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NORMAL INJECTIVE RESOLUTION OF GENERAL KRASNER HYPERMODULES

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Abstract. In this paper, we construct the concept of general Krasner hyperring based on the ring structures and the left general Krasner hypermodule based on the module structures. This study introduces the trivial left general Krasner hypermodules and proves that the trivial left general Krasner hypermodules are different from left Krasner hypermodules. We show that for any given general Krasner hyperring R and trivial left general Krasner hypermodules $A, B, {}_{\mathbf{R}}\mathbf{h}\text{om}(A, B)$ is a left general Krasner hypermodule and ${}_{\mathbf{R}}\mathbf{h}\text{om}(-, B)$, $({}_{\mathbf{R}}\mathbf{h}\text{om}(A, -))$ is an exact covariant functor (contravariant). Finally, we show that the category ${}_{\mathbf{R}}\mathbf{GKH}\text{mod}$ (left trivial general Krasner hypermodules and all homomorphisms) is an abelian category and trivial left general Krasner hypermodules have a normal injective resolution.

1. Introduction

The hyperstructure theory as an extension of classical structures was firstly introduced, by F. Marty in 1934 [12]. In hyperalgebraic system, the hyperproduct of elements is a set and so any algebraic system is a hyperalgebraic system. Marty extended the concept of groups to hypergroups and other researchers presented the hyperalgebraic concepts such as hyperring, hypermodule, hyperfield, hypergraph, polygroup,

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multiring, etc. Hyperstructures are applied in several branches of sciences such as artificial intelligence, chemistry and (hyper) complex network [6]. The foundations of the theory of hypermodules were provided by Massouros in 1988, when he introduced the concept of free hypermodules and cyclic hypermodules [13]. Recently, based on various types of morphisms, the authors introduced some categories consisting of these R-hypermodules such as the categories ${}_Rh\mathbf{mod}$, ${}_{R_s}h\mathbf{mod}$, ${}_R\mathcal{H}\mathbf{mod}$, and ${}_{R_s}\mathcal{H}\mathbf{mod}$ [1, 2, 3, 5, 7, 9, 8, 11, 10, 14, 15, 16, 18]. Bordbar et al. initiateed the study of the chains of hypermodules over a Krasner hyperring R, endowing first the set $Hom^n_R(M,N)$ of all normal homomorphisms between two R-hypermodules M and N with a structure of R-hypermodule. They, studied on the concepts of normal injectivity and projectivity of hypermodules over a Krasner hyperring R, characterizing them by the mean of chains of R-hypermodules [4]. Regarding these points, we introduce the concept of trivial left general Krasner hypermodules, normal injective general Krasner R-hypermodules and construct the trivial left general Krasner hypermodules based on any given non-empty set. We extend the concept of modules to (trivial) left general Krasner hypermodules and investigate their properties. The main motivation of this work is to show that the category of trivial left general Krasner hypermodules is an abelian category. In this regard, we prove that any trivial general Krasner R-hypermodule, where R is a commutative ring can be embedded as a general Krasner R-subhypermodule of a normal injective general Krasner R-hypermodule. Indeed, we show that every trivial general Krasner R-hypermodule, where R is commutative has a normal injective resolution. We investigate that under certain conditions for any given general Krasner hyperring R and trivial left general Krasner hypermodules $A, B, {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(A,B)$ is a left general Krasner hypermodule and ${}_{\mathbf{R}}\mathbf{h}\mathrm{om}(A,-)$ is an exact covariant functor. At the end, we present fundamental strong R-isomorphism theorems in left general Krasner hypermodules to construct the quotient of left general Krasner hypermodule and seek the homological properties of left general Krasner hypermodules.

2. Preliminaries

In what follows, we recall some results from [1, 6], that are needed in our work.

Let R be a nonempty set and $\mathcal{P}^*(R) = \{S \mid \emptyset \neq S \subseteq R\}$. Every map $+_R : R \times R \longrightarrow \mathcal{P}^*(R)$, is said to be a *hyperoperation*, a hyperstructure

$(R, +_R)$ is called a *hypergroupoid* and for all nonempty subsets A, B of R, $A +_R B = \bigcup_{a \in A, b \in B} (a +_R b)$(generally, the singleton $\{x\}$ is identified with its member x, so $x +_R B = \{x\} +_R B$, where $x \in R$). Recall that a *hypergroupoid* $(R, +_R)$ is called a *semihypergroup*, if for all $x, y, z \in R, (x +_R y) +_R z = x +_R (y +_R z)$ and a semihypergroup $(R, +_R)$ is called a *hypergroup*, if for $x \in R, x +_R R = R +_R x = R$ (*reproduction axiom*). A commutative hypergroup $(R, +_R)$ (for all $x, y \in R, x +_R y = y +_R x$) is called a *canonical hypergroup* provided that

- (i) there exists a unique element $0_R \in R$ such that for all $x \in R, 0_R +_R x = x +_R 0_R = \{x\}$,
- (ii) for all $x \in R$, there exists a unique element $-x \in R$ such that $0_R \in \big(x +_R (-x)\big) \cap \big((-x) +_R x\big)$,
- (iii) for all $x, y, z \in R, x \in y +_R z$ implies $y \in x +_R (-z)$ and $z \in x +_R (-y)$,

and we will denote it by $(R, +_R, 0_R)$. A system $(R, +_R, 0_R, \cdot_R)$ is called a *general Krasner hyperring* whenever

- (i) $(R, +_R, 0_R)$ is a canonical hypergroup,
- (ii) $(R, \cdot_R)$ is a semihypergroup such that for all $x \in R$, $x \cdot_R 0_R = 0_R \cdot_R x = \{0_R\}$,
- (iii) for all $x, y, z \in R$, we have $x \cdot_R (y +_R z) \subseteq (x \cdot_R y) +_R (x \cdot_R z)$ and $(y +_R z) \cdot_R x \subseteq (y \cdot_R x) +_R (z \cdot_R x)$.

