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A SURVEY ON THE FUSIBLE PROPERTY OF SKEW PBW EXTENSIONS

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Dedicated to the memory of Professor V. A. Artamonov

Abstract. We present several results that establish the fusible and the regular left fusible properties of the family of noncommutative rings known as skew Poincaré-Birkhoff-Witt extensions. Our treatment is based on the recent works of Ghashghaei and McGovern [13], and Koşan and Matczuk [31] concerning the left fusibleness and the regular left fusibleness of skew polynomial rings of automorphism type. Since the results formulated in this paper can be applied to algebraic structures more general than skew polynomial rings, our contribution to the theory of fusibleness is to cover more families of rings of interest in branches as quantum groups, noncommutative algebraic geometry and noncommutative differential geometry. We provide illustrative examples of the ideas developed here.

1. Introduction

For every topological space with certain restrictions, two non-trivial algebraic structures can be associated: the ring of continuous bounded functions with real values, and the ring of all continuous functions with real values defined on the space. A first approach to these rings of functions associated with topological spaces was presented by Gillman and Jerison [15]. They studied the algebraic properties of the ring of

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continuous functions at real value $C(X)$ and the ring of continuous bounded functions to real value denoted as $C^*(X)$, both rings defined on an arbitrary topological space X, and showed that a space is completely regular and Hausdorff if the family of all zero sets forms a basis for the closed subsets of a topology. This fact together with another results allow to show that for the ring of continuous functions at real value $C(X)$, every zero-divisor element can be written as the sum of a zero-divisor element and a non zero-divisor (also called *regular*) element.

Since the sum of two zero-divisors need not be a zero-divisor, Faith and Pillay [10], Theorem 1.12, characterized those commutative rings for which the set of zero-divisors is an ideal. Now, as is well known, if the set of left zero-divisors in a ring R is not a left ideal, then there exists a left zero-divisor which can be expressed as the sum of a left zero-divisor and a non-left zero-divisor in R. This fact motivated Ghashghaei and McGovern [13] to investigate the class of rings in which every element can be written as the sum of a left zero-divisor and a non-left zero-divisor ([13], p. 1151). Rings with this type of decomposition were defined by them in their paper as *fusible rings*. In the case of the noncommutative rings of polynomial type introduced by Ore [45], the *skew polynomial rings* or *Ore extensions*, they showed that if R is a left fusible ring with an automorphism σ, then $R[x; \sigma]$ (the skew polynomial ring of *automorphism type*) is also left fusible ([13], Proposition 2.9). Now, more recently, in 2019, Koşan and Matczuk [31] defined a more general class of fusible rings, the *regular fusible rings* (see Definition 2.6) that enriches this new theory, and proved that if R is a regular left fusible ring with an automorphism σ, then $R[x; \sigma]$ is also regular left fusible. As a matter of fact, Koşan and Matczuk answer two of the fourth questions asked by Ghashghaei and McGovern [13], and established interesting results such as a characterization of semiprime left Goldie rings in terms of this new class of fusible rings.

With the above results in mind and motivated for the recent development of the theory about the fusible property for noncommutative rings of polynomial type, our aim in this paper is to show that the fusible and regular fusible properties hold in several families of rings more general than skew polynomial rings of automorphism type. With this objective, our point of view is based on some ideas developed in the context of the *skew PBW extensions* (PBW denotes Poincaré-Birkhoff-Witt). These noncommutative algebraic structures were introduced by

Gallego and Lezama [12] as a natural generalization of *PBW extensions* defined by Bell and Goodearl [5]. Since its introduction, ring and homological properties of skew PBW extensions have been studied by several authors (c.f., Artamonov [4], Hamidizadeh et al., [17], Hashemi et al., [18], [19], [20], Lezama et al., [34], [36], [37], Louzari and Reyes [39], Niño et al., [42] and [43]), and, in fact, a book that includes several of the works carried out for these extensions have been published recently, see Fajardo et al., [11]. In Sections 3 and 5, we will say some words about the relation of skew PBW extensions with respect to skew polynomial rings and its importance in the framework of noncommutative rings having PBW bases.

The paper is organized as follows. In Section 2, we make a brief description of some facts about the fusible and regular fusible properties. We also recall the key results about this property in the skew polynomial rings setting with the aim of motivating the results we want to generalize. Next, in Section 3, we consider some facts about skew PBW extensions, and then we proceed in Section 4 to formulate sufficient conditions to guarantee that these extensions are fusible (see Theorems 4.1, 4.2, 4.3, and 4.4). Section 5 contains a detailed list of examples of skew PBW extensions which are not skew polynomial rings of automorphism type, so the fusibleness of these examples follow from the above theorems. The results presented in this paper are new in the theory and cover more families of rings, so that our work can be considered as a contribution to the study of the fusible property of algebraic structures.

Throughout this article, the word ring means an associative ring (not necessarily commutative) with identity. The letters k and $\Bbbk$ denote a commutative ring and a field, respectively. $\mathbb{R}$ and $\mathbb{C}$ denote, as usual, the set of real and complex numbers, respectively. Recall that a *reduced* ring is a ring without nonzero nilpotent elements, and in an *Abelian* ring, every idempotent is central. Of course, reduced rings are Abelian.

2. Fusible and regular fusible rings

In this section, we present the notion of *fusible ring* introduced by Ghashghaei and McGovern [13]. We recall some key facts about this concept and present a more general and recent notion introduced by Koşan and Matczuk [31], the *regular fusible ring*. For both notions, we present different algebraic objects that illustrate them.

For the next definition, following Ghashghaei and McGovern [13], if b is an element of a ring R, $\mathrm{Ann}_l(b) = \{a \in R \mid ab = 0\}$ and $\mathrm{Ann}_r(b) = \{a \in R \mid ba = 0\}$ denotes the *left annihilator and the right annihilator ideals of* $b \in R$, respectively. If $\mathrm{Ann}_r(b) \neq 0$, then b is called a *left zero-divisor*; in other case, b is a *non-left zero-divisor*. For these authors, $Z_l(R)$ and $Z_l^*(R)$ denote *the set of left zero-divisors and non-left zero-divisors of* R, respectively.

Definition 2.1 ([13], Definition 2.1)**.** Let R be a ring. We call a non-zero element $a \in R$ *left fusible* if it can be expressed as the sum of a left zero-divisor and a non-left zero-divisor in R. R is said to be *left fusible* if every non-zero element of R is left fusible. *Right fusible* rings are defined analogously. A ring R which is both right and left fusible is called *fusible ring*.

We present some examples of fusible rings. These are adapted from [13], Remark 2.2. Consider R a ring.

Example 2.2.
(i) Recall that $r \in R$ is called a *regular element* if r is not a zero-divisor. Thus, every regular element has a trivial left fusible representation given by $r = 0 + r$; hence, this element is left fusible. Analogously, it is right fusible. A ring where every element is regular is known as a *domain*. In this way, every domain is a fusible ring.
(ii) An element $e \in R$ is said to be *idempotent* if $e^2 = e$. Every idempotent element has a left fusible representation given by $e = (1 - e) + (2e - 1)$. Therefore, every Boolean ring (a ring R is said to *Boolean* if $a^2 = a$, for all $a \in R$) is a fusible ring.
(iii) A ring R is said to be *special almost clean* if each element a of R can be decomposed as the sum of a regular element $r \in R$ and an idempotent $e \in R$ with $aR \cap eR = 0$ (see [1], p. 851). Thus, it is clear that every special almost clean ring is a fusible ring.

Remark 2.3. Some algebraic properties of the ring of continuous functions at real value $C(X, \mathbb{R})$ and the ring of continuous bounded functions to real value denoted as $C^*(X, \mathbb{R})$, both rings defined on an arbitrary topological space X, were studied by Gillman and Jerison [15]. There, it was shown that for any topological space X, $C(X, A)$ is a fusible ring, where A is a subring of $\mathbb{R}$ (see [13], Theorem 4.3).

Next, we present a very interesting fact that relates the left fusible property with the skew polynomial rings of automorphism type. Definition and properties of these noncommutative rings can be found in Goodearl and Warfield [16] or McConnell and Robson [40].

Proposition 2.4 ([13], Proposition 2.9)**.** *If R is a left fusible ring and σ is a ring automorphism of R, then $R[x;\sigma]$ is a left fusible ring.*

The following example illustrates that the previous proposition need not be true for any skew polynomial ring; besides, this shows that there exist right fusible elements which are not left fusible.

Example 2.5 ([13], Example 2.10)**.** Let R be a domain and σ be a ring endomorphism of R which is not injective. Let $r \in R$ which satisfies $\sigma(r) = 0$, with $r \neq 0$, so $xr = \sigma(r)x = 0$. The set of zero-divisors of $R[x, \sigma]$ is represented by the ideal $\langle x \rangle$. Since the set of left zero-divisors of $R[x;\sigma]$ is a left ideal, then $R[x;\sigma]$ is not a left fusible ring. By definition, it follows that x is a non-right zero-divisor element, and hence x is a right fusible element but not a left fusible element.

The next definition presents the concept of *regular left fusible ring*. This class of rings was introduced by Koşan and Matczuk [31], in order to continue with the study of the fusible property of noncommutative rings. Before presenting Definition 2.6, we have to say some words about the notation.

