^*(N')$.*

*Proof.* It is clear by Lemma 2.12. □

**Proposition 2.14.** *Let $N$, $N'$ be submodules of an $R$-module $M$ and $N' \subseteq rad(N)$. Then $\nu^*(N') \in \gamma(\nu^*(N))$.*

*Proof.* Suppose $N' \subseteq rad(N)$. Since $\nu^*(N) = \nu^*(rad(N))$, then $\nu^*(N) \subseteq \nu^*(N')$. Thus $\nu^*(N') \in \gamma(\nu^*(N))$ by Lemma 2.12. □

**Theorem 2.15.** *Let radical submodules of an $R$-module $M$ satisfy ACC. Then every subtractive subspace of $\eta^*(M)$ is of the form $\gamma(\nu^*(N))$ for some submodule $N$ of $M$.*

*Proof.* Suppose $\Delta$ is a subtractive subspace of $\eta^*(M)$. If $\nu^*(M) \in \Delta$, then $\Delta = \eta^*(M) = \gamma(\nu^*(M))$. So assume that $\nu^*(M) \notin \Delta$. Let $A$ be the collection of all radical submodules $N$ of $M$ such that $\nu^*(N) \in \Delta$, and note that $A \neq \emptyset$ since $\nu^*(N) = \nu^*(rad(N))$ for every submodule $N$ of $M$. Now choose $N'$ to be a maximal element of $A$. To see that $\Delta = \gamma(\nu^*(N'))$, let $\nu^*(N'') \in \Delta$, where $N''$ is a submodule of $M$. If $S = rad(N' + N'')$, then $\nu^*(S) = \nu^*(N' + N'') = \nu^*(N') \cap \nu^*(N'') \in \Delta$.

Since $S$ is a radical submodule of $M$, then $N'' \subseteq S = N' = rad(N')$. Hence $\nu^*(N'') \in \gamma(\nu^*(N'))$ by Lemma 2.12. Thus $\Delta \subseteq \gamma(\nu^*(N'))$. Since $\nu^*(N') \in \Delta$, then $\gamma(\nu^*(N')) \subseteq \Delta$. Thus $\Delta = \gamma(\nu^*(N'))$. □

**Lemma 2.16.** *Let $M$ be an $R$-module and $\{N_i\}_{i\in I}$ be submodules of $M$. Then $\nu^*(\sum_{i\in I} N_i) = \sum_{i\in I} \nu^*(N_i)$.*

*Proof.* For $Q \in \mathcal{X}$ we have $Q \in \sum_{i\in I} \nu^*(N_i)$ if and only if $Q \in \nu^*(N_i)$ for every $i \in I$ iff $N_i \subseteq rad(Q)$ for each $i \in I$ iff $\sum_{i\in I} N_i \subseteq rad(Q)$ iff $Q \in \nu^*(\sum_{i\in I} N_i)$. □

**Theorem 2.17.** *Let $M$ be an $R$-module and $\{N_i\}_{i=1}^n$ be submodules of $M$. Then $\gamma(\sum_{i=1}^n \eta^*(R) * \nu^*(N_i)) = \gamma(\nu^*(\sum_{i=1}^n N_i))$.*