A general Krasner hyperring $(R, +_R, 0_R, \cdot_R)$ is called *commutative* (with unit element), if for all $x, y \in R$, $x \cdot_R y = y \cdot_R x$ (if there exists an element $1 \in R$ such that for all $x \in R, 1 \cdot_R x = x \cdot_R 1 = \{x\}$). For a given general Krasner hyperring $(R, +_R, 0_R, \cdot_R)$, a canonical hypergroup $(A, +_A, 0_A)$ together with a left external multiplication $* : R \times A \longrightarrow \mathcal{P}^*(A)$, is called a *left general Krasner hypermodule* over general Krasner R $\big($we say that it is a general Krasner R-hypermodule and denote it by $(A, +_A, 0_A, *)\big)$, if for all $r, s \in R$ and for all $a, b \in A$,

- (i) $r * (a +_A b) \subseteq (r * a) +_A (r * b)$,
- (ii) $(r +_R s) * a \subseteq (r * a) +_A (s * a)$,
- (iii) $(r \cdot_R s) * a \subseteq r * (s * a)$,
- (iv) $0_R * a = \{0_A\}$.

A general Krasner R-hypermodule A is called unitary if there exists $1_R \in R$ such that for all $a \in A$ we have $1_R * a = \{a\}$. A map $f : A \to A'$ is called a (an inclusion) strong or good R-homomorphism of general Krasner R-hypermodules if, for all $x, y \in A$ and for all $r \in R$, $(f(x +_A y) \subseteq f(x) +_{A'} f(y))$ $f(x +_A y) = f(x) +_{A'} f(y)$ and

$(f(r * x) \subseteq r *' f(x))$ $f(r * x) = r *' f(x)$. A map $f : A \to A'$ is called a weak R-homomorphism of general Krasner R-hypermodules, if for all $x, y \in A$ and for all $r \in R$, $f(x +_A y) \cap (f(x) +_{A'} f(y)) \neq \emptyset$ and $f(r * x) \cap (r *' f(x)) \neq \emptyset$. A nonempty subset B of A is said to be an R-subhypermodule of A (is denoted by $B \leq A$), if for all $x, y \in B$ and for all $r \in R$, $x +_A (-y) \subseteq B$ and $r * x \subseteq B$, also B is called a normal general R-subhypermodule (is denote by $B \trianglelefteq A$), if for all $x \in A$, we have $x +_A B +_A (-x) \subseteq B$.

3. Construction of general Krasner hyperrings

We extend the concept of the ring to general Krasner hyperring and construct general Krasner hyperring on any given non-empty set R, where $|R| \geq 4$. In what follows, we construct a general Krasner hyperring using the concept of the ring structures.

Theorem 3.1. *Assume $(R, +, 0, \cdot)$ is a ring. Then there exist hyperoperations "$+_R$" and "$\cdot_R$" on R such that $(R, +_R, 0, \cdot_R)$ is a commutative general Krasner hyperring.*

Proof. Let $x, y \in R$. Define "$+_R$" and "$\cdot_R$" on R by

$$x +_R y = \begin{cases} x, & y = 0, \\ R, & x = -y (x \neq 0), \\ R \setminus \{0\}, & \text{otherwise}, \end{cases} \quad x \cdot_R y = \begin{cases} 0, & y = 0, \\ \{x \cdot y, 0\}, & \text{otherwise} \end{cases},$$

where $x +_R y = y +_R x$ and $x \cdot_R y = y \cdot_R x$. Routine computations, show that $(R, +_R, 0, \cdot_R)$ is a general Krasner hyperring, while it is not a Krasner hyperring. □

In what follows, we construct a canonical hypergroup on any given non-empty set.

Theorem 3.2. *Suppose R is a set with $|R| \geq 4$. Then there exists a hyperoperation "$+_R$" on R and $0_R \in R$ such that $(R, +_R, 0_R)$ is a canonical hypergroup.*

Proof. Let R be an arbitrary set, with $|R| \geq 4$, fixed $a_0 = 0_R \in R$ and $C_3 = \{a_1, a_2, a_3\} \subseteq R$. Now, for all $a_i, a_j \in R$, we define "$+_R$" on R as follows:

$$a_i +_R a_j = \begin{cases} R \setminus C_3 & i = j \neq 0, \\ R \setminus (C_3 \cup \{0_R\}) & i \neq j \geq 4, \\ C_3 \setminus \{a_i, a_j\} & 1 \leq i \neq j \leq 3, \\ \{a_i\} & 1 \leq i \leq 3 \text{ and } j \geq 4, \end{cases}$$

where, for all $a_i \in R$, $a_i +_R 0_R = \{a_i\}$. It is easy to verify that $(R, +_R, 0_R)$ is a canonical hypergroup. □

Example 3.3. Consider $R = \{0_R, a_1, a_2, a_3, a_4, a_5\}$ and

$+_R$	0_R	a_1	a_2	a_3	a_4	a_5
0_R	0_R	a_1	a_2	a_3	a_4	a_5
a_1	a_1	T	a_3	a_2	a_1	a_1
a_2	a_2	a_3	T	a_1	a_2	a_2
a_3	a_3	a_2	a_1	T	a_3	a_3
a_4	a_4	a_1	a_2	a_3	T	T'
a_5	a_5	a_1	a_2	a_3	T'	T

.

Then $(R, +_R, 0_R)$ is a canonical hypergroup by Theorem 3.2, where $T = \{0_R, a_4, a_5\}$ and $T' = \{a_4, a_5\}$.

In the following theorem, we construct a commutative general Krasner hyperring on any given finite set $|R| \geq 4$.

