Journal of Algebraic Systems
Vol. 10, No. 1, (2022), pp 49-68

VALUED-POTENT (GENERAL) MULTIRINGS

M. HAMIDI*, A. A. TAVAKOLI AND R. AMERI

Abstract. This paper extends multirings to a novel concept as general multirings, investigates their properties and presents a special general multirings as notation of (m, n)-potent general multirings. This study analyzes the differences between class of multirings, general multirings and general hyperrings and constructs the class of (in)finite general multirings based on any given nonempty set. In final, we define the concept of hyperideals in general multirings and compare with hyperideals in other similar (hyper)structures.

1. Introduction

The theory of hyperstructures as a generalization of structures was introduced by Marty in 1934 [12], whence this theory works on sets instead of elements in algebraic systems. Hyperstructures were stronger in applications in world and were used in applied science specially in complex (hyper)networks [3, 4]. The notions of multigroups, multirings, multifields, and their corresponding reduced versions were introduced by Marshall in [14] and provided a convenient framework to study the reduced theory of quadratic forms and spaces of orderings. A multiring is just a ring with multivalued addition and the idea of a multiring is very natural. Multirings are considered in spaces of signs, also known as abstract real spectra and objects which arised naturally

DOI: 10.22044/JAS.2021.10499.1517.
MSC(2010): Primary: 20N20, 16Y20; Secondary: 20N20, 16Y20.
Keywords: (General) Multiring, (m, n)-Potent (general) multiring, General hyperring, Hyperideal.
Received: 24 January 2021, Accepted: 19 July 2021.
*Corresponding author.

in the study of constructible sets in real geometry. Fundamental relations are one of the main tools in algebraic hyperstructures theory in such a way that convert hyperstructures to structures. A fundamental relation is the smallest equivalence relation on a hyperstructure so that the quotient of hyperstructure via this relation is a corresponding (fundamental)structure. Some researchers worked on fundamental relations on hypergroups and hyperrings [2, 5, 6, 8, 9, 10, 11]. M. Hamidi et al. constructed multigroups and multiring on every non–empty set and introduced a relation on multirings (as a smallest strongly regular equivalence relation) in such a way that the quotient of multirings on this relation is a Boolean ring with identity[1].

In this paper, we try to generalize the concept of multirings to general multirings, to describe their properties and their differences in multirings and general hyperrings. This paper works on construction of general multirings and shows that this class of hyperstructures have some identity elements, while have a unique zero element. It is natural to question as to what is the relationships between elements whence are considered in a same set with respect to algebraic operations. Since any operation at most connects two elements, we need to extend more elements in defined axioms. It motivates us to introduce the concept of two algebraic hyperoperations in an underlying set. So the main motivation is to introduce some identity elements with respect to algebraic hyperproduct and to consider the differences between other hyperstructures and structures. We obtained some theorems and corollaries such that in a specially conditions are similar to corresponded theorems in (non-associative)rings, so we conclude that general multirings are a generalization of (non-associative)rings. Because in general multirings the hyperproduct of any element with zero element necessarily is not zero element, so the concept of kernel of homomorphisms and so isomorphism theorems are different in corresponded results in other hyperstructures.

2. **Preliminaries**

In this section, we recall some definitions and results which are indispensable to our research paper from [7, 13, 15].

Let R be a non–empty set and $P^*(R) = \{S \mid \emptyset \neq S \subseteq R\}$. Every map $+ : R \times R \longrightarrow P^*(R)$ is called *hyperoperations*, for all $x, y \in R$, $+(x, y)$ is called the *hypersum* of x, y and hyperalgebraic system $(R, +)$ is called a *hypergroupoid*. For any two non–empty subsets A and B of R, $A \cdot B$ means $\bigcup\limits_{a\in A, b\in B} a \cdot b$. Recall that a *hypergroupoid* $(R, +)$ is called a *semihypergroup* if for any $x, y, z \in R, (x+y)+z = x+(y+z)$ and a semihypergroup $(R, +)$ is called a *hypergroup* if satisfies in the *reproduction axiom*, i.e. for any $x \in R, x + R = R + x = R$. A commutative hypergroup $(R, +)$(for all $x, y \in R, x+y = y+x$) is called a *canonical* hypergroup, provided that (i) it has a zero element 0 (i.e., $0 + x = x + 0 = \{x\}$, for every $x \in R$), (ii) every element has a unique inverse, (i.e., for all $x \in R$, there exists a unique $-x \in R$, such that $0 \in x + (-x)$, (iii) $x \in y + z$ implies $y \in x + (-z)$ and $z \in -y + x$ and we will denote it by $(R, +, -, 0)$. A system $(R, +, -, 0, \cdot, 1)$ is called a *multiring* if (i) $(R, +, -, 0)$ is a canonical hypergroup (commutative multigroup), (ii) $(R, \cdot, 1)$ is a commutative monoid (" $\cdot$ " is a binary operation on R which is commutative and associative and $x \cdot 1 = x$ for all $x \in R$,) (iii) $x \cdot 0 = 0$ for all $x \in R$, (iv) $x \cdot (y+z) \subseteq x \cdot y + x \cdot z$. A system $(R, +, \cdot)$ is called a *general hyperring* if (i) $(R, +)$ is a hypergroup, (ii) $(R, \cdot)$ is a semihypergroup and (iii) for all $x, y, z \in R$

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

A map $f : R \to R'$ is a multiring homomorphism if, for all $x, y \in R$ we have $(i) f(x + y) \subseteq f(x) + f(y), (ii) f(x \cdot y) = f(x) \cdot f(y), (iii)$ $f(-x) = -f(x), (iv) f(0) = 0$ and $f(1) = 1$. Let $(R, +, .)$ be a hyperring and ρ be an equivalence relation on R. Let $\frac{R}{\rho} = \{\rho(r) \mid r \in R\}$, be the set of all equivalence classes of R with respect to the relation ρ. Define hyperoperations $\oplus$ and $\otimes$ as follows:

$$\rho(a) \oplus \rho(b) = \{\rho(c) \mid c \in \rho(a) + \rho(b)\}, \rho(a) \otimes \rho(b) = \{\rho(c) \mid c \in \rho(a).\rho(b)\}.$$

In [15] it was proved that $(\frac{R}{\rho}, \oplus, \otimes)$ is a ring if and only if ρ is strongly regular. Let $\mathcal{U}$ denote the set of all finite sum of finite products of elements of R. It is defined a relation as γ on R by

$$a \ \gamma \ b \Longleftrightarrow \exists \ u \in \mathcal{U} : \{a, b\} \subseteq u.$$

Then γ^* is a smallest equivalence relation on R such that $(\frac{R}{\gamma^*}, \oplus, \otimes)$ is a ring (is called fundamental ring) and so γ^* is called a *fundamental relation* on R. In [1] Hamidi, et.al defined a fundamental relation κ on every multirings by, $x \kappa_{n,m} y$ if and only if there exist $z_1, z_2, \ldots, z_n \in R$ and $m \in \mathbb{N}$ in such a way that $x \in \sum_{i=1}^{n} z_i$ and $y \in \sum_{i=1}^{n} z_i^{k_i}$, where $k_i \in \{1, m\}$ and for all $u \in R, u^m = \underbrace{u \cdot u \cdot u \cdot \ldots \cdot u}_{(m-times)}$.