*Proof.* Assume $\nu^*(N') \in \gamma(\sum_{i=1}^n \eta^*(R) * \nu^*(N_i))$. Hence by Lemma 2.12, $\nu^*(\sum_{i=1}^n J_i N_i) \subseteq \nu^*(N')$ for some ideal $J_i$ of $R$. Since $\nu^*(\sum_{i=1}^n N_i) \subseteq \nu^*(\sum_{i=1}^n J_i N_i)$, then $\nu^*(\sum_{i=1}^n N_i) \subseteq \nu^*(N')$. So $\nu^*(N') \in \gamma(\nu^*(\sum_{i=1}^n N_i))$. Thus $\gamma(\sum_{i=1}^n \eta^*(R) * \nu^*(N_i)) \subseteq \gamma(\nu^*(\sum_{i=1}^n N_i))$. Now, we let $\nu^*(N') \in \gamma(\nu^*(\sum_{i=1}^n N_i))$. Then $\nu^*(\sum_{i=1}^n N_i) \subseteq \nu^*(N')$. By Lemma 2.16 we have $\nu^*(\sum_{i=1}^n N_i) = \sum_{i=1}^n \nu^*(N_i) \in \sum_{i=1}^n \eta^*(R) * \nu^*(N_i)$. Hence $\nu^*(N') \in \gamma(\sum_{i=1}^n \eta^*(R) * \nu^*(N_i))$. Thus $\gamma(\nu^*(\sum_{i=1}^n N_i)) \subseteq \gamma(i = 1^n \eta^*(R) * \nu^*(N_i))$. □

## 3. Subtractive Closure and Subtractive Bases

We define the Z*-radical of a submodule $N$ of $M$, denoted by $\sqrt[z^*]{N}$, to be the intersection of all members of $\nu^*(N)$. A submodule $N$ of $M$ is a Z*-radical submodule if $\sqrt[z^*]{N} = N$. An $R$-module $M$ is called Z*-radical if $\sqrt[z^*]{0_M} = 0$. Let $\mathcal{Y}$ be a subset of $\mathcal{X}$. The closure of $\mathcal{Y}$ in $\mathcal{X}$, denoted by $\overline{\mathcal{Y}}$, is the intersection of all closed subset of $\mathcal{X}$ containing $\mathcal{Y}$. Also $\xi(\mathcal{Y})$ is the intersection of all elements in $\mathcal{Y}$ (note that if $\mathcal{Y} = \emptyset$, then $\xi(\mathcal{Y}) = M$). It is easy to verify that, if $\mathcal{Y}_1, \mathcal{Y}_2 \subseteq \mathcal{X}$, then $\xi(\mathcal{Y}_1 \cup \mathcal{Y}_2) = \xi(\mathcal{Y}_1) \cap \xi(\mathcal{Y}_2)$.

**Lemma 3.1.** *Let $M$ be an $R$-module and $N$, $N'$ be submodules of $M$. If $\nu^*(N) \subseteq \nu^*(N')$, then $\sqrt[z^*]{N'} \subseteq \sqrt[z^*]{N}$. The converse is true if $N' \subseteq \sqrt[z^*]{N'}$.*

*Proof.* Suppose $\nu^*(N) \subseteq \nu^*(N')$. Hence $\xi(\nu^*(N')) \subseteq \xi(\nu^*(N))$ and so $\sqrt[z^*]{N'} \subseteq \sqrt[z^*]{N}$. Conversely, Suppose $Q \in \nu^*(N)$. Hence $\sqrt[z^*]{N'} \subseteq \sqrt[z^*]{N} \subseteq Q$. Thus $N' \subseteq rad(Q)$ and so $\nu^*(N) \subseteq \nu^*(N')$. □

**Lemma 3.2.** *Let $M$ be a finitely generated $R$-module. Then $\sqrt[z^*]{N} \neq M$ if and only if $\nu^*(N) \neq \emptyset$ if and only if $N \neq M$.*

*Proof.* Suppose $\sqrt[Z^*]{N} \neq M$. Hence $N \neq M$. Now assume $N \neq M$. Then $(N : M) \neq R$ and so $(N : M) \subseteq p$ for some prime ideal $p$ of $R$. Since $M$ is finitely generated, $M$ is primeful by [10, Theorem 2.2]. So there exists $Q \in Spec(M) \subseteq \mathcal{X}$ such that $N \subseteq rad(Q)$. Hence $Q \in \nu^*(N)$. Thus $\nu^*(N) \neq \emptyset$. If $\nu^*(N) \neq \emptyset$ and $Q \in \nu^*(N)$. Hence $N \subseteq rad(Q)$. Thus $\sqrt[Z^*]{N} \subseteq Q \neq M$. □