Theorem 3.4. *Assume R is a set and $4 \leq |R| \leq n$. Then there exist hyperoperations "$+_R$", "$\cdot_R$" on R, and $0_R \in R$ such that $(R, +_R, 0_R, \cdot_R)$ is a commutative general Krasner hyperring.*

Proof. Let $R = \{a_0, a_1, \ldots, a_n\}$ be an arbitrary set, $n \geq 4$, fixed $a_0 = 0_R \in R$ and $C_3 = \{a_1, a_2, a_3\} \subseteq R$. By Theorem 3.2, there exists a hyperoperation "$+_R$" on R and $0_R \in R$ such that $(R, +_R, 0_R)$ is a canonical hypergroup. Now for any $a_i, a_j \in R$, we define "$\cdot_R$" on R as follows:

$$a_i \cdot_R a_j = \begin{cases} 0 & i = 0, \\ a_j & i = 1, \\ a_i & i = j \neq n, \\ a_4 & i = k, j \geq k+1, 2 \leq k, \\ \{a_4, a_n\} & i = j = n, \end{cases}$$

and for each $0 \leq i \neq j$, $a_i \cdot_R a_j = a_j \cdot_R a_i$. One can see that $(R, +_R, 0_R, \cdot_R)$ satisfies in the definition of commutative general Krasner hyperring. conclude that $(R, +, -, 0, \cdot, 1)$ is not a . □

Example 3.5. Consider $R = \{0, 1, a_2, a_3, a_4, a_5, a_6\}$ and

$+_R$	0	1	a_2	a_3	a_4	a_5	a_6
0	0	1	a_2	a_3	a_4	a_5	a_6
1	1	T	a_3	a_2	1	1	1
a_2	a_2	a_3	T	1	a_2	a_2	a_2
a_3	a_3	a_2	1	T	a_3	a_3	a_3
a_4	a_4	1	a_2	a_3	T	T'	T'
a_5	a_5	1	a_2	a_3	T'	T	T'
a_6	a_6	1	a_2	a_3	T'	T'	T

,

$\cdot_R$	0	1	a_2	a_3	a_4	a_5	a_6
0	0	0	0	0	0	0	0
1	0	1	a_2	a_3	a_4	a_5	a_6
a_2	0	a_2	a_2	a_4	a_4	a_4	a_4
a_3	0	a_3	a_4	a_3	a_4	a_4	a_4
a_4	0	a_4	a_4	a_4	a_4	a_4	a_4
a_5	0	a_5	a_4	a_4	a_4	a_5	a_4
a_6	0	a_6	a_4	a_4	a_4	a_4	$\{a_6, a_4\}$

.

Then $(R, +, 0, \cdot, 1)$ is a commutative general Krasner hyperring (while it is not a general Krasner hyperring), by Theorem 3.4, where $T = \{0, a_4, a_5, a_6\}$ and $T' = \{a_4, a_5, a_6\}$.

Theorem 3.6. *Suppose $(R, +_R, 0_R, \cdot_R, 1_R)$ is a general Krasner hyperring with unit and $A, B \subseteq R$. Then*

(i) $-A = \{-a \mid a \in A\}$
(ii) $0_R \in A - A$;
(iii) $C \subseteq A +_R B$ *if and only if* $A \cap (C - B) \neq \emptyset$;
(iv) $-(-A) = A$;
(v) $0_R +_R A = A = A +_R 0$;
(vi) *if* $0_R \in A +_R B$, *then* $A \cap (-B) \neq \emptyset$, *and* $(-A) \cap B \neq \emptyset$;
(vii) $0_R \cdot_R A = A \cdot_R 0_R = 0_R$ *and* $1_R \cdot_R A = A \cdot_R 1_R = A$;
(viii) $0_R - A = -A$ *and* $A - 0_R = A$;
(ix) $A \subseteq B$ *if and only if* $-A \subseteq -B$.

Proof. (i) Let $B = -A$. Then $0_R \in A +_R B$ and so there exists $a \in A$ and $b \in B$ such that $0_R \in a +_R b$. It follows that $b \in -a +_R 0_R = \{-a\}$ or $-a = b$.

(ii) By definition, we have $A - A = \bigcup_{a,b \in A} (a - b) = \bigcup_{a,b \in A} (a +_R (-b))$, so $0_R \in A - A$.

(iii) Since $C \subseteq A +_R B$, for all $c \in C$, there exist $a \in A$ and $b \in B$ such that $c \in a +_R b$. It concludes that $a \in c - b$ and so $A \cap (C - B) \neq \emptyset$. The converse part is similar to.

(iv) By definition, $-(-A) = \bigcup_{a \in A} (-(-a))$, so we get that $-(-A) = A$.

(v) It is clear.

(vi) Sine $0_R \in A +_R B$, there exist $a \in A$ and $b \in B$ in such a way that $0_R \in a +_R b$. It implies that $a = -b$ and so $A \cap (-B) \neq \emptyset$.

(vii), (viii), and (ix) are immediate consequences of the definition. □

4. Construction of general Krasner R-hypermodule

In this section, we are ready to construct (general) Krasner R-hyperm odule based on any given non-empty set and so extend ring-modules to (general) Krasner R-hypermodules. Moreover, we introduce the quotient of general Krasner R-hypermodules and prove the strong R-Isomorphism Theorems.

Definition 4.1. Let $(A, +_A, *)$ be a (general) Krasner R-hypermodule. $(A, +_A, *)$ is called (*i*) an associative (general) Krasner R-hypermodule, if for all $r, s \in R, a \in A, (r \cdot_R s) * a = r * (s * a)$.

(*ii*) a trivial general Krasner R-hypermodule, if for all $r \in R$ and for all $a \in A$, we have $|r * a| = 1$, otherwise it is called a non-trivial general Krasner R-hypermodule.