Clearly, $\kappa_{1,1} = \triangle = \{(x, x) \mid x \in R\}$ and so

$$\kappa = \bigcup_{k \in \{1,m\}} \bigcup_{n \geq 1} \left(\kappa_{n,m} \cup \kappa_{n,m}^{-1}\right)$$

is a reflexive and symmetric relation. If κ^* is the transitive closure of κ, then κ^* is the smallest strongly regular relation on R such that R/κ^* is a Boolean ring.

Let us first survey some simple results on general multirings such that we will apply in the next sections.

3. Construction of general multirings

In this section, we introduce the concept of general multiring, investigate their properties and for a given arbitrary set constructed at least a multiring. We present some example of general multirings that are not general hyperring.

Definition 3.1. A system $(R, +, -, 0, \cdot, 1)$ is called a general multiring if

- (i) $(R, +, -, 0)$ is a multigroup,
- (ii) $(R, \cdot)$ is a semihypergroup,
- (iii) for all $x \in R, 0 \in \big(0 \cdot x\big) \cap \big(x \cdot 0\big)$ and $x \in \big(1 \cdot x\big) \cap \big(x \cdot 1\big)$,
- (iv) for all $x, y, z \in R$, $x\cdot(y+z) \subseteq x\cdot y+x\cdot z$ and $(x+y)\cdot z \subseteq x\cdot z+y\cdot z$.

Clearly, every multiring is a general multiring.

A general multiring $(R, +, -, 0, \cdot, 1)$ is called a $(+)$-commutative general multiring, if it is commutative with respect to hyperoperation "$+$", a $(\cdot)$-commutative general multiring, if it is commutative with respect to hyperoperation "$\cdot$" and a commutative general multiring, if it is commutative with respect to hyperoperations "$+$" and "$\cdot$".

Theorem 3.2. *Let* $(R, +, -, 0, \cdot, 1)$ *be a general multiring and* $a, b, c, d \in R$. *Then* $(a + b) \cdot (c + d) \subseteq a \cdot c + a \cdot d + b \cdot c + b \cdot d$.

Proof. Let $x \in (a+b)\cdot(c+d)$. Then there exists $y \in a+b$ and $z \in c+d$ such that

$$\begin{aligned} x \in y \cdot z &\subseteq (a + b) \cdot z \\ &\subseteq a \cdot z + b \cdot z \\ &\subseteq a \cdot (c + d) + b \cdot (c + d) \\ &\subseteq a \cdot c + a \cdot d + b \cdot c + b \cdot d. \end{aligned}$$

□

Example 3.3. (i) Let $R = \{0, 1, a, b\}$. Then $(R, +, -, 0, \cdot, 1)$ is a commutative general multiring as follows:

$+$	0	1	a	b
0	0	1	a	b
1	1	R	$\{1, a\}$	$\{1, b\}$
a	a	$\{1, a\}$	R	$\{a, b\}$
b	b	$\{b, 1\}$	$\{a, b\}$	R

and

$\cdot$	0	1	a	b
0	0	R	R	R
1	R	1	a	b
a	R	a	a	a
b	R	b	a	$\{a, b\}$

.

Since $1 \cdot (0 + 1) = \{1\} \subset R = 1.0 + 1.1$, we get that $(R, +, -, 0, \cdot, 1)$ is not a general hyperring.

(ii) Let $R = \{0, 1, a, b\}$. Then $(R, +, -, 0, \cdot, 1)$ is a commutative general multiring as follows:

$+$	0	1	a	b
0	0	1	a	b
1	1	R	$\{1, b\}$	$\{1, a\}$
a	a	$\{1, b\}$	$\{0, a\}$	1
b	b	$\{1, a\}$	1	$\{0, b\}$

,

$\cdot$	1	a	b		
0	0	0	0	0	
1	1	0	$\{1, 0\}$	$\{0, a\}$	$\{0, b\}$
a	0	$\{0, a\}$	$\{a, 0\}$	0	
b	0	$\{0, b\}$	0	$\{0, b\}$	

.

Since $1 \cdot (a + b) = \{0, 1\} \subset R = 1.a + 1.b$, we get that $(R, +, -, 0, \cdot, 1)$ is not a general hyperring.

(iii) Let $R = \{0, 1, 2, 3\}$. Then $(R, +, -, 0, \cdot, 1)$ is a $(\cdot)$-commutative general multiring as follows:

$+$	0	1	2	3
0	0	1	2	3
1	1	1	R	3
2	2	$\{0, 1, 2\}$	2	$\{2, 3\}$
3	3	$\{1, 3\}$	3	R

and

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	R	R
3	0	3	R	R

,

while it is a $(+)$-non-commutative general multiring and it is not a general hyperring.

(iv) Let $R = \{0, 1, 2, 3\}$. Then $(R, +, -, 0, \cdot, 1)$ is a $(+)$-non-commutative and a $(\cdot)$-non-commutative general multiring as follows:

$+$	0	1	2	3
0	0	1	2	3
1	1	1	R	3
2	2	$\{0, 1, 2\}$	2	$\{2, 3\}$
3	3	$\{1, 3\}$	3	R

and

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	R	R
3	0	3	$\{1, 2, 3\}$	R

.

Now, apply the concept of monoids and semihypergroups, then construct general multirings in the following.

Example 3.4. (i) Let $(G,*,1)$ be a monoid and $G^0 = G \cup \{0\}$, where $0 \notin G$. Define hyperoperations "$+$" and "$\cdot$" on G^0 as follows:

$$x+y=y+x=\begin{cases} G^0 & \text{if } x=y\neq 0,\\ G & \text{if } x\neq y \text{ and } x,y\in G\end{cases} \qquad \text{and } x\cdot y=\{x*y\},$$

in such a way that $x+0=0+x=x$ and $x\cdot 0=0\cdot x=G^0$. Clearly for all $x\in G^0$, $-x=x$ and some modifications show that $(G^0,+,-,0,\cdot,1)$ is a $(+)$-commutative general multiring.

(ii) Let $(H,\circ)$ be a semihypergroup and $R = H\cup\{0,1\}$, where $0,1\notin H$. Define hyperoperations "$+$" and "$\cdot$" on R as follows:

$$x+y=y+x=\begin{cases} R & \text{if } x=y\neq 0,\\ R\setminus\{0\} & \text{if } x\neq y \text{ and } x,y\in R\setminus\{0\} \text{and } x\cdot y = x\circ y\end{cases},$$

in such a way that $x+0=0+x=x$, $x\cdot 0=0\cdot x=R$ and $x=1\cdot x=x\cdot 1$. Some modifications and computations show that $(R,+,-,0,\cdot,1)$ is a $(+)$-commutative general multiring.

From now on, in general multiring $(R,+,-,0,\cdot,1)$, we will call 0 is zero element and 1 is an identity element of R. Let

$$\mathcal{O}_R=\{0\in R \mid 0 \text{ is a zero element of R}\}$$

and $\mathcal{I}_R=\{1\in R\mid 1 \text{ is an identity element of } R\}$, then we have the following results.

Theorem 3.5. *Let $(R,+,-,0,\cdot,1)$ be a general multiring. Then*

(i) $|\mathcal{O}_R|=1$ *and so* 0 *is unique zero element in R;*
(ii) $0\in x+y$ *implies that* $y=-x$;
(iii) $|\mathcal{I}_R|\geq 1$;
(iv) *for all* $x\in R$, $|0\cdot x|\geq 1$.