Following Koşan and Matczuk [31], for a nonempty subset $S \subseteq R$, $\mathrm{lann}_R(S)$ stands for the *left annihilator of S in R*, that is,

$$\mathrm{lann}_R(S) = \{b \in R \mid bS = 0\}.$$

An element $a \in R$ is a *left zero-divisor* if $\mathrm{lann}_R(a) \neq 0$. The elements which are not left zero-divisors are said to be *left regular*. With this notation, the left zero-divisors considered by Koşan and Matczuk are right zero-divisors in the meaning of Ghashghaei and McGovern [13], so, in Definition 2.6, left fusible rings of Koşan and Matczuk are right fusible rings in the language of Ghashghaei and McGovern.

Definition 2.6 ([31], Definition 2.1)**.** A ring R is said to be *regular left fusible* if for any non-zero element $r \in R$, there exists a regular (i.e., left and right regular) element $s \in R$ such that the element sr is left fusible, i.e., $sr = c + w$, where c is a left zero-divisor and w is left regular. *Regular right fusible* rings are defined analogously. A ring which is both right and regular left fusible is called *regular fusible ring*.

Since every left regular element is left fusible, it is enough to observe that for every non-zero left zero-divisor a, there exists a regular element s in R such that the element sa has a left fusible representation. Now, every left fusible ring is regular left fusible ring. If $r = a + w$ is a left fusible representation, in particular, it is a regular left fusible representation taking $s = 1$. Nevertheless, the following example shows that the class of left fusible rings is strictly smaller than the class of regular left fusible rings.

Example 2.7 ([31], Example 2.2)**.** Consider the $\Bbbk$-algebra $R = \Bbbk\langle x, y\rangle$ subject to the relation $x^2 = 0$. Cohn [8] showed that the set of all left zero-divisors of R is equal to xR and hence, the sum of any two left zero-divisors is a left zero-divisor. Thus R is not a left fusible ring. Furthermore, y is a regular element of R and $ry = 0 + ry$ is a left fusible decomposition of ry, for any non-zero element $r \in R$. Therefore, R is a regular left fusible ring.

Now, we consider some important properties and results related to regular left fusible rings. We also present an interesting relationship of regular left fusible rings with Goldie's theory. The following statements are taken from [31]. Before, we present some definitions about localization in ring theory.

Definition 2.8 ([16] or [40])**.** Let R be a ring.

(i) A *multiplicative set* in R is a subset $S \subseteq R$ such that $1 \in S$ and S is closed under multiplication. If S is a multiplicative set in R, then S is a *left Ore set* if it satisfies that for all $r_1 \in R$, $s_1 \in R$, there exist $r_2 \in R$ and $s_2 \in S$ such that $r_1 s_2 = s_1 r_2$ (this property is commonly known as *the Ore's condition*). If S is a left Ore set, then $S^{-1}R$ is called the *left Ore localization* of R by S.

(ii) We recall that if S is the set of regular elements of R, then $Q_l(R)$ is called the *classical ring of left quotients* of R: if d is invertible in $Q_l(R)$, for every $d \in S$, each $q \in Q_l(R)$ can be factored $q = d^{-1}a$, for some $d \in S$ and $a \in R$. *Classical ring of right quotients* are defined similarly and it is denoted by $Q_r(R)$.

Proposition 2.9 ([31], Proposition 2.8)**.** *For R a ring, the following assertions hold:* (1) *If R is left fusible, then so is $S^{-1}R$.* (2) *If R is regular left fusible, then $Q_l(R)$ is left fusible.* (3) *If $S^{-1}R$ is regular left fusible, then R is regular left fusible.*

From [31], Corollary 2.9, we know that if a ring R has left quotient ring $Q_l(R)$, then R is regular left fusible if and only if $Q_l(R)$ is left fusible if and only if $Q_l(R)$ is regular left fusible.

Two important concepts in the study of Goldie's theory are related to this new class of regular fusible rings. Let us recall some notions of ring theory. If R is a ring, a left ideal I of R is *essential*, denoted by $I \leq_e {}_RR$, if $J \cap I \neq 0$, for every nonzero left ideal J of R. The *left singular ideal* of R is $\mathrm{Sing}_l(R) := \{b \in R \mid \mathrm{lann}_R(b) <_e {}_RR\}$. One can check that $\mathrm{Sing}_l(R)$ is a two-sided ideal of R. R is said to be *left nonsingular* if $\mathrm{Sing}_l(R) = 0$. On the other hand, a ring R is said to be *unit-regular* if for each $b \in R$, there exists a unit $u \in R$ such that $bub = b$. From Camillo and Khurana [7], Theorem 1, we know that R is unit-regular if and only if every element b of R can be written as $b = e + u$ such that $bR \cap eR = 0$, where e is an idempotent and u is a unit in R. This shows that every element in a unit-regular ring R can be expressed as a left (right) zero-divisor and a unit. Some examples of unit-regular rings are semisimple rings and strongly regular rings including, of course, the commutative von Neumann regular rings.

Proposition 2.10 ([31], Lemma 2.6 and Proposition 2.13)**.** *For R a ring, the following assertions hold:* (1) *If R is regular left fusible, then R is left nonsingular.* (2) *If R is unit-regular, then R is regular left fusible.*

From the above propositions and Goldie's theorem, a characterization of semiprime left Goldie rings in terms of regular left fusible rings were presented by Koşan and Matczuk [31]. Before, recall that a *semiprime ring* R is a ring in which the zero ideal is a semiprime ideal, that is, the zero ideal is an intersection of prime ideals (a prime ideal in a ring R is any proper ideal P of R such that, whenever I and J are ideals of R with $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$). For more details about semiprime rings, see Goodearl and Warfield [16] or McConnell and Robson [40].

Proposition 2.11 ([31], Theorem 2.14)**.** *If R is a semiprime ring, then R is left Goldie if and only if R is regular left fusible and has finite left Goldie* (*uniform*) *dimension.*

The next result generalizes Proposition 2.4, from left fusible rings to regular left fusible rings.

Proposition 2.12 ([31], Proposition 2.20)**.** *If R is a regular left fusible ring and σ is a ring automorphism of R, then $R[x;\sigma]$ is regular left fusible.*

In the next section, we present the *skew PBW extensions.* We recall some definitions and properties that are very useful to guarantee the fusibleness of noncommutative rings more general than skew polynomial rings of automorphism type.

3. Skew PBW extensions

As we said in the Introduction, the *skew PBW extensions* were introduced by Gallego and Lezama [12] with the aim of generalizing the *PBW extensions* defined by Bell and Goodearl [5]. During the last years, different authors have been shown that skew PBW extensions include remarkable examples of algebraic structures of interest in several branches of mathematics and theoretical physics. Briefly, we list some of these examples (see [53], for a detailed reference of every family of algebras): skew polynomial rings of injective type defined by Ore [45], almost normalizing extensions defined by McConnell and Robson [40], algebras of solvable type introduced by Kandri-Rody and Weispfenning, diffusion algebras [25], 3-dimensional skew polynomial algebras defined by Bell and Smith (cf. [55]), classes of algebras such as some types of Auslander-Gorenstein rings, some Calabi-Yau and skew Calabi-Yau algebras [52], and families of Artin-Schelter regular algebras.

Definition 3.1 ([12], Definition 1)**.** Let R and A be rings. We say that A is a *skew PBW extension* (also known as *σ-PBW extension*) *over* R, which is denoted by $A := \sigma(R)\langle x_1, \dots, x_n\rangle$, if the following conditions hold:

(i) R (the ring of coefficients) is a subring of A sharing the same multiplicative identity element.

(ii) there exist elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis
$$\mathrm{Mon}(A) := \{x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\},$$
and $x_1^0 \cdots x_n^0 := 1 \in \mathrm{Mon}(A)$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For any elements $1 \leq i, j \leq n$, there exists an element $d_{i,j} \in R \setminus \{0\}$ such that
$$x_j x_i - d_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$$
(i.e., there exist elements $r_0^{(i,j)}, r_1^{(i,j)}, \dots, r_n^{(i,j)}$ of R with $x_j x_i - d_{i,j} x_i x_j = r_0^{(i,j)} + \sum_{l=1}^{n} r_l^{(i,j)} x_l$).

Since $\mathrm{Mon}(A)$ is a left R-basis of A, the elements $c_{i,r}$ and $d_{i,j}$ are unique. Detailed examples of skew PBW extensions can be consulted in [22], Section 3, and [54], Section 5.

The next proposition establishes the relationship between skew polynomial rings [45] and skew PBW extensions.

Proposition 3.2 ([12], Proposition 3)**.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension. For each $1 \leq i \leq n$, there exist an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r)x_i + \delta_i(r)$, for each $r \in R$. We write $\Sigma := \{\sigma_1, \dots, \sigma_n\}$, and $\Delta := \{\delta_1, \dots, \delta_n\}$.*

Definition 3.3 ([12], Definition 4)**.** Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension.

(i) A is called *quasi-commutative* if the conditions (iii) and (iv) presented in the Definition 3.1 are replaced by the following: (iii') For every $1 \leq i \leq n$ and $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r}x_i$. (iv') For every $1 \leq i, j \leq n$ there exists $d_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = d_{i,j}x_i x_j$.

(ii) A is said to be *bijective* if σ_i is bijective, for each $1 \leq i \leq n$, and $d_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

(iii) A is called of *endomorphism type* if $\delta_i = 0$, for every $i = 1, \dots, n$. If we also have that every element of Σ is a bijective function, then A is said to be of *automorphism type*.