**Lemma 3.3.** *Let $M$ be an $R$-module. If $Q \in \mathcal{X}$ and $N$ is a submodule of $M$ such that $rad(Q) \cap N = rad(Q \cap N)$, then $N \subseteq Q$ or $Q \cap N$ is a primary-like submodule of $N$.*

*Proof.* Suppose $N \nsubseteq Q$, $n \in N$ and $rn \in Q \cap N$ such that $r \notin (Q \cap N : N)$. Then $rn \in Q$ and $r \notin (Q : M)$. Since $Q$ is primary-like, we have $n \in rad(Q)$. Thus $n \in rad(Q \cap N)$. □

**Lemma 3.4.** *Let $M$ be a $Z^*$-radical $R$-module such that every submodule $N$ of $M$ is finitely generated and $N \subseteq \sqrt[Z^*]{N}$. If for every $Q \in \mathcal{X}$, $rad(Q) \cap N = rad(Q \cap N)$, then every direct summand of $M$ is a $Z^*$-radical submodule of $M$.*

*Proof.* Suppose that $N$ is a direct summand of $M$ and $N \subset \sqrt[Z^*]{N}$. Hence $M = N \bigoplus N'$ for some submodule $N'$ of $M$. Therefore there exists $m = (n, n') \in \sqrt[Z^*]{N} \backslash N$. So $0 \neq (0, n') \in \sqrt[Z^*]{N}$. Since $M/N \cong N'$, there is a one-to-one correspondence between the primary-like submodules of $N'$ which satisfy the primeful property and the primary-like submodules of $M/N$ satisfying the primeful property. Since $(0, n') \in \sqrt[Z^*]{N}$, $(0, n')$ belongs to every primary-like submodule of the module $N'$ which satisfies the primeful property. Let $Q \in \mathcal{X}$. Then we show that $(0, n') \in Q$. If $N' \subseteq Q$, then $(0, n') \in Q$ because $(0, n') \in N'$. Suppose $N' \nsubseteq Q$. Hence by Lemma 3.3 and [10, Theorem 2.2], $Q \cap N' \in Spec_L(N')$. Thus $(0, n') \in Q \cap N' \subseteq Q$ and so $n' \in \sqrt[Z^*]{0_M} = 0$, a contradiction. □

Let $M$ be an $R$-module and $\{N_i\}_{i=1}^n$ be submodules of $M$. If $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$ we recall the following definitions.

(1) $\Delta$ is a subtractive generating set of $\eta^*(M)$ if $\eta^*(M) = \gamma(\sum_{i \in I} \eta^*(R) * \nu^*(N_i))$.
(2) $\Delta$ is a subtractive linearly independent set of $\eta^*(M)$ if $\nu^*(0) \notin \Delta$ and $\gamma(\nu^*(N_i)) \cap \gamma(\sum_{j \neq i} \eta^*(R) * \nu^*(N_j)) = \{\nu^*(0)\}$ for each $i$, $(1 \leq i \leq n)$.
(3) $\Delta$ is a subtractive linearly independent generating set of $\eta^*(M)$ if $\Delta$ satisfies both conditions (1) and (2).

**Lemma 3.5.** *Let $M$ be an $R$-module and $\{N_i\}_{i=1}^n$ be submodules of $M$. If $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$, then the following hold.*

(1) *$\Delta$ is a subtractive generating set of $\eta^*(M)$ iff* $\sqrt[z^*]{\sum_{i=1}^n N_i} = M$.
(2) *If $M$ is a finitely generated, then $\Delta$ is a subtractive generating set of $\eta^*(M)$ iff* $\sum_{i=1}^n N_i = M$.