Proof. By definition, it is straightforward. □

Let $(R,+,-,0_R,\cdot,1_R)$ be a general multiring, $n\in\mathbb{Z}$ and $x\in R$. Denote

$$nx=\begin{cases}\underbrace{x+\ldots+x,}_{n\ times} & \text{if } n>0\\ 0_R, & \text{if } n=0\\ \underbrace{(-x)+\ldots+(-x),}_{n\ times} & \text{if } n<0\end{cases},\quad x^n=\begin{cases}\underbrace{x\cdot\ldots\cdot x,}_{n\ times} & \text{if } n>0\\ \mathcal{I}_R, & \text{if } n=0\\ \underbrace{(-x)\cdot\ldots\cdot(-x),}_{n\ times} & \text{if } n<0\end{cases},$$

for $n=1, nx=x^n=\{x\}$ and for all $x,y\in R, x-y=x+(-y)$.

Example 3.6. (i) Consider the monoid $(\mathbb{N},\cdot,1)$. Then

$$(\mathbb{N}\cup\{0\},+,0,\cdot,1)$$

is a multiring as follows:

$$m+n=n+m=\begin{cases}\{0,1,2,\ldots,\} & \text{if } m=n\neq 0,\\ \{1,2,\ldots,\} & \text{if } m\neq n, m,n\in\mathbb{N}.\end{cases}$$

and $m\cdot n=\{mn\}$, in such a way that $m+0=0+m=m$ and $m\cdot 0=0\cdot m=0$. Since

$$2\cdot(3+4)=2\cdot\{1,2,3,\ldots\}=\{2,4,6,\ldots\}\subset\{1,2,3,\ldots\}=2\cdot 3+2\cdot 4,$$

we get that $(\mathbb{N}\cup\{0\},+,0,\cdot,1)$ is a general multiring, while is not a general hyperring. By some computing, for all $n\geq 2$ we have $2(-n)=2n=\mathbb{N}\cup\{0\}$.

(ii) Let $R=\{0,1\}$. Then $(R,+,-,0,\cdot,1)$ is a commutative general multiring as follows:

$$\begin{array}{c|cc} + & 0 & 1\\ \hline 0 & 0 & 1\\ 1 & 1 & 0\end{array}\quad \textit{and}\quad \begin{array}{c|cc} \cdot & 0 & 1\\ \hline 0 & 0 & \{0,1\}\\ 1 & \{0,1\} & \{0,1\}\end{array}.$$

Clearly $(R,+,-,0,\cdot,1)$ is not a multiring ($|0\cdot 1|\geq 1$) and since $0\cdot(1+1)=\{0\}\subseteq\{0,1\}=0\cdot 1+0\cdot 1$, we get that $(R,+,-,0,\cdot,1)$ is not a general hyperring. Then

$$n1=\begin{cases}0 & \text{if n is an even}\\ 1 & \text{if n is an odd}\end{cases}\quad, 1^n=1^0=0^0=\{0,1\}=\mathcal{I}_R, 1^{-1}=1,$$

for all $n\geq 2, 1^{-n}=\{0,1\}$ and for all $0\neq n\in\mathbb{Z}, 0^n=\{0\}$.

Theorem 3.7. *Let $(R,+,-,0,\cdot,1)$ be a general multiring and $x,y\in R$. Then*

(i) $-0=0$, *so* $0+0=0$ *and* $0=0-0$;
(ii) $-(-x)=x$;
(iii) $0\in x-x$;
(iv) *if* $x=y$, *then* $-x=-y$.

Proof. (i) Let $-0=x\in R$. Then $0\in 0+x=\{x\}$ and so $x=0$.

(ii) Let $x\in R$. By definition,

$$0\in x+(-x) \text{ so } x\in 0+(-(-x))=\{-(-x)\}.$$

It follows that $-(-(x))=x$.

(iii) For all $x\in R$, we have $0\in x+(-x)=x-x$, hence $0\in x-x$.

(iv) Let $-x=a$. Then $0\in x+a=y+a$ and so $y\in 0-a=\{-a\}$ It follows that $y=-a$ and by item (ii) $-y=-(-a)=a=-x$. □

Theorem 3.8. *Let $(R,+,-,0,\cdot,1)$ be a general multiring and $A,B\subseteq R$. Then*

(i) $-A=\{-a\mid a\in A\}$

(*ii*) $0 \in A - A$;
(*iii*) *if* $C \subseteq A + B$, *then* $A \cap (C - B) \neq \emptyset$;
(*iv*) $-(-A) = A$;
(*v*) $0 + A = A = A + 0$;
(*vi*) *if* $0 \in A + B$, *then* $A \cap (-B) \neq \emptyset(-A \cap B) \neq \emptyset)$;
(*vii*) $0 \in (0 \cdot A) \cap (A \cdot 0)$ *and* $A \subseteq (1 \cdot A) \cap (A \cdot 1)$;
(*viii*) $0 - A = -A$ *and* $A - 0 = A$;
(*ix*) *if* $A \subseteq B$, *then* $-A \subseteq -B$.

Proof. (*i*) Let $-A = B$. Then $0 \in A + B$ and so there exist $a \in A$ and $b \in B$ such that $0 \in a + b$. It follows that $b \in -a + 0 = \{-a\}$ or $-a = b$.

(*ii*) By definition, we have $A - A = \bigcup\limits_{a,b \in A} (a - b) = \bigcup\limits_{a,b \in A} (a + (-b))$, so by Theorem 3.7, $0 \in A - A$.

(*iii*) Since $C \subseteq A + B$, for all $c \in C$, there exist $a \in A$ and $b \in B$ such that $c \in a + b$. It concludes that $a \in c - b$ and so $A \cap (C - B) \neq \emptyset$.

(*iv*) By definition, $-(-A) = \bigcup\limits_{a \in A} (-(-a))$ and by Theorem 3.7, we get that $-(-A) = A$.

(*v*) It is clear.

(*vi*) Sine $0 \in A + B$, there exist $a \in A$ and $b \in B$ in such a way that $0 \in a + b$. It implies that $a = -b$ and so $A \cap (-B) \neq \emptyset$.

(*vii*), (*viii*), (*ix*) It is immediate by definition. □

Corollary 3.9. *Let* $(R, +, -, 0, \cdot, 1)$ *be a general multiring and* $x, y \in R$. *If for all* $z \in R, |z - z| = 1$, *then* $x + y = x + z$ *implies that* $y = z$.

Proof. Let $z \in R$. By Theorem 3.7 (*iii*), $|z - z| = 1$ implies that $z - z = \{0\}$. If $x + y = x + z$, then $-x + (x + y) = -x + (x + z)$ and so $(-x + x) + y = (-x + x) + z$. Hence $y = z$. □

Example 3.10. Let $R = \{0, a, 1\}$. Then $(R, +, -, 0, \cdot, 1)$ is a general multiring as follows:

$+$	0	1	a
0	0	1	a
1	1	$\{0, a\}$	1
a	a	1	$\{0, a\}$

and

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	$\{0, a\}$

.

One can see that $1 + 0 = 1 + a$, while $0 \neq a$. So in general multirings, the cancellation property, necessarily is not valid.

Theorem 3.11. *Let* $(R, +, -, 0, \cdot, 1)$ *be a general multiring,* $x, y \in R$ *and* $A \subseteq R$. *Then*

(i) *if* $-1 = 1$*, then for all* $x \in R$*, we have* $-(1 \cdot x) \cap (1 \cdot x) \neq \emptyset$*;*
(ii) *if* $-1 = 1$ *and for all* $x \in R$ *we have* $|1 \cdot x| = 1$*, then* $-x = x$*;*
(iii) *if* $-1 = 1$*, then for all* $x \in R$ *there exists* $y \in R$ *such that* $x \in y + 1$ *and* $y \in x + 1$;

Proof. (i) Let $x \in R$. Since $0 \in 1 + 1$, we get that

$$0 \in 0 \cdot x \subseteq (1 + 1) \cdot x \subseteq 1 \cdot x + 1 \cdot x.$$

Thus there exists $y \in 1 \cdot x$ and $z \in 1 \cdot x$ such that $0 \in y + z$. It follows that $z = -y$ and by Theorem 3.8, $-(1 \cdot x) \cap (1 \cdot x) \neq \emptyset$.