Remark 3.4. If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a quasi-commutative skew PBW extension over a ring R, then A is isomorphic to an iterated Ore extension of endomorphism type ([38], Theorem 2.3). Nevertheless, skew PBW extensions of endomorphism type are more general than iterated Ore extensions. With the aim of illustrating the differences between these structures, we consider the situations with two and three indeterminates.

If we take the iterated Ore extension of endomorphism type given by $R[x; \sigma_x][y; \sigma_y]$, by definition (see [16] or [40]), for any element $r \in R$, we have the following relations: $xr = \sigma_x(r)x$, $yr = \sigma_y(r)y$, and $yx = \sigma_y(x)y$. On the other hand, if we have $\sigma(R)\langle x, y\rangle$ a skew PBW extension of endomorphism type over R, then for any $r \in R$, by Definition 3.1, we have the relations $xr = \sigma_1(r)x$, $yr = \sigma_2(r)y$, and $yx = d_{1,2}xy + r_0 + r_1x + r_2y$, for some elements $d_{1,2}, r_0, r_1$ and r_2 belong to R. When we compare the defining relations of both algebraic structures, it is clear which one of them is more general.

Now, if we have the iterated Ore extension $R[x; \sigma_x][y; \sigma_y][z; \sigma_z]$, then for any $r \in R$, $xr = \sigma_x(r)x$, $yr = \sigma_y(r)y$, $zr = \sigma_z(r)z$, $yx = \sigma_y(x)y$, $zx = \sigma_z(x)z$, $zy = \sigma_z(y)z$. On the other hand, for the skew PBW extension of automorphism type $\sigma(R)\langle x, y, z\rangle$, we have the relations given by $xr = \sigma_1(r)x$, $yr = \sigma_2(r)y$, $zr = \sigma_3(r)z$, $yx = d_{1,2}xy + r_0 + r_1x + r_2y + r_3z$, $zx = d_{1,3}xz + r_0' + r_1'x + r_2'y + r_3'z$, and $zy = d_{2,3}yz + r_0'' + r_1''x + r_2''y + r_3''z$, for some elements $d_{1,2}, d_{1,3}, d_{2,3}, r_0, r_0', r_0'', r_1, r_1', r_1'', r_2, r_2', r_2'', r_3, r_3', r_3''$ of

R. As we can see, as the number of indeterminates increases, the differences between both algebraic structures are more remarkable.

These differences between skew polynomial rings of automorphism type and skew PBW extensions of automorphism type will be key in our Theorems 4.1 and 4.2.

The following definition presents some facts about the way elements are written in skew PBW extensions.

Definition 3.5 ([12], Section 3)**.** Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension.

(i) Consider the families Σ and Δ in Proposition 3.2. Throughout the paper, for any element $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write $\sigma^{\alpha} := \sigma_1^{\alpha_1} \circ \cdots \circ \sigma_n^{\alpha_n}$, $\delta^{\alpha} = \delta_1^{\alpha_1} \circ \cdots \circ \delta_n^{\alpha_n}$, where $\circ$ denotes composition, and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

(ii) Let $\succeq$ be a total order defined on $\mathrm{Mon}(A)$. If $x^{\alpha} \succeq x^{\beta}$ but $x^{\alpha} \neq x^{\beta}$, we will write $x^{\alpha} \succ x^{\beta}$. If f is a non-zero element of A, then f can be expressed uniquely as $f = a_0 + a_1X_1 + \cdots + a_mX_m$, with $a_i \in R$, and $X_m \succ \cdots \succ X_1$ (eventually, we will use expressions as $f = a_0 + a_1Y_1 + \cdots + a_mY_m$, with $a_i \in R$, and $Y_m \succ \cdots \succ Y_1$). With this notation, we define $\mathrm{lm}(f) := X_m$, the *leading monomial* of f; $\mathrm{lc}(f) := a_m$, the *leading coefficient* of f; $\mathrm{lt}(f) := a_mX_m$, the *leading term* of f; $\exp(f) := \exp(X_m)$, the *order* of f. Note that $\deg(f) := \max\{\deg(X_i)\}_{i=1}^{m}$. Finally, if $f = 0$, then $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$. We also consider $X \succ 0$ for any $X \in \mathrm{Mon}(A)$. Thus, we extend $\succeq$ to $\mathrm{Mon}(A) \cup \{0\}$.

Following [12], Definition 11, if $\succeq$ is a total order on $\mathrm{Mon}(A)$, we say that $\succeq$ is a *monomial order* on $\mathrm{Mon}(A)$ if the following conditions hold:

- For every $x^{\beta}, x^{\alpha}, x^{\gamma}, x^{\lambda} \in \mathrm{Mon}(A)$, $x^{\beta} \succeq x^{\alpha}$ implies that $\mathrm{lm}(x^{\gamma}x^{\beta}x^{\lambda}) \succeq \mathrm{lm}(x^{\gamma}x^{\alpha}x^{\lambda})$ (the total order is compatible with multiplication).
- $x^{\alpha} \succeq 1$, for every $x^{\alpha} \in \mathrm{Mon}(A)$.
- $\succeq$ is degree compatible, i.e., $|\beta| \succeq |\alpha| \Rightarrow x^{\beta} \succeq x^{\alpha}$.

The next proposition is very useful when one need to make some computations with elements of skew PBW extensions.

Proposition 3.6 ([12], Theorem 7)**.** *If A is a polynomial ring with coefficients in R with respect to the set of indeterminates $\{x_1, \dots, x_n\}$, then A is a skew PBW extension of R if and only if the following conditions hold:*

(1) *for each* $x^\alpha \in \mathrm{Mon}(A)$ *and every* $0 \neq r \in R$, *there exist unique elements* $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$, $p_{\alpha,r} \in A$, *such that* $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, *where* $p_{\alpha,r} = 0$, *or* $\deg(p_{\alpha,r}) < |\alpha|$ *if* $p_{\alpha,r} \neq 0$. *If* r *is left invertible, so is* r_α.

(2) *For each* $x^\alpha, x^\beta \in \mathrm{Mon}(A)$, *there exist unique elements* $c_{\alpha,\beta} \in R$ *and* $p_{\alpha,\beta} \in A$ *such that* $x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$, *where* $c_{\alpha,\beta}$ *is left invertible,* $p_{\alpha,\beta} = 0$, *or* $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ *if* $p_{\alpha,\beta} \neq 0$.

Next, we recall the Hilbert's Basis theorem for skew PBW extensions.

Proposition 3.7 ([38], Corollary 2.4)**.** (*Hilbert Basis Theorem*). *If* $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ *is a bijective skew PBW extension over a left Noetherian ring* R, *then* A *is also a left Noetherian ring. Analogously for the right case.*

Finally, in this section, we recall that a *filtered ring* is a ring R with a family $F(R) = \{F_n(R) \mid n \in \mathbb{Z}\}$ of additive subgroups of R, where we have the ascending chain $\cdots \subset F_{n-1}(R) \subset F_n(R) \subset \cdots$ such that $1 \in F_0(R)$ and $F_n(R)F_m(R) \subseteq F_{n+m}R$, for all $n, m \in \mathbb{Z}$. From a filtered ring R, it is possible to construct its associated graded ring $G(R)$ taking $G(R)_n := F_n(R)/F_{n-1}(R)$ (see McConnell and Robson [40] for more details).

The next result establishes that skew PBW extensions are filtered rings and describes its associated graded ring.

Proposition 3.8 ([38], Theorem 2.2)**.** *If* $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ *is a skew PBW extension over* R, *then* A *is a filtered ring with filtration given by* $F_m(A) := R$, *if* $m = 0$, *and* $\{f \in A \mid \deg(f) \leq m\}$, *if* $m \geq 1$. *The corresponding graded ring* $G(A)$ *is a quasi-commutative skew PBW extension of* R. *Moreover, if* A *is bijective, then* $G(A)$ *is a quasi-commutative bijective skew PBW extension of* R.

For the next section, we also need two results about localization of these extensions.

Proposition 3.9 ([35], Lemma 2.6)**.** *Let* $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ *be a skew PBW extension over* R, *and let* S *be the set of regular elements of* R *such that* $\sigma_i(S) = S$, *for every* $1 \leq i \leq n$, *where* σ_i *is considered as in Proposition 3.2.*

(1) *If* $S^{-1}R$ *exists, then* $S^{-1}A$ *exists and it is a bijective skew PBW extension of* $S^{-1}R$ *with* $S^{-1}A = \sigma(S^{-1}R)\langle x'_1, \dots, x'_n\rangle$, *where* $x'_i := \frac{x_i}{1}$ *and the system of constants of* $S^{-1}R$ *is given by* $c'_{i,j} := \frac{c_{i,j}}{1}$, $c'_{i,\frac{r}{s}} := \frac{\sigma_i(r)}{\sigma_i(s)}$, *for all* $1 \leq i, j \leq n$.