*Proof.* (1) By Theorem 2.17, $\gamma(\sum_{i=1}^n \eta^*(R)*\nu^*(N_i)) = \gamma(\nu^*(\sum_{i=1}^n N_i))$. So $\Delta$ is a subtractive generating set of $\eta^*(M)$ if and only if $\eta^*(M) = \gamma(\sum_{i=1}^n \eta^*(R)*\nu^*(N_i)) = \gamma(\nu^*(\sum_{i=1}^n N_i))$ iff $\nu^*(\sum_{i=1}^n N_i) \subseteq \nu^*(M) = \emptyset$ iff $\nu^*(\sum_{i=1}^n N_i) = \emptyset = \nu^*(M)$ iff $\sqrt[z^*]{\sum_{i=1}^n N_i} = M$.
(2) By (1) $\Delta$ is a subtractive generating set of $\eta^*(M)$ iff $\sqrt[z^*]{\sum_{i=1}^n N_i} = M$. Since $M$ is finitely generated, by Lemma 3.2 $\Delta$ is a subtractive generating set of $\eta^*(M)$ iff $\sum_{i=1}^n N_i = M$. □

**Theorem 3.6.** *Let $M$, $M'$ be $R$-modules and $f : \eta^*(M) \to \eta^*(M')$ be an $\eta^*(R)$-isomorphism. If $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$ is a subtractive linearly independent set of $\eta^*(M)$, then $\{f(\nu^*(N_1)), \cdots, f(\nu^*(N_n))\}$ is a subtractive linearly independent set of $\eta^*(M')$.*

*Proof.* Since $f$ is an isomorphism, $f(\nu^*(0)) = \nu^*(0)$. Hence $\nu^*(0) \notin \{f(\nu^*(N_1)), \cdots, f(\nu^*(N_n))\}$ because $\nu^*(0) \notin \Delta$. Now, suppose that there exists $1 \leq i \leq n$ such that

$$\nu^*(N') \in \gamma(f(\nu^*(N_i))) \cap \gamma(\textstyle\sum_{j\neq i} \eta^*(R)\nu^*(N_j)).$$

Since $f$ is surjective, $\nu^*(N') = f(\nu^*(N))$ for some submodule $N$ of $M$. Hence $f(\nu^*(N_i)) \subseteq f(\nu^*(N))$ and $f(\nu^*(\sum_{j\neq i} I_j N_j)) \subseteq f(\nu^*(N))$. By Lemma 2.7, $\nu^*(N_i) \subseteq \nu^*(N)$ and $\sum_{j\neq i} \nu^*(I_j N_j) \subseteq \nu^*(N)$. Thus $\nu^*(N) \in \gamma(\nu^*(N_i)) \cap \gamma(\sum_{j\neq i} \eta^*(R)*\nu^*(N_j))$. This implies that $\nu^*(N) = \nu^*(0)$. Therefore $f(\nu^*(N)) = \nu^*(0)$ and so $\nu^*(N') = \nu^*(0)$. Thus $\{f(\nu^*(N_1)), \cdots, f(\nu^*(N_n))\}$ is a subtractive linearly independent set of $\eta^*(M')$. □

For the remainder of this section, we assume that all modules are multiplication. So that $\nu^*(N) = \{Q \in \mathcal{X} \mid \sqrt{(N : M)} \subseteq \sqrt{(Q : M)}\}$ for every submodule $N$ of an $R$-module $M$.

**Lemma 3.7.** *Let $M$ be an $R$-module and $\mathcal{Y} \subseteq \mathcal{X}$. If $\mid \mathcal{X} \mid < \infty$, then $\nu^*(\xi(\mathcal{Y})) = \overline{\mathcal{Y}}$. In particular, $\mathcal{Y}$ is closed if and only if $\nu^*(\xi(\mathcal{Y})) = \mathcal{Y}$.*