(ii) By item (i), the proof is obtained.

(iii) Let $x \in R$. Since $x = x + 0 \subseteq x + (1 + 1) = (x + 1) + 1$, there exists $y \in (x + 1)$ such that $x \in y + 1$ and so $y \in x + 1$. □

Theorem 3.12. *Let* $(R, +, -, 0, \cdot, 1)$ *be a general multiring and* $x, y \in R$*. Then* $-(x \cdot y) \cap (-x) \cdot y \neq \emptyset$ *and* $-(x \cdot y) \cap x \cdot (-y) \neq \emptyset$*.*

Proof. Let $x, y \in R$. Then

$$0 \in 0 \cdot y = (x - x) \cdot y \subseteq x \cdot y + ((-x) \cdot y) = ((-x) \cdot y) + (x \cdot y).$$

Using Theorem 3.8 (vi), we get that $-(x \cdot y) \cap (-x) \cdot y \neq \emptyset$. In a similar a way we can see that $-(x \cdot y) \cap x \cdot (-y) \neq \emptyset$. □

Corollary 3.13. *Let* $(R, +, -, 0, \cdot, 1)$ *be a general multiring,* $x, y \in R$*. Then*

(i) *for all* $n, m \in \mathbb{N}$, $x^{n+m} = x^m \cdot x^n$*;*
(ii) *for all* $n \in \mathbb{N}$, $(n + 1)x = nx + x$ *and* $x^n = x^{n-1} \cdot x$*;*
(iii) *if* R *is a* $(+)$*-commutative general multiring, then* $(x \cdot y)^n = x^n \cdot y^n$;
(iv) $(x^{-1})^{-1} = x$*;*
(v) $(x.y)^{-1} \cap x^{-1}.y \neq \emptyset$ *and* $(x.y)^{-1} \cap x.y^{-1} \neq \emptyset$*.*

Example 3.14. (i) Let $R = \{0, 1\}$. Then $(R, +, -, 0, \cdot, 1)$ is a general multiring as follows:

$+$	0	1
0	0	1
1	1	0

and

$\cdot$	0	1
0	$\{0, 1\}$	$\{0, 1\}$
1	$\{0, 1\}$	$\{0, 1\}$

.

It shows that in general $0 \cdot 0 \neq 0$.

(ii) Let $R = \{0, 1, a\}$. Then $(R, +, -, 0, \cdot, 1)$ is a general multiring as follows:

$+$	0	1	a
0	0	1	a
1	1	1	R
a	a	R	$\{1, a\}$

and

$\cdot$	0	1	a
0	0	0	R
1	0	1	a
a	R	a	R

.

Clearly $-a = \{1\} \neq R = a \cdot a = (-1) \cdot a$. It shows that in general $-x \neq (-1) \cdot x$, where $x, y \in R$.

Theorem 3.15. *Let $(R, +, -, 0, \cdot, 1)$ be a general multiring.*

(i) *For all $a, b \in R$, we have $(a \cdot b) \cap (a \cdot b - (a \cdot 0)) \neq \emptyset$.*

(ii) *If there exists $0 \neq a \in R$, in such a way that $0 \cdot a = 0$, then $0 \cdot 0 = 0$.*

(iii) *If there exists $0 \neq a \in R$, in such a way that $a \cdot 0 = R$, then for all $r \in R$, there exists $x, y \in a \cdot a$ such that $r \in x - y$.*

Proof. (i) Let $a, b \in R$. $a \cdot b = a \cdot (b + 0) \subseteq a \cdot b + a \cdot 0$. Now by Theorem 3.8, we have $(a \cdot b) \cap (a \cdot b - (a \cdot 0)) \neq \emptyset$.

(ii) For all $a \in R$, By Theorem 3.7, we have $0 \in a - a$. If there exists $0 \neq a \in R$ in such a way that $0 \cdot a = 0$, then

$$0 \cdot 0 \subseteq 0 \cdot (a - a) \subseteq 0 \cdot a - 0 \cdot a \subseteq 0 - 0 = 0.$$

It concludes that $0 \cdot 0 = 0$.

(iii) Since exists $0 \neq a \in R$, in such a way that $a \cdot 0 = R$, for all $r \in R$ we get that $r \in a \cdot 0 \subseteq a \cdot (a - a) \subseteq a \cdot a - a \cdot a$. It follows that there exists $x, y \in a \cdot a$ such that $r \in x - y$. □

In the following, we construct commutative general multirings on every non-empty finite sets. Indeed, show that for any $4 \leq \zeta \in \mathbb{N}$, there exists a general multiring $(R, +, -, 0, \cdot, 1)$ of order ζ.

Example 3.16. (i) Let $R = \{a_0, a_1, \ldots, a_n\}$ and $4 \leq |R| \leq n$. Fixed $a_0 = 0 \in R$ and $C_3 = \{a_1, a_2, a_3\} \subseteq R$. Now for all $a_i, a_j \in R$, we define a hyperoperation "$+$" on R as follows:

$$a_i + a_j = \begin{cases} R \setminus C_3 & i = j \neq 0, \\ R \setminus (C_3 \cup \{0\}) & i \neq j \geq 4, \\ C_3 \setminus \{a_i, a_j\} & 1 \leq i \neq j \leq 3, \\ a_i & 1 \leq i \leq 3 \text{ and } j \geq 4 \end{cases},$$

where for all $a_i, a_j \in R$, $a_i + a_j = a_j + a_i$ and $a_i + 0 = \{a_i\}$. By a manipulation it is easy to verify that $(R, +, -, 0)$ is a commutative multigroup. Now for all $a_i, a_j \in R$, we define a hyperoperation "$\cdot$" on R as follows:

$$a_i \cdot a_j = \begin{cases} 0 & i = 0, \\ a_j & i = 1, \\ a_i & i = j \neq n, \\ a_4 & i = k, j \geq k+1, 2 \leq k, \\ \{a_4, a_n\} & i = j = n, \end{cases} \quad \text{and for all } \ 0 \leq i \neq j,$$

$a_i \cdot a_j = a_j \cdot a_i$. Some modifications and computations show that $(R, +, -, 0, \cdot, 1)$ is a commutative general multiring. Since

$$a_2 \cdot (a_1 + a_1) = a_2 \cdot (R \setminus C_3) = \{0, a_4\}$$

and $a_2 \cdot a_1 + a_2 \cdot a_1 = R \setminus C_3$, we get that $a_2 \cdot (a_1 + a_1) \subset a_2 \cdot a_1 + a_2 \cdot a_1$ and conclude that $(R, +, -, 0, \cdot, 1)$ is not a general hyperring.