(2) *If RS^{-1} exists, then AS^{-1} exists and it is a bijective skew PBW extension of RS^{-1} with $AS^{-1} = \sigma(RS^{-1})\langle x_1', \dots, x_n'\rangle$, where $x_i'' := \frac{x_i}{1}$ and the system of constants of $S^{-1}R$ is given by $c''_{i,j} := \frac{c_{i,j}}{1}$, $c''_{i,\frac{r}{s}} := \frac{\sigma_i(r)}{\sigma_i(s)}$, for all $1 \leq i, j \leq n$.*

Proposition 3.10 ([35], Proposition 4.6)**.** *Let R be a ring such that any left regular element is regular, and let S be the set of regular elements of R such that $S^{-1}R$ exists. Then,*

(1) *$Q_l(R)$ exists if and only if $Q_l(S^{-1}R)$ exists. In such case, we have that $Q_l(R) \cong Q_l(S^{-1}R)$.*
(2) *R is semiprime left Goldie if and only if $S^{-1}R$ is semiprime left Goldie.*

The right side version of the proposition holds.

Now, we proceed to formulate several results to guarantee the fusibleness of skew PBW extensions. The next section contains the original results of the paper.

4. Fusible property in skew PBW extensions

We start with Theorem 4.1, which extends Proposition 2.4 formulated for skew polynomial rings of automorphism type. As we saw in Example 2.5, the fusibleness of skew polynomial ring does not hold in general.

Theorem 4.1. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of automorphism type over a left fusible ring R, then A is left fusible.*

Proof. Suppose that R is a left fusible ring. Let

$$f = a_0 + a_1X_1 + \cdots + a_mX_m$$

be a non-zero element of $Z_l(A)$, where $X_1 \prec X_2 \prec \cdots \prec X_m$ and every $a_i \neq 0$, for $i = 1, \dots, m$ (this can be realized by using Definition 3.5 (ii)). We will prove the assertion by considering two cases depending of the value of the constant element a_0: (i) $a_0 = 0$ and (ii) $a_0 \neq 0$.

Case (i). Since R is left fusible, then there exist elements $z_1 \in Z_l(R)$ and $z_1' \in Z_l^*(R)$ with $a_1 = z_1 + z_1'$, whence

$$f = z_1X_1 + (z_1'X_1 + \cdots + a_mX_m).$$

The idea is to show that $z_1X_1 \in Z_l(A)$ and $z_1'X_1 + \cdots + a_mX_m \in Z_l^*(A)$. Having in mind that $z_1 \in Z_l(R)$, there exists a non-zero element r of R with $z_1r = 0$. Since A is of automorphism type, there exists a non-zero element $s \in R$ with $\sigma^{\alpha_1}(s) = r$, where $\alpha_1 = \exp(X_1)$, which implies that $z_1X_1s = z_1\sigma^{\alpha_1}(s)X_1 = z_1rX_1 = 0$, that is, $z_1X_1 \in Z_l(A)$.

Now, to prove that $z_1'X_1 + \cdots + a_mX_m \in Z_l^*(A)$, if we suppose that $z_1'X_1+\cdots+a_mX_m \in Z_l(A)$, then there exists a non-zero element $b \in R$ with $(z_1'X_1 + \cdots + a_mX_m)b = 0$, whence

$$z_1'X_1b + \cdots + a_mX_mb = z_1'\sigma^{\alpha_1}(b)X_1 + \cdots + a_m\sigma^{\alpha_m}(b)X_m = 0,$$

and since $X_1 \prec \cdots \prec X_m$, necessarily $z_1'\sigma^{\alpha_1}(b)X_1 = 0$, i.e., $z_1'\sigma^{\alpha_1}(b) = 0$. By assumption, A is a skew PBW extension of automorphism type, so $\sigma^{\alpha_1}(b)$ is a non-zero element of R, which means that z_1' belongs to $Z_l(R)$, which is false, since we saw that z_1' is an element of $Z_l^*(R)$.

Therefore, f is the sum of an element of $Z_l(A)$ and an element of $Z_l^*(A)$, which means that A is a left fusible ring.

Case (ii). Again, since R is left fusible, there exist elements $z_0 \in Z_l(R)$ and $z_0' \in Z_l^*(R)$ with $a_0 = z_0 + z_0'$, whence

$$f = z_0 + (z_0' + a_1X_1 + \cdots + a_mX_m).$$

Once more again, the idea is to show that $z_0' + a_1X_1 + \cdots + a_mX_m$ belongs to $Z_l^*(A)$. By contradiction, if this is not the case and

$$z_0' + a_1X_1 + \cdots + a_mX_m$$

is an element of $Z_l(A)$, then there exists a non-zero element b of R such that $(z_0' + a_1X_1 + \cdots + a_mX_m)b = 0$. By using Proposition 3.6 (i), we obtain

$$\begin{aligned}(z_0' + a_1X_1 + \cdots + a_mX_m)b &= z_0'b + a_1X_1b + \cdots + a_mX_mb\\ &= z_0'b + a_1\sigma^{\alpha_1}(b)X_1 + \cdots + a_m\sigma^{\alpha_m}(b)X_m.\end{aligned}$$

Since A is of automorphism type, the only constant element appearing in this expression is $z_0'b$, so necessarily $z_0'b = 0$, i.e., $z_0' \in Z_l(R)$. Of course, this fact is a contradiction, and hence f is the sum of an element of $Z_l(A)$ and an element of $Z_l^*(A)$, whence A is a left fusible ring. □

With respect to regular left fusibleness of skew polynomial rings of automorphism type, Proposition 2.12 establishes that if a ring R is regular left fusible with an automorphism σ, then $R[x;\sigma]$ is regular left fusible. The natural generalization of this fact for skew PBW extensions of automorphism type is presented in the next theorem; its proof is similar to the presented for Theorem 4.1, if we remember that left fusible for Ghashghaei and McGovern corresponds to right fusible for Koşan and Matczuk.

Theorem 4.2. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of automorphism type over a regular left fusible ring R, then A is regular left fusible.*

Unfortunately, for the moment, we are unable to formulate an assertion as Theorems 4.1 and 4.2 for general skew PBW extensions not only of automorphism type (of course, this is the same situation with skew polynomial rings where there are non-trivial derivations). However, if we impose another ring-theoretic conditions for the ring of coefficients, we can assert the fusibleness and regular fusibleness of a considerable number of noncommutative rings (see Section 5). This is what we are going to do in the Theorems 4.3 and 4.4.

Theorem 4.3. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a bijective skew PBW extension over a semiprime Noetherian ring R, then A is a regular left fusible ring.*

Proof. From Proposition 3.8, we know that

$$G(A) \cong R[x_1; \sigma_1] \cdots [x_n; \sigma_n].$$

Since R is Noetherian and σ_i is bijective, for every $1 \leq i \leq n$, $G(A)$ is a Noetherian ring ([16], Corollary 2.7). By Proposition 3.7, A is also a left Noetherian ring, whence A is left Goldie. On the other hand, as R is semiprime and left Noetherian, then $G(A)$ is semiprime ([33], Proposition 3.6), and hence A is also semiprime ([35], Proposition 4.7). In this way, A is a left Goldie semiprime ring, so that Proposition 2.11 implies that A is regular left fusible. □

Since every left Noetherian ring is a left Goldie, but not necessarily the converse is true, the following result extends Theorem 4.3.

Theorem 4.4. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a bijective skew PBW extension over a semiprime left Goldie ring R, then A is a regular left fusible ring.*

Proof. Let S be the set of regular elements of R. By Goldie's theorem, we have that $Q_l(R) = S^{-1}R$ exists and it is a left Artinian ring. Making use of Proposition 3.9, it follows that $S^{-1}A$ exists and it is a bijective extension of $Q_l(R)$, that is, $S^{-1}A = \sigma(Q_l(R))\,\langle x_1, \dots, x_n\rangle$. Since $Q_l(R)$ is left Artinian, we have that $Q_l(R)$ is left Noetherian, and so Proposition 3.7 implies that $S^{-1}A$ is left Noetherian, and thus also left Goldie. On the other hand, having in mind that $G(\sigma(Q_l(R))\,\langle x_1, \dots, x_n\rangle)$ is a quasi-commutative extension of the semiprime left Goldie ring $Q_l(R)$, then $G(S^{-1}A)$ is a semiprime ring, and so it follows that $S^{-1}A$ is semiprime ([35], Proposition 4.7). Therefore, $S^{-1}A$ is semiprime left Goldie, and Proposition 3.10 (ii) allows us to conclude that A is a semiprime left Goldie ring. Finally, since A is a semiprime left Goldie ring, we obtain that A is a regular left fusible ring as a consequence of Proposition 2.11. □

Remark 4.5. In [48], the second author considered the question about Goldie dimension of skew PBW extensions. For example, we know that for $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ a bijective skew PBW extension over R, (i) if R is a right Noetherian domain, then the right Goldie dimension of A is 1 ([48], Proposition 3.2). (ii) If R is a prime right Goldie ring, then the right uniform dimension of A is less or equal than right uniform dimension of R ([48], Theorem 3.5). In this way, thinking about Proposition 2.11, it is clear the importance of right regular fusibleness of skew PBW extensions under the assumptions established in both results (i) and (ii). (iii) For M a nonsingular right R-module M, if either R is a right Noetherian ring or M is a Noetherian module, then the right uniform dimension of the R-module M is the same as the right uniform dimension of the A-module $M \otimes_R A$. This result, together with [13], Proposition 2.11, which states that every left fusible ring is right nonsingular, allow us to conclude that when $M = A$ is left fusible, then the right uniform dimension of A as an R-module is the same as the right uniform dimension of the A-module A.