*Proof.* Suppose $Q \in \mathcal{Y}$. Hence $\xi(\mathcal{Y}) \subseteq Q$. Therefore $\sqrt{(Q : M)} \supseteq \sqrt{(\xi(\mathcal{Y}) : M)}$. Since $M$ is multiplication, $Q \in \nu^*(\xi(\mathcal{Y}))$ and so $\mathcal{Y} \subseteq \nu^*(\xi(\mathcal{Y}))$. Next, let $\nu^*(N)$ be any closed subset of $\mathcal{X}$ containing $\mathcal{Y}$. Then $\sqrt{(Q : M)} \supseteq \sqrt{(N : M)}$ for every $Q \in \mathcal{Y}$ so that $\sqrt{(\xi(\mathcal{Y}) : M)} \supseteq \sqrt{(N : M)}$ since $\mid \mathcal{X} \mid < \infty$. Hence, for every $Q' \in \nu^*(\xi(\mathcal{Y}))$ we have $\sqrt{(Q' : M)} \supseteq \sqrt{(\xi(\mathcal{Y}) : M)} \supseteq \sqrt{(N : M)}$. Then $\nu^*(\xi(\mathcal{Y})) \subseteq \nu^*(N)$.

Thus $\nu^*(\xi(\mathcal{Y}))$ is the smallest closed subset of $\mathcal{X}$ containing $\mathcal{Y}$, hence $\nu^*(\xi(\mathcal{Y})) = \overline{\mathcal{Y}}$. □

**Lemma 3.8.** *Let $M$ be an $R$-module and $N$ be a submodule of $M$. If $\mid \mathcal{X} \mid < \infty$, then $\nu^*(\xi(\nu^*(N))) = \nu^*(\sqrt[z^*]{N}) = \nu^*(N)$.*

*Proof.* It is clear by Lemma 3.7. □

**Lemma 3.9.** *Let $M$ be an $R$-module and $N$, $N'$ be submodules of $M$. Then the following hold.*

1. *$\nu^*(N) \cup \nu^*(N') = \nu^*(N \cap N')$.*
2. *If $\mid \mathcal{X} \mid < \infty$, then $\sqrt[z^*]{\sqrt[z^*]{N}} = \sqrt[z^*]{N}$.*
3. *$\sqrt[z^*]{N \cap N'} = \sqrt[z^*]{N} \cap \sqrt[z^*]{N'}$.*

*Proof.* (1) Since $M$ is multiplication, we have $\nu^*(N) = \{Q \in \mathcal{X} \mid \sqrt{(N:M)} \subseteq \sqrt{(Q:M)}\}$ for a submodule $N$ of $M$. Hence the assertion follows from the fact that $(Q : M)$ is a primary ideal for $Q \in \mathcal{X}$.
(2) $\nu^*(\sqrt[z^*]{N}) = \nu^*(N)$, by Lemma 3.8. Therefore $\xi(\nu^*(\sqrt[z^*]{N})) = \xi(\nu^*(N))$. Thus $\sqrt[z^*]{\sqrt[z^*]{N}} = \sqrt[z^*]{N}$.
(3) $\sqrt[z^*]{N \cap N'} = \xi(\nu^*(N \cap N')) = \xi(\nu^*(N) \cup \nu^*(N')) = \xi(\nu^*(N)) \cap \xi(\nu^*(N')) = \sqrt[z^*]{N} \cap \sqrt[z^*]{N'}$, by (1). □

**Lemma 3.10.** *Let $M$ be an $R$-module such that $\mid \mathcal{X} \mid < \infty$ and for every submodule $K$ of $M$, $K \subseteq \sqrt[z^*]{K}$. If $N$, $N'$ are submodules of $M$, then $\gamma(\nu^*(N)) \cap \gamma(\nu^*(N')) = \gamma(\nu^*(\sqrt[z^*]{N} \cap \sqrt[z^*]{N'}))$.*