Theorem 4.2. *Assume $(A, +_A, *)$ is a general Krasner R-hypermodule. Then for all $r \in R$ and $a \in A$:*

(*i*) $0_A \in r * 0_A$;
(*ii*) $-(r * a) \cap (-r * a) \neq \emptyset$;
(*iii*) *if $(A, +_A, *)$ is a trivial general Krasner R-hypermodule, then* $-(r * a) = (-r) * a$;
(*iv*) $-(r * a) \cap (r * (-a)) \neq \emptyset$;
(*v*) *if $(A, +_A, *)$ is a trivial general Krasner R-hypermodule, then* $-(r * a) = r * (-a)$;
(*vi*) *if $(A, +_A, *)$ is a trivial general Krasner R-hypermodule, then* $0_A = r * 0_A$.

Proof. (*i*) Let $r \in R$. Then $(r \cdot_R 0_R) * 0_A \subseteq r * (0_R * 0_A)$. It follows that $\{0_A\} = 0_R * 0_A \subseteq r * 0_A$.

(*ii*) Let $r \in R$ and $a \in A$. Then

$$0_A \in 0_R * a = (r - r) * a \subseteq (r * a) +_A (-r * a).$$

Using Theorem 3.6, we get that $-(r * a) \cap ((-r) * a) \neq \emptyset$.

(*iii*) Let $r \in R$ and $a \in A$. Since $|-(r * a)| = |(-r) * a| = 1$, by item (*ii*),we get that $-(r * a) = (-r) * a$.

(*iv*), (*v*) and (*vi*) are proved similarly.

□

Example 4.3. Set $R = \{0, 1, a\}$. Then $(R, +_R, \cdot_R)$ is a commutative general Krasner hyperring and $(R, +_R, *)$ is an associative non-trivial general Krasner R-hypermodule.

$+_R$	0	1	a
0	0	1	a
1	1	$\{0, a\}$	1
a	a	1	$\{0, a\}$

,

$\cdot_R$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	$\{0, a\}$

and

$*$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	$\{0, a\}$

.

Example 4.4. (*i*) Let $R = \{0, 1, 2, 3\}$. Then $(R, +_R, \cdot_R)$ is a commutative general Krasner hyperring with unit, $(R, +'_R)$ is a canonical hypergroup and so it is a non-trivial general Krasner R-hypermodule as follows.

$+_R$	0	1	2	3
0	0	1	2	3
1	1	1	R	$\{1,3\}$
2	2	R	2	$\{2,3\}$
3	3	$\{1,3\}$	$\{2,3\}$	R

,

$\cdot_R$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	R	R
3	0	3	R	R

and

$+'_R$	0	1	2	3
0	0	1	2	3
1	1	R	$\{1,2,3\}$	$\{1,2,3\}$
2	2	$\{1,2,3\}$	R	$\{1,2,3\}$
3	3	$\{1,2,3\}$	$\{1,2,3\}$	R

,

where for all $x \in R$, $0 * x = x * 0 = \{0\}$, and for all $x, y \in R \setminus \{0\}$, $x * y = R$.

(ii) Let $R = \{0, 1, a, b\}$. Then $(R, +_R, \cdot_R)$ is a commutative general Krasner hyperring with unit, $(R, +'_R)$ is a canonical hypergroup and so it is a non-trivial general Krasner R-hypermodule as follows.

$+_R$	0	1	a	b
0	0	1	a	b
1	1	R	$\{1,b\}$	$\{1,a\}$
a	a	$\{1,b\}$	$\{0,a\}$	1
b	b	$\{1,a\}$	1	$\{0,b\}$

,

$\cdot_R$	0	1	a	b
0	0	0	0	0
1	0	1	$\{1,a\}$	$\{1,b\}$
a	0	$\{1,a\}$	a	R
b	0	$\{1,b\}$	R	b

and

$+'_R$	0	1	a	b
0	0	1	a	b
1	1	$\{1,0\}$	$\{a,b\}$	$\{a,b\}$
a	a	$\{a,b\}$	$R \setminus \{0\}$	R
b	b	$\{a,b\}$	R	$R \setminus \{0\}$

,

where for all $x \in R$, $0 * x = x * 0 = \{0\}$ and for all $x, y \in R \setminus \{0\}$, $x * y = R$.

Example 4.5. Consider the ring $(\mathbb{Z}, +, .)$ and $A = \{\pi, e, a, b, c\}$. Then $(A, +_A, *)$ is a non-trivial general Krasner R-hypermodule as follows.

$+_A$	π	e	a	b	c
π	$\{\pi, e\}$	π	a	b	c
e	π	e	a	b	c
a	a	a	$\{\pi, e\}$	c	b
b	b	b	c	$\{\pi, e\}$	a
c	c	c	b	a	$\{\pi, e\}$

and

$*$	e	a	b	c	π
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-2	$\{e, \pi\}$	$\{e, \pi\}$	$\{e, \pi\}$	$\{e, \pi\}$	$\{e, \pi\}$
-1	$\{e, \pi\}$	$\{a, \pi\}$	$\{b, \pi\}$	$\{c, \pi\}$	$\{e, \pi\}$
0	e	e	e	e	e
1	$\{e, \pi\}$	$\{a, \pi\}$	$\{b, \pi\}$	$\{c, \pi\}$	$\{e, \pi\}$
2	$\{e, \pi\}$	$\{e, \pi\}$	$\{e, \pi\}$	$\{e, \pi\}$	$\{e, \pi\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		

.

Remark 4.6. Suppose $(R, +_R, 0_R, \cdot_R)$ is a general Krasner hyperring and $(A, +_A, *)$ is a general Krasner R-hypermodule. Example 4.5, shows that for any given $r \in R$, necessarily $r * 0_A \neq 0_A$.

The following theorem shows the existence of general Krasner R-hypermodule.