Finally, in this section, another approach to the fusibleness of skew PBW extensions can be done considering the ring-theoretic notion of *Rickart ring*. The reason for this point of view is that some works have been realized in this line of thinking for these extensions (see Hashemi et al., [19], [20], and Reyes et al., [44], [47], [50] and [53]), and also the following fact proved by Ghashghaei and McGovern [13], Corollary 2.15: if R is an Abelian ring which is a left (right) Rickart ring, then R is fusible. In this way, if we guarantee that skew PBW extensions are Abelian and Rickart rings, then the fusibleness property holds. Next, we will say some words about this approach.

A ring R is a *left p.p.* (*principally projective*)-ring (also called *left Rickart ring*) if any principal left ideal of R is projective. McGovern [41], Proposition 16, showed that every commutative p.p.-ring is almost clean. This result was considered by Akalan and Vas [1], Theorem 4.1, who proved that for an Abelian ring R, R is a left p.p.-ring if and only if R special almost clean. In this way, it is clear that, as we said above, if R is an Abelian ring which is left (right) p.p.-ring, then R is fusible.

The next definition contains some ring-theoretic notions that allow to guarantee that skew PBW extensions are Rickart and Abelian rings. We want to say some words about the key notion in this definition: *Armendariz ring*.

Briefly, a ring R is called *Armendariz* (the term was introduced by Rege and Chhawchharia [46]) if for polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m$ of $R[x]$ which satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$, for every i, j. The importance of this notion lies in its natural and useful role in understanding the relation between the annihilators of the ring R and the annihilators of the polynomial ring $R[x]$. For example, Armendariz [3], Lemma 1, showed that a reduced ring always satisfies this condition. For skew polynomial rings, the notion of Armendariz has been also studied by several authors, as we can see in [2], [3], [23], [24], [30], [32], and [46].

Definition 4.6. Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension.

(i) ([18], Definition 3.7 (2); [44], Definitions 3.4 and 3.5) We say that R is a (Σ, Δ)-*Armendariz ring* if for polynomials
$$f = a_0 + a_1X_1 + \cdots + a_mX_m$$
and $g = b_0 + b_1Y_1 + \cdots + b_tY_t$ in A, the equality $fg = 0$ implies $a_iX_ib_jY_j = 0$, for every i, j. R is a (Σ, Δ)-*weak Armendariz ring* if for linear polynomials $f = a_0 + a_1x_1 + \cdots + a_nx_n$ and $g = b_0 + b_1x_1 + \cdots + b_nx_n$ in A, the equality $fg = 0$ implies $a_ix_ib_jx_j = 0$, for every $1 \le i, j \le n$.

(ii) ([50], Definitions 3.1 and 3.2) R is said to be a Σ-*skew Armendariz* ring, if for elements $f = \sum_{i=0}^{m} a_iX_i$ and $g = \sum_{j=0}^{t} b_jY_j$ in A, the equality $fg = 0$ implies $a_i\sigma^{\alpha_i}(b_j) = 0$, for all $0 \le i \le m$ and $0 \le j \le t$, where $\alpha_i = \exp(X_i)$. R is a *weak* Σ-*skew Armendariz* ring if for elements $f = \sum_{i=0}^{n} a_ix_i$ and $g = \sum_{j=0}^{n} b_jx_j$ in A ($x_0 := 1$), the equality $fg = 0$ implies $a_i\sigma_i(b_j) = 0$, for all $0 \le i, j \le n$ ($\sigma_0 := \mathrm{id}_R$).

(iii) ([53], Definitions 4.1 and 4.2) R is a *skew-Armendariz* ring if for polynomials $f = a_0 + a_1X_1 + \cdots + a_mX_m$ and
$$g = b_0 + b_1Y_1 + \cdots + b_tY_t$$
in A, $fg = 0$ implies $a_0b_k = 0$, for each $0 \le k \le t$. R is a *weak skew-Armendariz* ring if for linear polynomials
$$f = a_0 + a_1x_1 + \cdots + a_nx_n,$$
and $g = b_0 + b_1x_1 + \cdots + b_nx_n$ in A, $fg = 0$ implies $a_0b_k = 0$, for every $0 \le k \le n$.

Several relations and examples between above skew Armendariz notions can be found in [53]. We will only remember some facts of interest for the paper. This is the content of the following remark.

Remark 4.7. Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a ring R.

(i) ([44], Theorems 3.13 and 4.2) If R is (Σ, Δ)-Armendariz, then A is an Abelian ring. Besides, R is a p.p.-ring if and only if A is a p.p.-ring.
(ii) ([50], Corollary 3.10 and Theorem 5.3) If R is a weak Σ-skew Armendariz ring, then A is Abelian. Besides, if A bijective, then R is a Rickart ring if and only if A is a Rickart ring.
(iii) ([53], Proposition 4.10 and Theorem 5.6) If R is skew-Armendariz, then A is Abelian. In the case that A is bijective, then R is a p.p.-ring if and only if A is a p.p.-ring.

From the above results mentioned in Remark 4.7, and the fact that for an Abelian and left (right) Rickart ring one can assert its fusibleness, then it is clear that for all Armendariz notions in Definition 4.6, the skew PBW extensions satisfying all of them will be fusible rings.

5. Examples

The importance of our results is appreciated when we can apply them to algebraic structures (in terms of generators and relations) more general than those considered by Ghashghaei and McGovern [13] and Koşan and Matczuk [31]. In this way, our aim in this section is to provide several examples of noncommutative rings which are skew PBW extensions (not only of automorphism type) but not skew polynomial rings of automorphism type. Of course, our list of examples is not exhaustive, so another algebraic structures can be found in papers such as [22] or [38]. For all examples in this section, our Theorems 4.1, 4.2, 4.3, and 4.4 can be illustrated considering that the skew PBW extensions satisfy the conditions established in each one of them. Since domains are fusible rings, if we consider these extensions over rings not necessarily domains but fusible rings, then our results will become more important. Below we say some words about this fact.

About the family of Weyl algebras $A_n(\Bbbk)$, in the literature it is common to find several characterizations of these algebras as rings of differential operators. Surely, the most beautiful and excellent treatment about Weyl algebras is presented by Coutinho [9]. Briefly, the nth Weyl algebra $A_n(\Bbbk)$ over $\Bbbk$ is the $\Bbbk$-algebra generated by the $2n$ indeterminates $x_1, \dots, x_n, y_1, \dots, y_n$ where

$$\begin{aligned} &x_jx_i = x_ix_j, \quad y_jy_i = y_iy_j, && 1 \leq i < j \leq n, \\ &y_jx_i = x_iy_j + \delta_{ij}, \quad \delta_{ij} \text{ is the Kronecker delta}, && 1 \leq i, j \leq n. \end{aligned}$$

From the relations defining the Weyl algebras, it follows that these cannot be expressed as skew polynomial rings of automorphism type (since the algebra is simple) but skew polynomial rings with non-trivial derivations. Of course, all these algebras are examples of skew PBW extensions over $\Bbbk$, that is, $A_n(\Bbbk) \cong \sigma(\Bbbk)\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle$ ([38], Section 3.1). In this way, if we change $\Bbbk$ by a left fusible or regular left fusible R, then the fusibleness of $A_n(R)$ will be hold.

Different deformations of Weyl algebras have been introduced in the literature. We recall some of them.

Following Goodearl and Warfield [16], p. 36, for an element $q \in \Bbbk \setminus \{0\}$, $A_1^q(\Bbbk)$ denotes the $\Bbbk$-algebra presented by two generators x and y and one relation $xy - qyx = 1$, which is known as a *quantized Weyl algebra* over $\Bbbk$. Note that $A_1^q(\Bbbk) = A_1(\Bbbk) = \Bbbk[y][x; d/dy]$, when $q = 1$. If $q \neq 1$, then $A_1^q(\Bbbk) = \Bbbk[y][x; \sigma, \delta]$, where σ is the $\Bbbk$-algebra automorphism given by $\sigma(f(y)) = f(qy)$, and δ is the *q-difference operator* (also known as *Eulerian derivative*)

$$\delta(f(y)) = \frac{f(qy) - f(y)}{qy - y} = \frac{\alpha(f) - f}{\alpha(y) - y},$$

as it is mentioned in [16], Exercise 2N, so this algebra is not a skew polynomial ring of automorphism type. By a direct computation, we can prove that $A_1^q(\Bbbk) \cong \sigma(\Bbbk)\langle x, y\rangle$, and since $\Bbbk$ is trivially satisfies the assumptions in our Theorems 4.1 and 4.2, 4.3, and 4.4, then $A_1^q(\Bbbk)$ is left fusible and regular left fusible. Again, if we change $\Bbbk$ by a left fusible or regular left fusible R, then we will obtain the fusibleness of $A_n(R)$.

A generalization of $A_1^q(\Bbbk)$ is given by the *additive analogue of the Weyl algebra* $A_q(q_1, \dots, q_n)$. For non-zero elements $q_1, \dots, q_n \in \Bbbk$, this algebra is generated by the indeterminates $x_1, \dots, x_n$ and $y_1, \dots, y_n$ satisfying the relations $x_jx_i = x_ix_j$, $y_jy_i = y_iy_j$, for every $1 \leq i, j \leq n$, $y_ix_j = x_jy_i$, for all $i \neq j$, and $y_ix_i = q_ix_iy_i + 1$, for $1 \leq i \leq n$. It is clear from these definitions that these algebras are not skew polynomial rings of automorphism type. In [38], Section 3.5, it was shown that these algebras are skew PBW extensions over $\Bbbk$. The left fusibleness and regular left fusibleness of $A_q(q_1, \dots, q_n)$ is clear. If we change $\Bbbk$ by a left fusible or regular left fusible R, then the fusibleness of $A_n(R)$ will be hold.