*Proof.* Suppose $\nu^*(N'') \in \gamma(\nu^*(N)) \cap \gamma(\nu^*(N'))$. So $\nu^*(N'') \in \gamma(\nu^*(N))$ and $\nu^*(N'') \in \gamma(\nu^*(N'))$. Hence $\nu^*(N) \subseteq \nu^*(N'')$ and $\nu^*(N') \subseteq \nu^*(N'')$. By Lemma 3.1, $\sqrt[z^*]{N''} \subseteq \sqrt[z^*]{N}$ and $\sqrt[z^*]{N''} \subseteq \sqrt[z^*]{N'}$. Therefore $\sqrt[z^*]{N''} \subseteq \sqrt[z^*]{N} \cap \sqrt[z^*]{N'}$. So $\nu^*(\sqrt[z^*]{N} \cap \sqrt[z^*]{N'}) \subseteq \nu^*(\sqrt[z^*]{N''})$. Thus $\nu^*(N'') \in \gamma(\nu^*(\sqrt[z^*]{N} \cap \sqrt[z^*]{N'}))$. For the reverse inclusion, let $\nu^*(N'') \in \gamma(\nu^*(\sqrt[z^*]{N} \cap \sqrt[z^*]{N'}))$. Then $\nu^*(\sqrt[z^*]{N} \cap \sqrt[z^*]{N'}) \subseteq \nu^*(N'')$. Hence by Lemma 3.1 and Lemma 3.9 $\sqrt[z^*]{N''} \subseteq \sqrt[z^*]{\sqrt[z^*]{N} \cap \sqrt[z^*]{N'}} = \sqrt[z^*]{\sqrt[z^*]{N}} \cap \sqrt[z^*]{\sqrt[z^*]{N'}} = \sqrt[z^*]{N} \cap \sqrt[z^*]{N'}$. Thus $\sqrt[z^*]{N''} \subseteq \sqrt[z^*]{N}$ and $\sqrt[z^*]{N''} \subseteq \sqrt[z^*]{N'}$. By Lemma 3.1, $\nu^*(N) \subseteq \nu^*(N'')$ and $\nu^*(N') \subseteq \nu^*(N'')$. Thus $\nu^*(N'') \in \gamma(\nu^*(N)) \cap \gamma(\nu^*(N'))$ □

**Lemma 3.11.** *Let $M$ be an $R$-module such that $\mid \mathcal{X} \mid < \infty$ and for every submodule $N$ of $M$, $N \subseteq \sqrt[z^*]{N}$. If $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$, then $\Delta$ is a subtractive linearly independent set of $\eta^*(M)$ if and only if $\nu^*(0) \notin \Delta$ and $\sqrt[z^*]{N_i} \cap \sqrt[z^*]{\sum_{j \neq i} N_j} = \sqrt[z^*]{0}$, for each $i$, $(1 \leq i \leq n)$.*

*Proof.* Suppose $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$. Therefore $\gamma(\sum_{i=1}^n \eta^*(R) * \nu^*(N_i)) = \gamma(\nu^*(\sum_{i=1}^n N_i))$ by Theorem 2.17. Thus $\Delta$ is a subtractive linearly independent set of $\eta^*(M)$ if and only if $\nu^*(0) \notin \Delta$ and $\gamma(\nu^*(N_i)) \cap \gamma(\nu^*(\sum_{j\neq i} N_j)) = \{\nu^*(0)\}$ for each $i$, $(1 \leq i \leq n)$. Therefore $\gamma(\nu^*(\sqrt[Z^*]{N_i} \cap \sqrt[Z^*]{\sum_{j\neq i} N_j})) = \{\nu^*(0)\}$ for each $i$, $(1 \leq i \leq n)$ by Lemma 3.10, so $\nu^*(\sqrt[Z^*]{N_i} \cap \sqrt[Z^*]{\sum_{j\neq i} N_j}) = \nu^*(0)$ for each $i$, $(1 \leq i \leq n)$. Thus by Lemma 3.1 and Lemma 3.9 we have $\sqrt[Z^*]{N_i} \cap \sqrt[Z^*]{\sum_{j\neq i} N_j} = \sqrt[Z^*]{0}$ for each $i$, $(1 \leq i \leq n)$. □