(*ii*) Let $R = \{a_0, a_1, \ldots, a_n\}$ and $4 \leq |R|$. Fixed $a_0 = 0 \in R$ and $C_3 = \{a_1, a_2, a_3\} \subseteq R$. Now for all $a_i, a_j \in R$, we define a hyperoperation "$+$" on R in similar to item (*i*), and a hyperoperation "$\cdot$" on R as follows:

$$a_i \cdot a_j = \begin{cases} 0 & i = 0, \\ a_j & i = 1, \\ a_i & i = j, \\ a_4 & i = k, j \geq k+1, 2 \leq k, \end{cases} \qquad \text{and for all } \ 0 \leq i \neq j,$$

$a_i \cdot a_j = a_j \cdot a_i$. Some modifications and computations show that $(R, +, -, 0, \cdot, 1)$ is a commutative general multiring. Since

$$a_2 \cdot (a_1 + a_1) = a_2 \cdot (R \setminus C_3) = \{0, a_4\}$$

and $a_2 \cdot a_1 + a_2 \cdot a_1 = R \setminus C_3$, we get that $a_2 \cdot (a_1 + a_1) \subset a_2 \cdot a_1 + a_2 \cdot a_1$ and conclude that $(R, +, -, 0, \cdot, 1)$ is not a general hyperring.

3.1. (m, n)**-Potent general multirings.** In this subsection, we applied the concept of general multirings and presented a special class of general multirings as (m, n)-potent general multiring, where $m, n \geq 2$. It is shown that every Boolean general multiring is a $(+)$-commutative general multiring, while it is not a commutative general multiring.

Definition 3.17. Let $(R, +, -, 0, \cdot, 1)$ be a general multiring and $m, n \geq 2$. Then R is said to be an (m, n)-potent general multiring, if for all $x \in R$, we have $0 \in mx$ and $x \in x^n$. If $m = n = 2$, we will call $(R, +, -, 0, \cdot, 1)$ is a Boolean general multiring.

Example 3.18. (*i*) Let $R = \{0, a, 1\}$. Then $(R, +, -, 0, \cdot, 1)$ is a Boolean general multiring as follows:

$+$	0	1	a
0	0	1	a
1	1	R	$\{1, a\}$
a	a	$\{1, a\}$	R

and

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	$\{1, a\}$

.

Since $a \cdot (a + 1) = \{1, a\} \subseteq a \cdot a + a \cdot 1 = R$, we get that it is not a general hyperring.

(ii) Let $R=\{0,a,1\}$. Then $(R,+,-,0,\cdot,1)$ is a $(3,2)$-potent general multiring as follows:

$+$	0	a	1
0	0	a	1
a	a	$\{1,a\}$	R
1	1	R	$\{1,a\}$

and

$\cdot$	0	a	1
0	0	0	0
a	0	$\{1,a\}$	$\{1,a\}$
1	0	$\{1,a\}$	$\{1,a\}$

.

Since $0\notin a+a$, we get $(R,+,-,0,\cdot,1)$ is not a Boolean general multiring.

Theorem 3.19. *Let $(R,+,-,0,\cdot,1)$ be an (m,n)-potent general multiring, where $m,n\in\mathbb{N}$ and $x\in R$. Then*

(i) for all $k\in\mathbb{N}$, we have $x\in x^{k(n-1)+1}$;
(ii) $x\in(m+1)x\cap x^n$;
(iii) $0\in mx\cap 2mx$;
(iv) $0\in mx^n$;
(v) $x\in(m+1)x^n$.

Proof. (i) Let $x\in R$. Then

$$x\in x^n=x^{n-1}\cdot x\subseteq x^{n-1}\cdot x^n=x^{2n-1}=x^{2n-2}\cdot x\subseteq x^{2n-2}\cdot x^n=x^{3n-2}.$$

So by induction, we get that for all $k\geq 1, x\in x^{k(n-1)+1}$.

(ii) Since R is an (m,n)-potent general multiring, for all $x\in R$ we get that $x\in x^n$ and $0\in mx$. So by Corollary 3.13, we have $x\in x+0\subseteq x+mx=(m+1)x\cap x^n$.

(iii) Since for all $x\in R, mx\subseteq mx$ and $0\in mx$, we have $0+mx\subseteq mx+mx=2mx$ and so $0\in mx\cap 2mx$.

$(iv),(v)$ For all $x\in R$, we have $0\in mx\subseteq mx^n$. So $x\in(m{+}1)x^n$. □

Corollary 3.20. *Let $(R,+,-,0,\cdot,1)$ be an (m,n)-potent general multiring, where $m,n\in\mathbb{N}$. Then for all $k\in\mathbb{N}$, R is an $(km,(n-1)k+1)$-potent general multiring.*

Theorem 3.21. *Let $(R,+,-,0,\cdot,1)$ be an (m,n)-potent general multiring, where $m,n\in\mathbb{N}$ and $x\in R$. Then*

(i) $R=\bigcup\limits_{x\in R}x^n$.
(ii) $R=\bigcup\limits_{x\in R}(m+1)x$.

Proof. It is straightforward. □

Theorem 3.22. *Let $(R,+,-,0,\cdot,1)$ be a Boolean general multiring.*

(i) For all $x\in R$, we have $-x=x$.
(ii) R is a $(+)$-commutative general multiring.

Proof. (i) Let $x \in R$. Since $0 \in x+x$, by Theorem 3.5 (ii), we get that $-x = x$.

(ii) Let $x, y \in R$. If $z \in x+y$, then by the item (i), $y \in -x+z = x+z$. It follows that $x \in y - z = y + z$ and so $z \in -y + x = y + x$. Thus $x + y \subseteq y + x$ and so $x + y = y + x$. □

Example 3.23. Let $R = \{0, 1, a, b\}$. Then $(R, +, -, 0, \cdot, 1)$ is a Boolean general multiring, while it is a $(\cdot)$-non-commutative general multiring as follows:

$+$	0	1	a	b
0	0	1	a	b
1	1	R	$\{1,a\}$	$\{b,1\}$
a	a	$\{1,a\}$	R	$\{b,a\}$
b	b	$\{b,1\}$	$\{b,a\}$	R

and

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	a
b	0	b	$\{a,b\}$	$\{a,b\}$

.

This example shows that Boolean general multirings are not necessarily, commutative general multirings.

In the following, we present some examples for constructing (m, n)-potent general multirings and for any prime p and $k \in \mathbb{N}$, show that there exists at least a general multiring R in such a way that $|R| \in \{p + 1, p^k + 1\}$.

Example 3.24. (i) Let p be a prime and $R = \mathbb{Z}_p \cup \{\sqrt{p}\}$. Define hyperoperations "$+_{\sqrt{p}}$" and "$\cdot_{\sqrt{p}}$" on R as follows:

$$x +_{\sqrt{p}} y = y +_{\sqrt{p}} x = \begin{cases} \{\bar{0}, \sqrt{p}\} & x = -y \text{ or } x = y = \sqrt{p}, \\ x + y & x, y \in \mathbb{Z}_p, x \neq -y \\ y & (x = \sqrt{p} \text{ and } y \notin \{\bar{0}, \sqrt{p}\}) \text{ or } x = \bar{0} \end{cases}$$

and

$$x \cdot_{\sqrt{p}} y = y \cdot_{\sqrt{p}} x = \begin{cases} x.y & x, y \in \mathbb{Z}_p, \\ \sqrt{p} & x \in \mathbb{Z}_p \smallsetminus \{\bar{0}\}, y = \sqrt{p}, \\ \bar{0} & x = \bar{0}, y = \sqrt{p}, \\ \{\bar{0}, \sqrt{p}\} & x = y = \sqrt{p} \end{cases}.$$

Some modifications and computations show that $(R, +_{\sqrt{p}}, -_{\sqrt{p}}, \bar{0}, \cdot_{\sqrt{p}}, \bar{1})$ is a (p, p)-potent general multiring. Since for all $x \neq -y$, $\sqrt{p} \cdot_{\sqrt{p}} (x +_{\sqrt{p}} y) = \sqrt{p} \subseteq \{0, \sqrt{p}\} = \sqrt{p} \cdot_{\sqrt{p}} x +_{\sqrt{p}} \sqrt{p} \cdot_{\sqrt{p}} y$, we get that $(R, +, -, 0, \cdot, 1)$ is not a general hyperring.