Another deformation of Weyl algebras was introduced by Giaquinto and Zhang [14] with the aim of studying the Jordan Hecke symmetry as a quantization of the usual second Weyl algebra. By definition, the The *quantum Weyl algebra* $A_2(J_{a,b})$ is the $\Bbbk$-algebra generated by the variables $x_1, x_2, \partial_1, \partial_2$, with relations (depending on parameters $a, b \in \Bbbk$)

$$
\begin{aligned}
x_1x_2 &= x_2x_1 + ax_1^2, & \partial_2\partial_1 &= \partial_1\partial_2 + b\partial_2^2\\
\partial_1x_1 &= 1 + x_1\partial_1 + ax_1\partial_2, & \partial_1x_2 &= -ax_1\partial_1 - abx_1\partial_2 + x_2\partial_1 + bx_2\partial_2\\
\partial_2x_1 &= x_1\partial_2, & \partial_2x_2 &= 1 - bx_1\partial_2 + x_2\partial_2.
\end{aligned}
$$

Note that if $a = b = 0$, then $A_2(J_{0,0})$ is precisely the second Weyl algebra $A_2(\Bbbk)$. From the defining relations, one can see that this algebra is not a skew polynomial ring of automorphism type but a skew PBW extension over the commutative polynomial ring $\Bbbk[x_1, \partial_2]$ ([49], Example 2(i)). Once more again, if we change $\Bbbk$ by a left fusible or regular left fusible ring R, then the fusibleness of $A_2(J_{a,b})$ will be guaranteed.

With respect to the universal enveloping algebras of finite-dimensional Lie algebras, we recall briefly their definition over fields. If $\mathfrak{g}$ is a finite dimensional Lie algebra over $\Bbbk$ with basis $\{x_1, \dots, x_n\}$, then the *universal enveloping algebra* of $\mathfrak{g}$, denoted by $U(\mathfrak{g})$, is the algebra generated by $x_1, \dots, x_n$ subject to the relations $x_ir - rx_i = 0 \in \Bbbk$, for every element $r \in \Bbbk$, and $x_ix_j - x_jx_i = [x_i, x_j] \in \mathfrak{g}$, where

$$[x_i, x_j] \subseteq \Bbbk + \Bbbk x_1 + \cdots + \Bbbk x_n,$$

for all $1 \leq i, j \leq n$. Since these enveloping algebras are PBW extensions over $\Bbbk$ in the sense of Bell and Goodearl [5] (note that these authors presented another examples of enveloping rings related to enveloping universal algebras), all of them are skew PBW extensions also over the field $\Bbbk$ ([38], Section 3.1). As is well-known, in general, these algebras are not skew polynomial rings even including non-zero trivial derivations. In this way, their left fusibleness and regular left fusibleness can be deduced from Theorems 4.1, 4.2, 4.3, and 4.4. The same result will be obtained if we change $\Bbbk$ by a left fusible or regular left fusible ring.

Of course, some enveloping algebras can be expressed as skew polynomial rings; however, in these rings the derivations are non-trivial. Let us see an example of this situation.

Following Goodearl and Warfield, [16], p. 40, the standard basis for the Lie algebra $\mathfrak{sl}(\Bbbk)$ is labelled $\{e, f, h\}$, where $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$. In this way, the enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$ is the $\Bbbk$-algebra presented by three generators e, f, h and three relations $ef - fe = h$, $he - eh = 2e$, and $hf - fh = -2f$. If R is the subalgebra of $U(\mathfrak{sl}_2(\Bbbk))$ generated by e and h, then

$$R = \Bbbk[e][h; \delta_1] = \Bbbk[h][e; \sigma_1],$$

where $\Bbbk[e]$ and $\Bbbk[h]$ are commutative polynomial rings, δ_1 denotes the derivation $2e(d/de)$ on $\Bbbk[e]$, and σ_1 is the $\Bbbk$-algebra automorphism of $\Bbbk[h]$ with $\sigma_1(h) = h - 2$. Thus,

$$U(\mathfrak{sl}_2(\Bbbk)) = \Bbbk[e][h; \delta_1][f; \sigma_2, \delta_2] = \Bbbk[h][e; \sigma_1][f; \sigma_2, \delta_2],$$

where $\sigma_2(e) = e$, $\sigma_2(h) = h + 2$, $\delta_2(e) = -h$, and $\delta_2(h) = 0$ ([16], Exercise 2S). Hence, it is clear that the left fusibleness and regular left fusibleness of $U(\mathfrak{sl}_2(\Bbbk))$ does not follow from Ghashghaei and McGovern [13], Proposition 2.9, and Koşan and Matczuk [31], Proposition 2.20. However, since $U(\mathfrak{sl}_2(\Bbbk)) \cong \sigma(\Bbbk)\langle e, f, h\rangle$, by our four theorems above, both properties about fusibleness hold for this algebra. The same situation if we take this ring over a left fusible or regular left fusible ring.

Some deformations of universal enveloping algebras (also known as quantum groups) can be considered as skew PBW extensions. Let us see two examples.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\Bbbk$ with basis $x_1, \dots, x_n$ and $U(\mathfrak{g})$ its enveloping algebra. The homogenized enveloping algebra of $\mathfrak{g}$ is $\mathcal{A}(\mathfrak{g}) := T(\mathfrak{g} \oplus \Bbbk z)/\langle R\rangle$, where $T(\mathfrak{g} \oplus \Bbbk z)$ denotes the tensor algebra, z is a new indeterminate, and R is spanned by the union of sets $\{z \otimes x - x \otimes z \mid x \in \mathfrak{g}\}$ and $\{x \otimes y - y \otimes x - [x, y] \otimes z \mid x, y \in \mathfrak{g}\}$. The algebra $A(\mathfrak{g})$ is a skew PBW extension over $\Bbbk[z]$, whence $A(\mathfrak{g})$ is left fusible and regular left fusible.

From [16], p. 41, for $\Bbbk$ an arbitrary field, if q is an element of $\Bbbk$ with $q \neq \pm 1$, the *quantized enveloping algebra* of $\mathfrak{sl}_2(\Bbbk)$ corresponding to the choice of q is the $\Bbbk$-algebra $U_q(\mathfrak{sl}_2(\Bbbk))$ presented by the generators E, F, K, K^{-1} and the relations $KK^{-1} = K^{-1}K = 1$, $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$, $KE = q^2EK$, and $KF = q^{-2}FK$. From [16], Exercise 2T, we know that $U_q(\mathfrak{sl}_2(\Bbbk))$ can be expressed as an iterated skew polynomial ring of the form $\Bbbk[E][K^{\pm 1}; \sigma_1][F; \sigma_2, \delta_2]$ ([16], Exercise 2T), so this algebra is not of automorphism type. As it was observed in [38] p. 1216, $U_q(\mathfrak{sl}_2(\Bbbk)) = \sigma(\Bbbk[K, K^{-1}])\langle E, F\rangle$, so our theorems above guarantee that $U_q(\mathfrak{sl}_2(\Bbbk))$ is left fusible and regular left fusible. Of course, if we take a regular left fusible R, since $R[K, K^{-1}]$ is also regular left fusible ([31], Proposition 2.20), our results guarantee that $U_q(\mathfrak{sl}_2(\Bbbk))$ will be regular left fusible.

Next, we present another examples of quantum algebras which are not skew polynomial rings of automorphism type but skew PBW extensions satisfying the conditions imposed in Theorems 4.1, 4.2, 4.3, and 4.4.

The Jordan Algebra introduced by Jordan [29] is the free $\Bbbk$-algebra $\mathcal{J}$ defined by $\mathcal{J} := \Bbbk\{x, y\}/\langle yx - xy - y^2\rangle$. It is immediate to see that this algebra is not a skew polynomial ring of automorphism type but an easy computation shows that $\mathcal{J} \cong \sigma(\Bbbk[y])\langle x\rangle$. In this way, $\mathcal{J}$ is not a skew PBW extension of automorphism type, so our Theorems 4.1 and 4.2 cannot be applied. Nevertheless, since the commutative polynomial ring $\Bbbk[y]$ satisfies the conditions imposed in Theorems 4.3 and 4.4, both results guarantee that $\mathcal{J}$ is left fusible and regular left fusible.

By definition, the *q-Heisenberg algebra* is the $\Bbbk$-algebra $\mathbf{h}_n(q)$ generated over $\Bbbk$ by $x_i,\ y_i,\ z_i$, for $1 \leq i \leq n$, subject to the relations

$$\begin{aligned} x_ix_j = x_jx_i, \quad y_iy_j = y_jy_i, \quad z_jz_i = z_iz_j, & \qquad 1 \leq i < j \leq n,\\ x_iz_i - qz_ix_i = z_iy_i - qy_iz_i = x_iy_i - q^{-1}y_ix_i + z_i = 0, & \qquad 1 \leq i \leq n,\\ x_iy_j = y_jx_i, \quad x_iz_j = z_jx_i, \quad y_iz_j = z_jy_i, & \qquad i \neq j, \end{aligned}$$

Again, it is clear that is neither a skew polynomial ring of automorphism type and a skew PBW extension of automorphism type, so left fusible and regular left fusible properties for $\mathbf{h}_n(q)$ follow from Theorems 4.3 and 4.4, since $\mathbf{h}_n(q) \cong \sigma(\Bbbk)\langle x_1, \dots, x_n\rangle$ ([54], Section 5.3).