**Lemma 3.12.** *Let $M$ be a $Z^*$-radical $R$-module such that $|\ \mathcal{X}\ | < \infty$ and for every submodule $N$ of $M$, $N \subseteq \sqrt[Z^*]{N}$. If $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$ is a subtractive linearly independent set of $\eta^*(M)$, then $\sum_{i=1}^n N_i$ is direct.*

*Proof.* By Lemma 3.11, $\sqrt[Z^*]{N_i} \cap \sqrt[Z^*]{\sum_{j\neq i} N_j} = \sqrt[Z^*]{0} = 0$ for each $i$, $(1 \leq i \leq n)$. By assumption $N_i \cap \sum_{j\neq i} N_j = 0$. Thus $\sum_{i=1}^n N_i$ is direct. □

**Theorem 3.13.** *Let $M$ be a Noetherian $Z^*$-radical $R$-module such that for every submodule $N$ of $M$ and $Q \in \mathcal{X}$, $N \subseteq \sqrt[Z^*]{N}$ and $rad(Q) \cap N = rad(Q \cap N)$. If $|\ \mathcal{X}\ | < \infty$, then $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n) \mid N_i \neq 0\}$ is a subtractive linearly independent set of $\eta^*(M)$ if and only if $M = \bigoplus_{i=1}^n N_i$.*

*Proof.* Suppose $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$ is a subtractive linearly independent set of $\eta^*(M)$. Hence by Lemma 3.5(2), $M = \sum_{i=1}^n N_i$. Thus by Lemma 3.12, $M = \bigoplus_{i=1}^n N_i$. Conversely, assume $M = \bigoplus_{i=1}^n N_i$. Hence by Lemma 3.5(2), $\Delta$ is a subtractive generating set of $\eta^*(M)$. Moreover, for every $i$, $(1 \leq i \leq n)$ we have $\sqrt[Z^*]{0} = 0 = N_i \cap \sum_{j\neq i} N_j = \sqrt[Z^*]{N_i} \cap \sqrt[Z^*]{\sum_{j\neq i} N_j}$ by Lemma 3.4. Since $N_i \neq 0$ for every $i$, $(1 \leq i \leq n)$ we have $\nu^*(0) \notin \Delta$. Thus $\Delta$ is a subtractive linearly independent set of $\eta^*(M)$ by Lemma 3.11. □

Let $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$ be a subtractive linearly independent set of $\eta^*(M)$. Assume that for some $j$, $(1 \leq j \leq n)$ there exist submodules $N'_{j_1}$ and $N'_{j_2}$ of $M$ such that $\Gamma = \{\nu^*(N_1), \cdots, \nu^*(N_{j-1}), \nu^*(N'_{j_1}), \nu^*(N'_{j_2}), \nu^*(N_{j+1}), \nu^*(N_n)\}$ is likewise a subtractive linearly independent set of $\eta^*(M)$. Then $\Gamma$ is said to be a simple refinement of $\Delta$. A subtractive linearly independent set $\Delta$ of $\eta^*(M)$ is said to be a subtractive basis if there does not exist a simple refinement of $\Delta$.

**Corollary 3.14.** *Let $M$ be a Noetherian $Z^*$-radical $R$-module such that for every submodule $N$ of $M$ and $Q \in \mathcal{X}$, $N \subseteq \sqrt[Z^*]{N}$ and $rad(Q) \cap N = rad(Q \cap N)$. If $|\mathcal{X}| < \infty$, then $\eta^*(M)$ has a subtractive basis.*

*Proof.* Since $M$ is Noetherian, it has a finite indecomposable direct sum decomposition such as $M = \bigoplus_{i=1}^{n} N_i$. Thus by Theorem 3.13 $\{\nu^*(N_i)\}_{i=1}^{n}$ is a subtractive basis for $M$. □