(ii) Let p be a prime, $k \in \mathbb{N}$ and $R = \mathbb{Z}_{p^k} \cup \{\sqrt{p}\}$. Define hyperoperations "$+_{\sqrt{p}}$" and "$\cdot_{\sqrt{p}}$" on R as follows:

$$x +_{\sqrt{p}} y = y +_{\sqrt{p}} x = \begin{cases} \{\overline{0}, \sqrt{p}\} & x = -y \text{ or } x = y = \sqrt{p}, \\ x + y & x, y \in \mathbb{Z}_{p^k}, x \neq -y, \\ y & x = \overline{0} \text{ or } (\ x = \sqrt{p} \text{ and } y \notin \{\overline{0}, \sqrt{p}\}) \end{cases}$$

and

$$x \cdot_{\sqrt{p}} y = y \cdot_{\sqrt{p}} x = \begin{cases} x.y & x, y \in \mathbb{Z}_{p^k}, \\ \sqrt{p} & x \in \mathbb{Z}_{p^k} \setminus \{mp\}, y = \sqrt{p}(m \in \mathbb{N}), \\ \overline{0} & x = mp, y = \sqrt{p}(m \in \mathbb{N}), \\ \{\overline{0}, \sqrt{p}\} & x = y = \sqrt{p} \end{cases}.$$

Some modifications and computations show that $(R, +_{\sqrt{p}}, -_{\sqrt{p}}, \overline{0}, \cdot_{\sqrt{p}}, \overline{1})$ is a general multiring.

4. Hyperideals on general multirings

In this section, introduce a concept of general submultirings and (maximal-prime)hyperideals and find some equivalent conditions for hyperideals in general multirings.

Definition 4.1. Let $(R, +, -, 0, \cdot, 1)$ be a general multiring and $\emptyset \neq I \subseteq R$. We say

(i) I is a general submultiring of R, if $(I, +, -, 0, \cdot, 1)$ is a general multiring;
(ii) I is a hyperideal of R, if $I - I = I$ and $(R \cdot I \cup I \cdot R) \subseteq I$.

We will denote the set of all hyperideals of general multiring R by $\mathcal{H}I(R)$.

Theorem 4.2. *Let $(R, +, -, 0, \cdot, 1)$ be a general multiring and I be a hyperideal of R. Then*

(i) *$0 \in I$.*
(ii) *if $1 \in I$, then $I = R$.*
(iii) *for all $r \in R, x \in I$ and $n \in \mathbb{N}$, we have $n(r \cdot x) \subseteq I$;*
(iv) *if $x \in I$, then $-x \in I$.*

Proof. (i) By definition, for all $x \in I$, we have $0 \in 0 \cdot x \subseteq I$.

(ii) For all $x \in R$, since $1 \in I$ and I is a hyperideal of R, we get that $x \in 1 \cdot x \subseteq I$. Thus $R = I$.

$(iii), (iv)$ It is obtained by definition. □

Theorem 4.3. *Let $(R, +, -, 0, \cdot, 1)$ be a general multiring and $\emptyset \neq I \subseteq R$. Then I is a hyperideal of R if and only if satisfies in the following conditions:*

(i) *for all $x, y \in I, x - y \subseteq I$;*
(ii) *for all $r \in R$ and $x \in I$, we have $(r \cdot x) \cup (x \cdot r) \subseteq I$.*

Proof. It is obvious. □

Theorem 4.4. *Let $(R, +, -, 0, \cdot, 1)$ be a general multiring and $\emptyset \neq I \subseteq R$. Then I is a general submultiring of R if and only if satisfies in the following conditions:*

(i) $1 \in I$;
(ii) *for all $x, y \in I, x - y \subseteq I$;*
(iii) *for all $x, y \in I, x \cdot y \subseteq I$.*

Proof. The proof is obtained by definition. □

Example 4.5. (i) Consider the general multiring which is defined in Example 3.3 (ii). If $I = \{0\}, J = \{0, a\}$ and $K = \{0, b\}$, then $I, J, K, R \in \mathcal{H}I(R)$, while $M = \{0, 1, b\}$ is not a hyperideal of R.

(ii) Let $R = \{0, 1, a, b\}$. Then $(R, +, -, 0, \cdot, 1)$ is a general multiring as follows.

$+$	0	1	a	b
0	0	1	a	b
1	1	R	$\{1, a\}$	$\{1, b\}$
a	a	$\{1, a\}$	R	$\{a, b\}$
b	b	$\{1, b\}$	$\{a, b\}$	R

and

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	a
b	0	b	a	$\{a, b\}$

.

Then $\mathcal{H}I(R) = \{I = \{0\}, J = R\}$.

Definition 4.6. Let R be a general multiring and $M \neq R$ be an arbitrary hyperideal of R.

(i) M is called a maximal hyperideal of R, if the only hyperideals containing M are M and R;
(ii) M is called a prime hyperideal of R, if for all $a, b \in R, a \cdot b \subseteq M$ implies that $a \in M$ or $b \in M$.

We will denote the set of all maximal hyperideals of R by $\mathcal{M}x(R)$ and the set of all prime hyperideals of R by $\mathcal{P}r(R)$.

Example 4.7. (i) Let $(R, +, -, 0, \cdot, 1)$ be a general multiring in Theorem 3.16. Then

$$\mathcal{H}I(R) = \{\{0\}, I = \{0, a_4, a_5, \ldots a_n\}, J = \{0, a_2, a_4, a_5, \ldots a_n\},$$
$$K = \{0, a_3, a_4, a_5, \ldots a_n\}, R\}.$$

Clearly $\mathcal{M}x(R) = \{J, K\}$. Since $a_2 \cdot a_3 = a_4 \subseteq I$, while $a_2 \notin I$ and $a_3 \notin I$, we get that $I \notin \mathcal{P}r(R)$.

(ii) Consider the general multiring in Example 4.5 (i). Clearly $\mathcal{M}x(R) = \{\{0, a\}, \{0, b\}\} = \mathcal{P}r(R)$.

(iii) Consider the general multiring in Example 3.24. Clearly $I = \{\overline{0}\} \in \mathcal{P}r(R)$, while $I \notin \mathcal{M}x(R)$. Thus $\mathcal{P}r(R) \nsubseteq \mathcal{M}x(R)$.