Theorem 4.7. *Assume $(R, +_R, \cdot_R)$ is a general Krasner hyperring (wiyh unit). Then there exists a left external multiplication $* : R \times R \longrightarrow \mathcal{P}^*(R)$ such that $(R, +_R, *)$ is a non-trivial (unitary) general Krasner R-hypermodule.*

Proof. For any $r, a \in R$, define $* : R \times R \to \mathcal{P}^*(R)$ by $r * a = r \cdot_R a$. One can see that $(R, +_R, *)$ is a general Krasner R-hypermodule. □

Remark 4.8. Since $(R, +_R, 0, \cdot_R)$ is a general Krasner hyperring, it is clear that trivial general Krasner R-hypermodules are not isomorphic to Krasner R-hypermodules, necessarily.

In the following, we introduce some general Krasner R-hypermodules differently from Theorem 4.7.

Example 4.9. (i) Let $R = \{0, 1, a\}$. Then $(R, +_R, \cdot_R)$ is a commutative general Krasner hyperring with unit, so it is a unitary non-trivial general Krasner R-hypermodule as follows.

$+_R$	0	1	a
0	0	1	a
1	1	1	R
a	a	R	a

,

$\cdot_R$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	$\{1,0\}$

and

$*$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	R

.

(ii) Let $R = \{0, 1, a\}$. Then $(R, +_R, \cdot_R)$ is a commutative general Krasner hyperring with unit, so it is a unitary non-trivial general Krasner R-hypermodule as follows.

$+_R$	0	1	a
0	0	1	a
1	1	R	$\{1,a\}$
a	a	$\{1,a\}$	R

,

$\cdot_R$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	$\{0,1\}$

and

$*$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	R

.

(iii) Let $R = \{0, 1, a\}$. Then $(R, +_R, \cdot_R)$ is a commutative general Krasner hyperring with unit, so it is a unitary non-trivial general Krasner R-hypermodule as follows.

$+_R$	0	1	a
0	0	1	a
1	1	R	$\{1,a\}$
a	a	$\{1,a\}$	R

,

$\cdot_R$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	$\{0,1\}$

and

$*$	0	1	a
0	0	0	0
1	0	$\{0,1\}$	$\{0,a\}$
a	0	$\{0,a\}$	$\{0,1\}$

.

(iv) Let $R = \{0, 1, a, b\}$. Then $(R, +_R, \cdot_R)$ is a commutative general Krasner hyperring with unit, $(R, +_R)$ is a canonical hypergroup and so it is a unitary non-trivial general Krasner R-hypermodule as follows.

$+_R$	0	1	a	b
0	0	1	a	b
1	1	R	$\{1,a\}$	$\{1,b\}$
a	a	$\{1,a\}$	R	$\{a,b\}$
b	b	$\{b,1\}$	$\{a,b\}$	R

,

$\cdot_R$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	R
b	0	b	R	b

and

$*$	0	1	a	b
0	0	0	0	0
1	0	$\{0,1\}$	$\{0,a\}$	$\{0,b\}$
a	0	$\{0,a\}$	$\{a,b\}$	R
b	0	$\{0,b\}$	R	$\{a,b\}$

.

(v) Let $R = \{0, 1, a, b\}$. Then $(R, +_R, \cdot_R)$ is a commutative general Krasner hyperring with unit, $(R, +)$ is a canonical hypergroup and so it is a unitary non-trivial general Krasner R-hypermodule as follows.

$+_R$	0	1	a	b
0	0	1	a	b
1	1	R	$\{1,b\}$	$\{1,a\}$
a	a	$\{1,b\}$	$\{0,a\}$	1
b	b	$\{1,a\}$	1	$\{0,b\}$

,

$\cdot_R$	0	1	a	b
0	0	0	0	0
1	0	1	$\{1,a\}$	$\{1,b\}$
a	0	$\{1,a\}$	a	R
b	0	$\{1,b\}$	R	b

and

$*$	0	1	a	b
0	0	0	0	0
1	0	$\{0,1\}$	$R\setminus\{0\}$	$R\setminus\{0\}$
a	0	$R\setminus\{0\}$	R	R
b	0	$R\setminus\{0\}$	R	R

.

In what follows, we construct non-trivial general Krasner R-hypermodule on any given non-empty set.

Theorem 4.10. *Suppose R is a nonempty set and $4 \le |R| \le n$. Then there exist hyperoperations "$+_R$", "$\cdot_R$", on R and $0_R \in R$ and a left external multiplication $* : R \times R \longrightarrow \mathcal{P}^*(R)$ such that $(R, +_R, *)$ is a general Krasner R-hypermodule.*

Proof. Applying Theorem 3.4, there exist hyperoperations "$+_R$" and "$\cdot_R$", on R and $0_R \in R$ such that $(R, +_R, 0_R, \cdot_R)$ is a general Krasner hyperring. Now by Theorem 4.7, there exists a left external multiplication $* : R \times R \to \mathcal{P}^*(R)$ by $r * a = r \cdot_R a$. One can see that $(R, +_R, *)$ is a general Krasner R-hypermodule. □

In the following theorem, using the concept of class of ring-modules, construct a class of arbitrary general Krasner R-hypermodules and for any ring-module (R, A), the associated non-trivial general Krasner hypermodule, will denote by general Krasner $R^{\uparrow}$-hypermodule.

Theorem 4.11. *Assume $(R, +_R, \cdot_R)$ is a ring and $(A, +'_A, \cdot'_A)$ is an R-module. Then there exist a left external multiplication*

$$* : R \times A \longrightarrow \mathcal{P}^*(A)$$

such that $(A, +'_A, *)$ is a general Krasner $R^{\uparrow}$-hypermodule.