Given any $q \in \Bbbk \setminus \{0\}$, for every field $\Bbbk$, the corresponding *quantized coordinate ring* of $M_2(\Bbbk)$ is the $\Bbbk$-algebra $O_q(M_2(\Bbbk))$ presented by four generators x_{11}, x_{12}, x_{21}, and x_{22} and the six relations

$$\begin{aligned} &x_{11}x_{12} = qx_{12}x_{11},\ x_{12}x_{22} = qx_{22}x_{12},\ x_{11}x_{21} = qx_{21}x_{11},\\ &x_{21}x_{22} = qx_{22}x_{21},\ x_{12}x_{21} = x_{21}x_{12}, \end{aligned}$$

and $x_{11}x_{22} - x_{22}x_{11} = (q - q^{-1})x_{12}x_{21}$. This algebra, also known as the *coordinate ring of quantum* 2×2 matrices over $\Bbbk$, or the 2×2 *quantum matrix algebra* over $\Bbbk$, can be expressed as the skew polynomial ring $\Bbbk[x_{11}][x_{12}; \sigma_{12}][x_{21}; \sigma_{21}][x_{22}; \sigma_{22}, \delta_{22}]$ ([16], Exercise 2V). Since $O_q(M_2(\Bbbk))$ is a skew PBW extension over $\Bbbk[x_{12}]$,

$$O_q(M_2(\Bbbk)) = \sigma(\Bbbk[x_{12}])\langle x_{11}, x_{21}, x_{22}\rangle$$

([38], p. 1216), then the left fusible and regular left fusible properties of $O_q(M_2(\Bbbk))$ are guaranteed.

Following Yamane [56], if q is a complex number such that $q^8 \neq 1$, the complex algebra R' generated by $e_{12}, e_{13}, e_{23}, f_{12}, f_{13}, f_{23}, k_1, k_2, l_1, l_2$ subject to the relations

$$
\begin{array}{lll}
e_{13}e_{12} = q^{-2}e_{12}e_{13}, & f_{13}f_{12} = q^{-2}f_{12}f_{13}, & \\
e_{23}e_{12} = q^{2}e_{12}e_{23} - qe_{13}, & f_{23}f_{12} = q^{2}f_{12}f_{23} - qf_{13}, & \\
e_{23}e_{13} = q^{-2}e_{13}e_{23}, & f_{23}f_{13} = q^{-2}f_{13}f_{23}, & \\
e_{12}f_{12} = f_{12}e_{12} + \dfrac{k_1^2 - l_1^2}{q^2 - q^{-2}}, & e_{12}k_1 = q^{-2}k_1e_{12}, & k_1f_{12} = q^{-2}f_{12}k_1, \\
e_{12}f_{13} = f_{13}e_{12} + qf_{23}k_1^2, & e_{12}k_2 = qk_2e_{12}, & k_2f_{12} = qf_{12}k_2, \\
e_{12}f_{23} = f_{23}e_{12}, & e_{13}k_1 = q^{-1}k_1e_{13}, & k_1f_{13} = q^{-1}f_{13}k_1, \\
e_{13}f_{12} = f_{12}e_{13} - q^{-1}l_1^2e_{23}, & e_{13}k_2 = q^{-1}k_2e_{13}, & k_2f_{13} = q^{-1}f_{13}k_2, \\
e_{13}f_{13} = f_{13}e_{13} - \dfrac{k_1^2k_2^2 - l_1^2l_2^2}{q^2 - q^{-2}}, & e_{23}k_1 = qk_1e_{23}, & k_1f_{23} = qf_{23}k_1, \\
e_{13}f_{23} = f_{23}e_{13} + qk_2^2e_{12}, & e_{23}k_2 = q^{-2}k_2e_{23}, & k_2f_{23} = q^{-2}f_{23}k_2, \\
e_{23}f_{12} = f_{12}e_{23}, & e_{12}l_1 = q^{2}l_1e_{12}, & l_1f_{12} = q^{2}f_{12}l_1, \\
e_{23}f_{13} = f_{13}e_{23} - q^{-1}f_{12}l_2^2, & e_{12}l_2 = q^{-1}l_2e_{12}, & l_2f_{12} = q^{-1}f_{12}l_2, \\
e_{23}f_{23} = f_{23}e_{23} + \dfrac{k_2^2 - l_2^2}{q^2 - q^{-2}}, & e_{13}l_1 = ql_1e_{13}, & l_1f_{13} = qf_{13}l_1, \\
e_{13}l_2 = ql_2e_{13}, & l_2f_{13} = qf_{13}l_2, & e_{23}l_1 = q^{-1}l_1e_{23}, \\
l_1f_{23} = q^{-1}f_{23}l_1, & e_{23}l_2 = q^{2}l_2e_{23}, & l_2f_{23} = q^{2}f_{23}l_2, \\
l_1k_1 = k_1l_1, & l_2k_1 = k_1l_2, & k_2k_1 = k_1k_2, \\
l_1k_2 = k_2l_1, & l_2k_2 = k_2l_2, & l_2l_1 = l_1l_2,
\end{array}
$$

is very important in the definition of the quantized enveloping algebra of $\mathfrak{sl}_3(\mathbb{C})$. From [38], Section 3.5, A is a skew PBW extension over the commutative polynomial ring $\mathbb{C}[l_1, l_2, k_1, k_2]$, so left fusible and regular left fusible properties hold for R'. The same results are obtained if we change $\mathbb{C}$ by a left fusible and regular left fusible, respectively.

Another family of rings which includes the universal enveloping algebra $U(\mathfrak{sl}(2, \Bbbk))$, the Dispin algebra $U(\mathfrak{osp}(1, 2))$ and the Woronowicz's algebra $W_\nu(\mathfrak{sl}(2, \Bbbk))$, is called the family of *3-dimensional skew polynomial algebras*. These algebras were introduced by Bell and Smith and are very important in noncommutative algebraic geometry, see [55]. Next, we present its definition and classification.

Definition 5.1 ([55], Definition C4.3). *A 3-dimensional skew polynomial algebra A* is a $\Bbbk$-algebra generated by the variables x, y, z restricted to relations $yz - \alpha zy = \lambda$, $zx - \beta xz = \mu$, and $xy - \gamma yx = \nu$, such that

(i) $\lambda, \mu, \nu \in \Bbbk + \Bbbk x + \Bbbk y + \Bbbk z$, and $\alpha, \beta, \gamma \in \Bbbk \setminus \{0\}$;
(ii) Standard monomials $\{x^i y^j z^l \mid i, j, l \geq 0\}$ are a $\Bbbk$-basis of the algebra.

From Definition 5.1, it is clear that these algebras are skew PBW extensions over the field $\Bbbk$, that is, $A \cong \sigma(\Bbbk)\langle x, y, z\rangle$ (see [22], Section 3.1.2 or [54], Section 5.2 for more details).

Proposition 5.2 ([55], Theorem C.4.3.1). *If A is a 3-dimensional skew polynomial algebra, then A is one of the following algebras:*

(1) *if $|\{\alpha, \beta, \gamma\}| = 3$, then A is defined by the relations $yz - \alpha zy = 0$, $zx - \beta xz = 0$, $xy - \gamma yx = 0$.*
(2) *if $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma = 1$, then A is one of the following algebras:*
  (i) $yz - zy = z$, $\quad zx - \beta xz = y$, $\quad xy - yx = x$;
  (ii) $yz - zy = z$, $\quad zx - \beta xz = b$, $\quad xy - yx = x$;
  (iii) $yz - zy = 0$, $\quad zx - \beta xz = y$, $\quad xy - yx = 0$;
  (iv) $yz - zy = 0$, $\quad zx - \beta xz = b$, $\quad xy - yx = 0$;
  (v) $yz - zy = az$, $\quad zx - \beta xz = 0$, $\quad xy - yx = x$;
  (vi) $yz - zy = z$, $\quad zx - \beta xz = 0$, $\quad xy - yx = 0$,
  where a, b are any elements of $\Bbbk$. All nonzero values of b give isomorphic algebras.
(3) *If $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma \neq 1$, then A is one of the following algebras:*
  (i) $yz - \alpha zy = 0$, $\quad zx - \beta xz = y + b$, $\quad xy - \alpha yx = 0$;
  (ii) $yz - \alpha zy = 0$, $\quad zx - \beta xz = b$, $\quad xy - \alpha yx = 0$.
  In this case, b is an arbitrary element of $\Bbbk$. Again, any nonzero values of b give isomorphic algebras.
(4) *If $\alpha = \beta = \gamma \neq 1$, then A is the algebra defined by the relations $yz - \alpha zy = a_1x + b_1$, $zx - \alpha xz = a_2y + b_2$, $xy - \alpha yx = a_3z + b_3$. If $a_i = 0$ $(i = 1, 2, 3)$, then all nonzero values of b_i give isomorphic algebras.*
(5) *If $\alpha = \beta = \gamma = 1$, then A is isomorphic to one of the following algebras:*
  (i) $yz - zy = x$, $\quad zx - xz = y$, $\quad xy - yx = z$;
  (ii) $yz - zy = 0$, $\quad zx - xz = 0$, $\quad xy - yx = z$;
  (iii) $yz - zy = 0$, $\quad zx - xz = 0$, $\quad xy - yx = b$;
  (iv) $yz - zy = -y$, $\quad zx - xz = x + y$, $\quad xy - yx = 0$;
  (v) $yz - zy = az$, $\quad zx - xz = z$, $\quad xy - yx = 0$;

Parameters $a, b \in \Bbbk$ are arbitrary, and all nonzero values of b generate isomorphic algebras.