Theorem 4.8. *Let p be a prime and $R = \mathbb{Z}_p \cup \{\sqrt{p}\}$. Then in general multiring $(R, +_{\sqrt{p}}, -, 0, \cdot_{\sqrt{p}}, 1)$, we have*

(i) *$\mathcal{H}I(R) = \{R, \{\overline{0}\}, \{\overline{0}, \sqrt{p}\}\}$;*

(ii) *$M = \{\overline{0}, \sqrt{p}\}$ is the only maximal hyperideal of R.*

Proof. We prove only (i) and the item (ii) is immediate. Let $I \in \mathcal{H}I(R) \setminus \{\{\overline{0}\}, R\}$. Since $\emptyset \neq I$ is a hyperideal of R, there exists $a \in I$ and so $\{a, 2a, 3a, \ldots, (p-1)a, \overline{0}\} \subseteq I$. In addition, for all $r \neq \sqrt{p}$ we have $r.\{a, 2a, 3a, \ldots, (p-1)a, \overline{0}\} \subseteq \{a, 2a, 3a, \ldots, (p-1)a, \overline{0}\}$. Also for $r = \sqrt{p}$, we have $r.\{a, 2a, 3a, \ldots, (p-1)a, \overline{0}\} \subseteq \{\sqrt{p}, \overline{0}\}$. Thus $I = \{\sqrt{p}, \overline{0}\}$. □

Theorem 4.9. *Let p be a prime, $k \in \mathbb{N}$ and $R = \mathbb{Z}_{p^k} \cup \{\sqrt{p}\}$. Then in the general multiring $(R, +_{\sqrt{p}}, -, 0, \cdot_{\sqrt{p}}, 1)$, we have*

(i) *if I be a nontrivial hyperideal of R, then $\sqrt{p} \in I$;*

(ii) *for all $1 \leq m \leq p^{k-1}$,*

$$I_p^{(m)} = \{m\overline{p}, 2m\overline{p}, \ldots, tm\overline{p}, \sqrt{p} \mid t \in \mathbb{N} \text{ is the smallest s.t } tm \overset{p^{k-1}}{\equiv} 0\}$$

is a hyperideal of R;

(iii) *for all $1 \leq m \leq p^{k-1}$, we have $|I_p^{(m)}| = 1 + \dfrac{p^{k-1}}{\gcd(m, p^{k-1})}$;*

(iv) *for all $1 \leq m, m' \leq p^{k-1}$, $I_p^{(m)} = I_p^{(m')}$ if and only if $\gcd(p^{k-1}, m) = \gcd(p^{k-1}, m')$.*

Proof. (i) Let $\overline{0} \neq x \in I$. Since I is a hyperideal of R and $\sqrt{p} \in R$, we get that $\sqrt{p} \cdot x \subseteq I$. On other hand for all $x \in I$, $\sqrt{p} \cdot x = \overline{0}, \sqrt{p}$ or $\{\overline{0}, \sqrt{p}\}$. If $\sqrt{p} \cdot x = \overline{0}$, then by definition there exists $m \in \mathbb{N}$ such that $x = mp$. Hence there is $n \in \mathbb{N}$ such that $\{\overline{0}, \sqrt{p}\} = nx \subseteq I$ and in any case $\sqrt{p} \in I$.

(ii) Let $1 \leq m \leq p^{k-1}$ and $x, y \in I_p^{(m)} \setminus \{\sqrt{p}\}$. Then there exists $1 \leq k_1, k_2 \leq t \in \mathbb{N}$ such that $x + y = (k_1 + k_2)(m\overline{p}) \subseteq I_p^{(m)}$, because of $\overline{0} \leq (k_1 + k_2)(m\overline{p}) \leq \overline{p^{k-1}}$. In addition for all $\overline{x} \in I_p^{(m)}$, $\sqrt{p} + \overline{x} = \{\overline{x}\} \subseteq I_p^{(m)}$ and $\sqrt{p} + \sqrt{p} = \{\overline{0}, \sqrt{p}\} \subseteq I_p^{(m)}$, imply that for all $x, y \in I_p^{(m)}$, $x + y \subseteq I_p^{(m)}$. Also for all $r \in R \setminus \{\sqrt{p}\}$ and $x \in I_p^{(m)} \setminus \{\sqrt{p}\}$

there exists $1 \leq k \leq t \in \mathbb{N}$ such that $r \cdot x = rk(m\overline{p}) \subseteq I_p^{(m)}$, because of $\overline{0} \leq (rkm)\overline{p} \leq \overline{p^{k-1}}$. On the other hand, $\sqrt{p} \cdot \overline{x} \subseteq \{\overline{0}, \sqrt{p}\}$, imply that for all $r \in R$ and $x \in I_p^{(m)}$, we have $r \cdot x \subseteq I_p^{(m)}$.

(iii) Let $1 \leq m \leq p^{k-1}$. Using item (i), $\sqrt{p} \in I_p^m$, so

$$|I_p^m| = 1 + |\{t \in \mathbb{N} \mid t \text{ is the smallest s.t } tm \equiv 0 \pmod{p^{k-1}}\}| = q.$$

Suppose $t \in \mathbb{Z}$ is the smallest such that $tm \equiv 0 \pmod{p^{k-1}}$. Thus $p^{k-1} \mid tm$. If $\gcd(p^{k-1}, m) = 1$, then $p^{k-1} \mid t$ and because t is the smallest, we obtain that $t = p^{k-1}$. But for $\gcd(p^{k-1}, m) = d \neq 1$, have $\dfrac{p^{k-1}}{d} \mid t$. Since $p^{k-1}m \equiv 0 \pmod{p^{k-1}}$ and $t \in \mathbb{N}$ is the smallest such that $tm \equiv 0 \pmod{p^{k-1}}$, we get that $\dfrac{p^{k-1}}{\gcd(m, p^{k-1})} = t$.

(iv) Let $1 \leq m, m' \leq p^{k-1}$. Then by item ($ii$), $I_p^{(m)} = I_p^{(m')}$ if and only if

$$1 + \frac{p^{k-1}}{\gcd(m, p^{k-1})} = 1 + \frac{p^{k-1}}{\gcd(m', p^{k-1})} \iff \gcd(p^{k-1}, m) = \gcd(p^{k-1}, m').$$

□

Theorem 4.10. *Let p be a prime, $k \in \mathbb{N}$ and $R = \mathbb{Z}_{p^k} \cup \{\sqrt{p}\}$. Then in the general multiring $(R, +_{\sqrt{p}}, -, 0, \cdot_{\sqrt{p}}, 1)$, we have*

- (i) $\mathcal{H}I(R) = \{R, \{\overline{0}\}, I_p^{(m)} \mid 1 \leq m \leq p^{k-1}\}$;
- (ii) $|\mathcal{H}I(R)| = k + 2$;
- (iii) $m \leq m'$ *if and only if* $I_p^{(m)} \supseteq I_p^{(m')}$, *where* $1 \leq m, m' \leq p^{k-1}$;
- (iv) $I_p^{(1)}$ *is the only maximal hyperideal of* R.

Proof. (i) Clearly $R, \{\overline{0}\} \in \mathcal{H}I(R)$. Let I be a nontrivial hyperideal of R, using Theorem 4.9(i), $\overline{0}, \sqrt{p} \in I$. Suppose that $0 \neq a \in I$. If $\gcd(a, p^k) = 1$, then there exist $s, s' \in \mathbb{Z}$ such that $1 = as + s'p^k$. It follows that $1 \in I$ and we get that $R = I$. But for $\gcd(a, p^k) = d \neq 1$, since p is a prime, there exist $1 \leq i \leq k$ in such a way that $d = p^i$, consequently $p^i \in I$.

(ii), (iii) It is obtained by (i). □

Corollary 4.11. *Let p be a prime, $k \in \mathbb{N}$ and $R = \mathbb{Z}_{p^k} \cup \{\sqrt{p}\}$. Then in the general multiring $(R, +_{\sqrt{p}}, -, 0, \cdot_{\sqrt{p}}, 1)$, we have the* **Hass** *tree HT of $\mathcal{H}I(R)$ in Figure 1.*

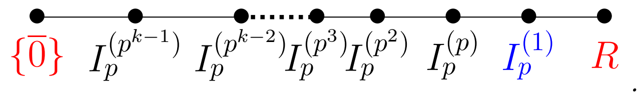

FIGURE 1. Chain of $(\mathcal{H}I(R), \subseteq)$.