Proof. Since every ring is a general Krasner hyperring and every abelian group is a canonical hypergroup, for all $r \in R$ and $a \in A$, define a map $* : R \times A \to \mathcal{P}^*(A)$ by $r * a = \{r \cdot'_A a, 0_A\}$. One can see that $(A, +'_A, *)$ is a general Krasner R-hypermodule. □

"$\cdot'_R$" on R and a left external multiplication $* : R \times A \longrightarrow \mathcal{P}^*(R)$ such that $(A, +', *)$ is a non-trivial general Krasner $R^{\uparrow}$-hypermodule.

Example 4.12. Consider the ring $(\mathbb{Z}, +, \cdot)$ and the Klein four-group $(A = \{e, a, b, c\}, +_A)$.

$*$	e	a	b	c
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-2	e	e	e	e
-1	e	$\{e, a\}$	$\{e, b\}$	$\{e, c\}$
0	e	e	e	e
1	e	$\{e, a\}$	$\{e, b\}$	$\{e, c\}$
2	e	e	e	e
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Then by Theorem 4.11, $(A, +_A, *)$ is a non-trivial general Krasner $\mathbb{Z}_5^{\uparrow}$-hypermodule.

Theorem 4.13. [17] *Suppose $(A, +_A)$ is a canonical hypergroup and B is an arbitrary canonical subhypergroup of A $(B \leq A)$. Then $(A/B, +_{A/B})$ is a canonical hypergroup, where $A/B = \{a +_A B \mid a \in A\}$ and $(a +_A B) +_{A/B} (a' +_A B) = \{t +_A B \mid t \in a +_A a'\}$.*

Let $(A, +_A, *)$ be a general Krasner R-hypermodule and $B \leq A$. Define $* : R \times A/B \longrightarrow \mathcal{P}^*(A/B)$ by $r * (a +_A B) = (r * a) +_A B$.

Theorem 4.14. *Assume $(A, +_A, *)$ is a general Krasner R-hypermodule and $B \leq A$. Then $(A/B, +_{A/B}, *)$ is a general Krasner R-hypermodule.*

Proof. It is trivial. □

Let $(A, +, *)$ and $(B, +, *)$ be general Krasner R-hypermodules and $f : A \longrightarrow B$ be a strong R-homomorphism. Define

$$Ker(f) := \{x \in A \mid f(x) = 0_B\}$$

and $Im(f) := \{y \in B \mid \exists\ x \in A : \ \ y = f(x)\}$. A strong R-homomorphism $f : A \longrightarrow B$ is called a strong R-monomorphism, if f is a one to one map, a strong R-epimorphism, if f is an onto map and a strong R-isomorphism, if f is a bijective map. Let $S \leq A$ and $x, y \in A$. Define $x \overset{S}{\sim} y \Leftrightarrow (x +_A (-y)) \cap S \neq \emptyset$.

Lemma 4.15. *Assume $f : (A, +_A, *) \to (B, +_B, *)$ is a strong homomorphism and $x, y \in A$. Then*

(i) *$f(0_A) = 0_B$ and for all $a \in A$, $f(-a) = -f(a)$;*
(ii) *$Im(f) \leq B$ and $Ker(f)$ is a canonical subhypergroup of A;*
(iii) *$f(x) = f(y)$ if and only if $x \overset{Ker(f)}{\sim} y$.*
(iv) *f is a strong R-monomorphism if and only if $Ker(f) = \{0_A\}$.*

(v) *if* $(B, +_B, *)$ *is a trivial general Krasner* R*-hypermodule, then* $Ker(f) \leq A$*;*

Proof. (i) Let $a \in A$. Then $f(0_A) = f(0_R * a) = 0_R * f(a) = 0_B$. In addition, $0_B \in f(0_A) \in f(a +_A (-a)) = f(a) +_B f(-a)$. Hence $f(-a) = -f(a)$.

(ii) Let $a \in Ker(f)$. Since $0_A \in a +_A (-a)$, we get that $0_B = f(0_A) \in f(a +_A (-a)) = f(a) +_B f(-a) = 0_B +_B f(-a) = f(-a)$, so for all $a \in Ker(f)$ we have $-a \in Ker(f)$. It follows that $Ker(f)$ is a canonical subhypergroup of A. Clearly $Im(f) \leq B$.

(iii) Let $x, y \in A$. Then we have $f(x) = f(y)$ if and only if $0_B \in f(x) +_B (-f(y))$ if and only if $0_B \in f(x +_A (-y))$ if and only if $\exists\, t \in x +_A (-y)$ such that $0_B = f(t)$ if and only if

$$(x +_A (-y)) \cap Ker(f) \neq \emptyset.$$

(iv) It is clear by (ii), (iii).

(v) Immediate by Theorem 4.2 (vi). □

Theorem 4.16. (*First Strong* R*-isomorphism Theorem*) *Let*

$$f : (A, +_A, *) \to (B, +_B, *)$$

be a strong homomorphism. If $(B, +_B, *)$ *is a trivial general Krasner* R*-hypermodule, then* $A/Ker(f) \cong Img(f)$*.*

Proof. Define $\varphi : A/Ker(f) \to Im(f)$ by $\varphi(a +_A Ker(f)) = f(a)$. Clearly φ is a strong R-isomorphism and so $A/Ker(f) \cong Img(f)$. □

Theorem 4.17. (*Second Strong* R*-isomorphism Theorem*) *Let* $(A, +_A, *)$ *be a general Krasner* R*-hypermodule,* $B_1 \leq A$ *and* $B_2 \leq A$*, where* $(B_1 + B_2)/B_2$ *is trivial. Then* $(B_1 + B_2)/B_2 \cong B_1/(B_1 \cap B_2)$*.*

Proof. Define $\varphi : B_1 \to ((B_1 + B_2)/B_2)$, by $\varphi(b_1) = b_1 + B_2$. Clearly φ is a strong R-homomorphism and so $(B_1 + B_2)/B_2 \cong B_1/(B_1 \cap B_2)$. □