As we said before, every 3-dimensional skew polynomial algebra is a skew PBW extension. Nevertheless, some of these algebras cannot be expressed as skew polynomial rings even in the case of non-trivial derivations. One of the possible illustrative examples of this fact can be the *Dispin algebra* $U(\mathfrak{o}sp(1,2))$, which is the enveloping algebra of the Lie superalgebra $\mathfrak{o}sp(1,2)$ ([55], Definition C4.1). By definition, Dispin algebra is generated by the indeterminates x, y, z over a field $\Bbbk$ satisfying the relations $yz - zy = z$, $zx + xz = y$ and $xy - yx = x$ (the algebra (b)(i) above with $\beta = -1$). Even without knowing exactly if the algebra is of the derivation type or not, the left fusibleness and regular left fusibleness of this algebra is guaranteed by Theorems 4.1 and 4.2. The same results are obtained if we change $\Bbbk$ by a left fusible or regular left fusible ring.

Related to skew polynomial rings, Jordan [27] introduced a subclass of these rings which he defined as *ambiskew polynomial rings*. Different levels of generality of the construction of ambiskew polynomial rings have been considered by Jordan in his papers (see Jordan and Wells [28] for a detailed description). However, from its definition we can see that ambiskew rings are skew polynomial rings of mixed type, and hence it is not possible to apply directly the results of Ghashghaei and McGovern [13], and Koşan and Matczuk [31]. From relations defining ambiskew polynomial rings, we can check that these rings are skew PBW extensions. In this way, if we consider ambiskew polynomial rings defined over a semiprime left Goldie ring, then Theorem 4.4 guarantees the regular left fusible property of these rings.

Different algebraic structures of remarkable importance in theoretical physics can be also expressed as skew PBW extensions. We present four of them.

The *Lie-deformed Heisenberg* is the free $\mathbb{C}$-algebra defined by the commutation relations

$$\begin{aligned} q_j(1+i\lambda_{jk})p_k - p_k(1-i\lambda_{jk})q_j &= i\hbar\delta_{jk} \\ [q_j, q_k] = [p_j, p_k] &= 0, \quad j, k = 1, 2, 3, \end{aligned}$$

where q_j, p_j are the position and momentum operators, and $\lambda_{jk} = \lambda_k\delta_{jk}$, with λ_k real parameters. If $\lambda_{jk} = 0$ one recovers the usual Heisenberg algebra. An easy computation shows that this algebra is a skew PBW extension over $\mathbb{C}$.

With the aim of obtaining bosonic representations of the Drinfield-Jimbo quantum algebras, Hayashi [21] considered the A_q^- algebra by using the free algebra $\mathbf{U}$. Following Berger [6], Example 2.7.7, this $\Bbbk$-algebra $\mathbf{U}$ is generated by the indeterminates $\omega_1, \dots, \omega_n, \psi_1, \dots, \psi_n$, and $\psi_1^*, \dots, \psi_n^*$, subject to the relations

$$\begin{aligned}
\psi_j\psi_i - \psi_i\psi_j = \psi_j^*\psi_i^* - \psi_i^*\psi_j^* = \omega_j\omega_i - \omega_i\omega_j = \psi_j^*\psi_i - \psi_i\psi_j^* = 0, &\quad 1 \le i < j \le n,\\
\omega_j\psi_i - q^{-\delta_{ij}}\psi_i\omega_j = \psi_j^*\omega_i - q^{-\delta_{ij}}\omega_i\psi_j^* = 0, &\quad 1 \le i, j \le n,\\
\psi_i^*\psi_i - q^2\psi_i\psi_i^* = -q^2\omega_i^2, &\quad 1 \le i \le n.
\end{aligned}$$

From [51], Section 3, we have that this algebra is a skew PBW extension over the commutative polynomial ring $\Bbbk[\omega_1, \dots, \omega_n]$, so left fusibleness and regular left fusibleness are properties of this algebra.

The *Non-Hermitian realization of a Lie deformed* defined by Jannussis et al., [26] is an important example of a non-canonical Heisenberg algebra considering the case of non-Hermitian (i.e., $\hbar = 1$) operators $A_j,\ B_k$, where the following relations are satisfied:

$$\begin{aligned}
A_j(1 + i\lambda_{jk})B_k - B_k(1 - i\lambda_{jk})A_j &= i\delta_{jk}\\
[A_j, B_k] &= 0 \quad (j \neq k)\\
[A_j, A_k] &= [B_j, B_k] = 0,
\end{aligned}$$

and,

$$\begin{aligned}
A_j^+(1 + i\lambda_{jk})B_k^+ - B_k^+(1 - i\lambda_{jk})A_j^+ &= i\delta_{jk}\\
[A_j^+, B_k^+] &= 0 \ (j \neq k),\\
[A_j^+, A_k^+] &= [B_j^+, B_k^+] = 0,
\end{aligned} \tag{5.1}$$

with $A_j \neq A_j^+,\ B_k \neq B_k^+\ (j, k = 1, 2, 3)$. If the operators $A_j,\ B_k$ are in the form $A_j = f_j(N_j + 1)a_j,\ B_k = a_k^+ f_k(N_k + 1)$, where $a_j,\ a_j^+$ are leader operators of the usual Heisenberg-Weyl algebra, with N_j the corresponding number operator ($N_j = a_j^+ a_j,\ \langle N_j \mid n_j\rangle = \langle n_j | n_j\rangle$), and the structure functions $f_j(N_j + 1)$ complex, then it is showed that A_j and B_k are given by

$$\begin{aligned}
A_j &= \sqrt{\frac{i}{1 + i\lambda_j}}\left(\frac{[(1 - i\lambda_j)/(1 + i\lambda_j)]^{N_j+1} - 1}{(1 - i\lambda_j)/(1 + i\lambda_j) - 1}\frac{1}{N_j + 1}\right)^{\frac{1}{2}} a_j\\
B_k &= \sqrt{\frac{i}{1 + i\lambda_k}}a_k^+\left(\frac{[(1 - i\lambda_k)/(1 + i\lambda_k)]^{N_k+1} - 1}{(1 - i\lambda_k)/(1 + i\lambda_k) - 1}\frac{1}{N_k + 1}\right)^{\frac{1}{2}}.
\end{aligned}$$

As one can show by some computations, this algebra is a skew PBW extension over $\mathbb{C}$ where the operators A_j and B_k are the indeterminates (see [51], Section 3, for a detailed description of the algorithms to show this fact).

Diffusion algebras arose in physics as a possible way to understand a large class of 1-dimensional stochastic process, see [25]. A diffusion algebra A with parameters $a_{ij} \in \mathbb{C} \setminus \{0\}, 1 \leq i, j \leq n$ is an algebra over $\mathbb{C}$ generated by variables $x_1, \dots, x_n$ subject to relations $a_{ij}x_ix_j - b_{ij}x_jx_i = r_jx_i - r_ix_j$, whenever $i < j$, $b_{ij}, r_i \in \mathbb{C}$, for all $i < j$, such that the indeterminates x's form a $\mathbb{C}$-basis of the algebra A. In the applications to physics, the parameters a_{ij} are strictly positive reals, and the parameters b_{ij} are positive reals as they are unnormalised measures of probability. As we can see, these algebras are not skew polynomial rings over $\mathbb{C}[x_1, \dots, x_n]$ but are skew PBW extensions over this ring (see [54], Section 5.3) satisfying the conditions imposed in theorems above, so we can assert their left fusibleness and regular left fusibleness. The same results are obtained if we change $\mathbb{C}$ by a left fusible and regular left fusible, respectively.

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A SURVEY ON THE FUSIBLE PROPERTY OF SKEW PBW EXTENSIONS

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مروری بر خاصیت گداختی توسیع‌های PBW اریب

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در این مقاله برخی نتایج مرتبط با خواص گداختی و گداختی چپ منظم خانواده‌ای از حلقه‌های ناجابه‌جایی موسوم به توسیع‌های PBW اریب را بیان خواهیم کرد. این نتایج بر پایه نتایج اخیر قشقایی و مک‌گاورن [۱۳]، و کوشان و ماتزوک [۳۱] در ارتباط با خاصیت گداختی چپ و گداختی چپ منظم حلقه‌های چندجمله‌ای اریب می‌باشند. از آن‌جایی که نتایج ارائه شده در این مقاله برای ساختارهای جبری بسیار کلی‌تری از حلقه‌های چندجمله‌ای اریب نیز می‌تواند مورد استفاده قرار گیرد، لذا نتایج ارائه شده در این مقاله که در ارتباط با خواص گداختی می‌باشند، خانواده‌ی زیادی از حلقه‌های مورد استفاده در گروه‌های کوانتوم، هندسه جبری ناجابه‌جایی و هندسه دیفرانسیل ناجابه‌جایی را پوشش می‌دهد. مثال‌هایی در ارتباط با ایده‌های تعمیم داده شده نیز ارائه خواهد شد.

کلمات کلیدی: خاصیت گداختی، حلقه چندجمله‌ای‌های اریب، توسیع PBW اریب.