Example 4.12. (i) Consider the general multiring $R = \mathbb{Z}_{16} \cup \{\sqrt{2}\}$. Computations show that

$$I_2^{(1)} = I_2^{(3)} = I_2^{(5)} = I_2^{(7)} = \{\overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}, \overline{12}, \overline{14}, \overline{0}, \sqrt{2}\},$$
$$I_2^{(2)} = I_2^{(6)} = \{\overline{4}, \overline{8}, \overline{12}, \overline{0}, \sqrt{2}\},$$
$$I_2^{(4)} = \{\overline{8}, \overline{0}, \sqrt{2}\}, I_2^{(8)} = \{\overline{0}, \sqrt{2}\}$$

and so $\mathcal{H}I(R) = \{I_2^{(1)}, I_2^{(2)}, I_2^{(4)}, I_2^{(8)}, \{\overline{0}\}, R\}$.

(ii) Consider the general multiring $R = \mathbb{Z}_{27} \cup \{\sqrt{3}\}$. Computations show that

$$I_3^{(1)} = I_3^{(2)} = I_3^{(4)} = I_3^{(5)} = I_3^{(7)} = I_3^{(8)} = \{\overline{3}, \overline{6}, \overline{9}, \ldots, \overline{24}, \overline{0}, \sqrt{3}\},$$
$$I_3^{(3)} = I_3^{(6)} = \{\overline{9}, \overline{18}, \overline{0}, \sqrt{3}\},$$
$$I_3^{(9)} = \{\overline{0}, \sqrt{3}\}$$

and so $\mathcal{H}I(R) = \{I_3^{(1)}, I_3^{(3)}, I_3^{(9)}, \{\overline{0}\}, R\}$.

In the following there are two problems, which we can not prove or disprove them.

Open Problem 4.13. Let $(R, +, -, 0, \cdot, 1)$ be a general multiring and $x, y \in R$. Then

(i) $\big((-1) \cdot x\big) \cap \big(1 \cdot (-x)\big) \neq \emptyset$;
(ii) $\{-x\} \cap \big((-1) \cdot x\big) \neq \emptyset$;
(iii) $((-x) \cdot y) \cap (x \cdot (-y)) \neq \emptyset$.

Open Problem 4.14. Let R be a general multiring. Then, $\mathcal{M}x(R) \subseteq \mathcal{P}r(R)$.

5. **Conclusion**

The current paper has defined the general multirings as a generalization of multirings and presented some properties in these hyperstructures. Also:

(i) Based on multigroups, general multirings are constructed.
(ii) Boolean general multiring is a $(+)$-commutative general multiring, while it is not a commutative general multiring.

(*iii*) It is proved that the Hass tree of hyperideals of special finite (m, n)–general multirings is a chain.

We hope that these results are helpful for further studies in general multiring theory. In our future studies, we hope to obtain more results regarding fuzzy general multiring, soft general multiring, tropical general multifield and their applications.

Acknowledgments

The authors are deeply grateful to the referees for a very careful reading of the paper and many valuable suggestions.

References

1. R. Ameri, M. Hamidi and A. A. Tavakoli, Boolean Rings Based on Multirings, *J. Sci .I. R. I.*, **32**(2) (2021), 159–168.
2. R. Ameri, T. Nozari, A New Characterization of Fundamental Relation on Hyperrings, *Int. J. Contemp. Math. Sciences*, **5**(15) (2010), 721–738.
3. J. Chvalina, S.H. Mayerova and A.D. Nezhad, General actions of hyperstructures and some applications, *An St Univ Ovidius Constanta*, **21**(1) (2013), 59–82.
4. P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory*, Klwer Academic Publishers, 2002.
5. B. Davvaz, T. Vougiouklis, Commutative rings obtained from hyperrings (Hv-rings) with α^*–relations, *Commun. Algebra.*, **35** (2007), 3307–3320.
6. V. L. Fotea, M. Jafarpour, S. S. Mousavi, The relation δ^n and multisemi–direct hyperproducts of hypergroups, *Commun. Algebra.*, **40** (2012), 3597–3608.
7. P. Galadki and M. Marshal, Orderings and Signatures of Higher Level on Multirings and Hyperfields, *J. K-Theory*, **10** (2012), 489–518.
8. P. Ghiasvand, S. Mirvakili and B. Davvaz, Boolean Rings Obtained from Hyperrings with $\eta^*_{1,m}$ Relations, *Iran. J. Sci. Technol Trans. Sci.*, **41**(1) (2017), 69–79.
9. M. Hamidi and A. R. Ashrafi, Fundamental relation and automorphism group of very thin H_v–groups, *Commun. Algebra.*, **45**(1) (2017), 130–140.
10. M. Hamidi, A. Borumand Saeid, V. Leoreanu, Divisible Groups Derived From Divisible Hypergroups, *U.P.B. Sci. Bull., Series A*, **79**(2) (2017), 59–70.
11. M. Hamidi, V. Leoreanu, ON (2-Closed) Regular Hypergroups, *U.P.B. Sci. Bull., Series A*, **80(4)** (2018), 173–186.
12. F. Marty, *Sur une generalization de la notion de groupe* 8*th Congres Math*, Scandinaves, Stockholm, (1934), 45–49.
13. M. Marshall, *Spaces of orderings and abstract real spectra*, Springer Lecture Notes in Math. 1636, (1996).
14. M. Marshall , Real reduced multirings and multifields, *J. Pure Appl. Algebra.*, **205** (2006), 452–468.
15. T. Vougiouklis, The fundamental Relations in Hyperrings, *The general th International congress in Algebraic Hyperstructures hyperfield Proc.4 and its Applications (AHA 1990) World Scientific*, 203–211.

Mohammad Hamidi
Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-4697, Tehran, Iran.
Email: m.hamidi@pnu.ac.ir

Ali Asghar Tavakoli
Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-4697, Tehran, Iran.
Email: tavakoli397@gmail.com

Reza Ameri
School of Mathematics, Statistics and Computer Sciences, University of Tehran, P.O. Box 14155-6455, Tehran, Iran.
Email: rameri@ut.ac.ir

Journal of Algebraic Systems

VALUED-POTENT (GENERAL) MULTIRINGS

M. HAMIDI, A. A. TAVAKOLI AND R. AMERI

چندحلقه های (عمومی) توان-ارزشی

محمد حمیدی[۱]، علی اصغر توکلی[۲]، رضا عامری[۳]

[۱،۲]دانشکده علوم، دانشگاه پیام نور، ایران

[۳]دانشکده علوم ریاضی، آمار و کامپیوتر، دانشگاه تهران، تهران، ایران

این مقاله چند حلقه ها را به مفهوم جدیدی تحت عنوان چندحلقه های عمومی توسعه می دهد، خواص آن ها را تحقیق می کند و یک نوع خاصی از چند حلقه های عمومی تحت عنوان نماد چند حلقه های عمومی (m, n)- توان را ارایه می دهد. این مطالعه تفاوت بین رده چندحلقه ها، چندحلقه های عمومی و ابرحلقه های عمومی را مورد تجزیه و تحلیل قرار می دهد و روی هر مجموعه غیر تهی داده شده دلخواه، چندحلقه های عمومی متناهی (غیرمتناهی) را می سازد. در پایان مفهوم ابر ایدآل ها در چندحلقه های عمومی راتعریف می کنیم و به مقایسه ابر ایدآل ها در چندحلقه های عمومی و سایر ابرساختارهای مشابه می پردازیم.

کلمات کلیدی: چندحلقه(عمومی)، چندحلقه(عمومی) (m, n)- توان، ابرحلقه عمومی، ابرایدآل.