Theorem 4.18. (*Third Strong* R*-isomorphism Theorem*) *Let* $(A, +_A, *)$ *be a general Krasner* R*-hypermodule,* $B_1 \leq A$ *and* $B_2 \leq A$*, where* $B_1 \subseteq B_2$ *and* A/B_2 *is trivial. Then* $(A/B_1)/(B_2/B_1) \cong (A/B_2)$*.*

Proof. Define $\varphi : (A/B_1) \to (A/B_2)$, by $\varphi(a +_A B_1) = a +_A B_2$. Clearly φ is a strong R-homomorphism and so $(A/B_1)/(B_2/B_1) \cong (A/B_2)$. □

5. Homological general Krasner R-hypermodules

In this section, we define the category of general Krasner R-hypermod ules and investigate some of its properties. In addition, we prove that the class of all inclusion R-homomorphisms, under some of conditions are general Krasner R-hypermodules. Finally, we introduce two functors on the class of all inclusion R-homomorphisms.

5.1. **Hom-functor.** The category whose objects are all general Krasner R-hypermodules and whose morphisms are all R-homomorphisms is denoted by ${}_{\mathbf{R}}\mathbf{GKH}\mathrm{mod}$. The class of all strong R-homomorphisms from $(A, +_A, *)$ into $(B, +_B, *)$ is denoted by ${}_{\mathbf{R}}\mathbf{h}\mathrm{om}^s(A, B)$, the class of all weak R-homomorphisms from A into B is denoted by ${}_{\mathbf{R}}\mathbf{h}\mathrm{om}^w(A, B)$ and the class of all inclusion R-homomorphisms from A into B is denoted by ${}_{\mathbf{R}}\mathbf{h}\mathrm{om}(A, B)$. For all $f, g \in {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(A, B)$ and $a \in A$, define "$+$" on ${}_{\mathbf{R}}\mathbf{h}\mathrm{om}(A, B)$ by

$$f + g = \{h \in {}_{\mathbf{R}}\mathbf{h}om(A, B) \mid h(a) \in f(a) +_B g(a)\},$$

$0_{A,B} : A \to B$ by $0_{A,B}(a) = 0_B$ and $1_A : A \to A$ by $1_A(a) = a$.

From now on, $(R, +_R, \cdot_R)$ is a general Krasner hyperring, unless otherwise specified. So we have the following results.

Theorem 5.1. *Assume $(A, +_A, *)$ and $(B, +_B, *)$ are two general Krasner R-hypermodules. Then*

- (i) *$0_{A,B} \in {}_{\mathbf{R}}\mathbf{h}om(A, B)$;*
- (ii) *if $(B, +_B, *)$ is a trivial general Krasner R-hypermodule, then $0_{A,B} \in {}_{\mathbf{R}}\mathbf{h}om^s(A, B)$;*
- (iii) *for all $f \in {}_{\mathbf{R}}\mathbf{h}om(A, B)$, we have $-f \in {}_{\mathbf{R}}\mathbf{h}om^w(A, B)$; $-f \in {}_{\mathbf{R}}\mathbf{h}om(A, B)$;*
- (iv) *if $(A, +_A, *)$ is a trivial general Krasner R-hypermodule, then for all $f \in {}_{\mathbf{R}}\mathbf{h}om(A, B)$, we have $-f \in {}_{\mathbf{R}}\mathbf{h}om(A, B)$;*
- (v) *for all $f \in {}_{\mathbf{R}}\mathbf{h}om(A, B)$, we have $0_B \in f(0_A)$.*

Proof. (i) Let $a, a' \in A$. Then

$$\begin{aligned} 0_{A,B}(a +_A a') &= \bigcup_{t \in a +_A a'} 0_{A,B}(t) = \bigcup\{0_B\} = \{0_B\} = 0_B +_B 0_B \\ &= 0_{A,B}(a) +_B 0_{A,B}(a'). \end{aligned}$$

Let $r \in R$. Since $0_B \in r * 0_B$, we get that $0_{A,B}(r * a) \subseteq r * 0_{A,B}(a)$, so $0_{A,B} \in {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(A, B)$.

(ii) It is obtained from part (i).

(iii) Let $x, y \in R$. Then by Lemma 4.15,

$$\begin{aligned} (-f)(x +_A y) &= f(-(x +_A y)) \\ &= f(-x - y) \\ &\subseteq f(-x) +_B f(-y) \\ &= (-f)(x) +_A (-f)(y). \end{aligned}$$

Let $r \in R$ and $a \in A$. By Theorem 4.2, $-(r * a) \cap (r * (-a)) \neq \emptyset$, so $f(-(r*a)) \cap f(r*(-a)) \neq \emptyset$. It concludes that $f(-(r*a)) \cap r*f(-a) \neq \emptyset$ and so $(-f)(r * a)) \cap (r * (-f)(a)) \neq \emptyset$ and so $-f \in {}_{\mathbf{R}}\mathbf{h}\mathrm{om}^w(A, B)$.

(iv) It is obtained from part (iii).

(v) It is clear by Lemma 4.15. □

Corollary 5.2. *Assume $(A, +_A, *)$ and $(B, +_B, *)$ are two general Krasner R-hypermodules. If $(A, +_A, *)$ is a trivial general Krasner R-hyper module, then $({}_{\mathbf{R}}\mathbf{h}om(A, B), +)$ is a canonical hypergroup.*

From now on, we consider R to be a commutative general Krasner hyperring, unless otherwise specified.