**Corollary 3.15.** *Let $M$ be a Noetherian $Z^*$-radical $R$-module such that $|\mathcal{X}| < \infty$ and for every submodule $N$ of $M$ and $Q \in \mathcal{X}$, $N \subseteq \sqrt[Z^*]{N}$ and $rad(Q) \cap N = rad(Q \cap N)$. If $N'$ is a direct summand of $M$ and $N''$ is a submodule of $M$ such that $\sqrt[Z^*]{N''} = N'$, then $N'' = N'$.*

*Proof.* By Lemma 3.4, $\sqrt[Z^*]{N'} = N'$. Hence $\sqrt[Z^*]{N''} = N' = \sqrt[Z^*]{N'}$. So by Lemma 3.1, $\nu^*(N') = \nu^*(N'')$. Hence by Theorem 3.13 $N''$ is a direct summand of $M$. Then by Lemma 3.4, $\sqrt[Z^*]{N''} = N''$. Thus $N'' = N'$. □

## Acknowledgments

The author would like to thank the anonymous reviewers for their careful reading and many helpful suggestions on this paper.

## References

1. J. Abuhlail, A dual zariski topology for modules, *Topology Appl.* **158** (2011), 457-467.
2. H. Ansari-Toroghy and R. Ovlyaee-Sarmazdeh, On the prime spectrum of a module and zariski topologies, *Comm. Algebra* **38** (2010), 4461-4475.
3. M. Behboodi and M. R. Haddadi, Classical zariski topology of modules and spectral spaces I, *Int. Elec. J. Algebra* **4** (2008), 104-130.
4. Z. A. El-Bast and P. F. Smith, Multiplication modules, *Comm. Algebra* **16** (1988), 755-779.
5. H. Fazaeli Moghimi and F. Rashedi, Primary-like submodules satisfying the primeful property, (to appear).
6. H. Fazaeli Moghimi and F. Rashedi, A module whose primary-like spectrum has the zariski-like topology, (submitted).
7. H. Fazaeli Moghimi and F. Rashedi, The zariski topolgy on the spectrum of primary-like submodules, (submitted).
8. J. S. Golan, The Theory of Semirings with Applications in Mathematics and Theoretical Computer Science, Pitman monographs and surveys in pure and applied mathematics, John Wiley and Sons, New York, 1992.
9. I. Kaplansky, *Commutative Rings*, University of Chicago press, Chicago, 1974.
10. C. P. Lu, A module whose prime spectrum has the surjective natural map, *Houston J. Math.* **33** (2007), 125-143.

11. C. P. Lu, Saturations of submodules, *Comm. Algebra* **31** (2003), 2655- 2673.
12. C. P. Lu, The zariski topology on the prime spectrum of a module, *Houston J. Math.* **25** (1999), 417-425.
13. R. L. McCasland, M. E. Moore and P. F. Smith, Subtractive bases of zariski spaces, *Houston J. Math.* **32** (2006), 689-701.
14. R. L. McCasland, M. E. Moore and P. F. Smith, An introduction to zariski spaces over zariski topologies, *Rocky Mountain J. Math.* **28** (1998), 1357-1369.
15. R. L. McCasland, M. E. Moore and P. F. Smith, On the spectrum of a module over a commutative ring, *Comm. Algebra* **25** (1997), 79-103.
16. A. Nikseresht and A. Azizi, Zariski spaces of modules, *J. Pure Appl. Algebra* **217** (2013), 1187-1194.

**Hosein Fazaeli Moghimi**
Department of Mathematics, Department of Mathematics, University of Birjand, P.O. Box 97175-615, Birjand, Iran.
Email: `hfazaeli@birjand.ac.ir`

**Fatemeh Rashedi**
Department of Mathematics, University of Birjand, P.O. Box 97175-615, Birjand, Iran.
Email: `fatemehrashedi@birjand.ac.ir`