Theorem 5.3. *Assume $(A, +_A, *)$ and $(B, +_B, *)$ are associative and trivial general Krasner R-hypermodules and $r \in R$. Then there exists a left external multiplication $* : R \times {}_{\mathbf{R}}\mathbf{h}om(A, B) \longrightarrow \mathcal{P}^*({}_{\mathbf{R}}\mathbf{h}om(A, B))$ such that $({}_{\mathbf{R}}\mathbf{h}om(A, B), +, *)$ is a general Krasner R-hypermodule.*

Proof. For all $r \in R$ and all $a \in A$, define

$$* : R \times {}_{\mathbf{R}}\mathbf{h}\text{om}(A, B) \longrightarrow \mathcal{P}^*({}_{\mathbf{R}}\mathbf{h}\text{om}(A, B))$$

by $r * f = f_r$, such that for all $a \in A$, $f_r(a) = r * f(a)$. Clearly for all $r \in R, f_r \in {}_{\mathbf{R}}\mathbf{h}\text{om}(A, B)$. By Corollary 5.2, $({}_{\mathbf{R}}\mathbf{h}\text{om}(A, B), +)$ is a canonical hypergroup. We claim that $({}_{\mathbf{R}}\mathbf{h}\text{om}(A, B), +)$ is a general Krasner R-hypermodule.

$$\begin{aligned}(1)(f+g)_r(a) &= (r*(f+g))(a) \subseteq r*(f(a)+g(a))\\ &\subseteq r*f(a)+r*g(a) = f_r(a)+g_r(a).\end{aligned}$$

$$(2)(r+s)(a) = (r+s)*f(a) \subseteq (r*f(a)+s*f(a)) = f_r(a)+f_s(a).$$

$$(3)\ f_{r.s}(a) = (r.s)*f(a) \subseteq r*(s*f(a)) = r*f_s(a).$$

$$(4)\ f_{0_R}(a) = 0_R * f(a) = 0_B.$$

Thus $({}_{\mathbf{R}}\mathbf{h}\text{om}(A, B), +, *)$ is a general Krasner R-hypermodule. □

From now on, we consider general Krasner R-hypermodules A, B, B' that satisfy the conditions in Theorem 5.3, unless otherwise specified. Let $g \in {}_{\mathbf{R}}\mathbf{h}\text{om}(B, B')$. Then it can be extend to a map

$$\begin{aligned}{}_{\mathbf{R}}\mathbf{h}om(A, g) :{}_{\mathbf{R}}\mathbf{h}om(A, B) &\longrightarrow {}_{\mathbf{R}}\mathbf{h}om(A, B'),\\ (f : A \longrightarrow B) &\longmapsto (g \circ f : A \longrightarrow B').\end{aligned}$$

Theorem 5.4. *Suppose A is a general Krasner R-hypermodule. Then ${}_{\mathbf{R}}\mathbf{h}om(A, -) : {}_{\mathbf{R}}\mathbf{GKH}mod \longrightarrow {}_{\mathbf{R}}\mathbf{GKH}mod$ is a covariant functor.*

Proof. Let $A, B, C, D \in {}_{\mathbf{R}}\mathbf{GKH}mod$ and $f \in {}_{\mathbf{R}}\mathbf{h}\text{om}(A, B)$. Then by Theorem 5.3, ${}_{\mathbf{R}}\mathbf{h}om(A, B) \in {}_{\mathbf{R}}\mathbf{GKH}mod$. In addition, for all $r \in R,\ f, f' \in {}_{\mathbf{R}}\mathbf{h}om(A, B)$ and $g \in {}_{\mathbf{R}}\mathbf{h}om(B, B')$, one can see that ${}_{\mathbf{R}}\mathbf{h}om(A, g)(f + f') \subseteq {}_{\mathbf{R}}\mathbf{h}om(A, g)(f) + {}_{\mathbf{R}}\mathbf{h}om(A, g)(f')$ and

$$_{\mathbf{R}}\mathbf{h}om(A,g)(r*f)\subseteq r*{}_{\mathbf{R}}\mathbf{h}om(A,g)(f),$$

so $_{\mathbf{R}}\mathbf{h}om(A,-):{}_{\mathbf{R}}\mathbf{GKH}mod\longrightarrow{}_{\mathbf{R}}\mathbf{GKH}mod$ is well defined. Moreover, $_{\mathbf{R}}\mathbf{h}om(A,1_B)(f)=1_B\circ f=f$. Now, if $h\in{}_{\mathbf{R}}\mathbf{h}\mathrm{om}(C,D)$, $g\in{}_{\mathbf{R}}\mathbf{h}\mathrm{om}(B,C)$, then

$$\begin{aligned}_{\mathbf{R}}\mathbf{h}om(A,h\circ g)(f)&=(h\circ g)\circ f\\&=h\circ(g\circ f)\\&=\big({}_{\mathbf{R}}\mathbf{h}om(A,h)\circ{}_{\mathbf{R}}\mathbf{h}om(A,g)\big)(f).\end{aligned}$$

It follows that $_{\mathbf{R}}\mathbf{h}om(A,-):{}_{\mathbf{R}}\mathbf{GKH}mod\longrightarrow{}_{\mathbf{R}}\mathbf{GKH}mod$ is a covariant functor. □

Theorem 5.5. *Assume A is a general Krasner R-hypermodule. Then $_{\mathbf{R}}\mathbf{h}om(-,B):{}_{\mathbf{R}}\mathbf{GKH}mod\longrightarrow{}_{\mathbf{R}}\mathbf{GKH}mod$ is a contravariant functor.*

Proof. It is similar to Theorem 5.4. □

5.2. **Category of general Krasner R-hypermodules.**
In this subsection, we show that $_{\mathbf{R}}\mathbf{GKH}$mod is an abelian category.

Definition 5.6. A complex (abbreviating chain complex) in $_{\mathbf{R}}\mathbf{GKH}mod$ is a sequence of general Krasner R-hypermodules and strong R-homomor phisms

$$(\mathbf{C},d):\ldots\longrightarrow C_{i+1}\xrightarrow{d_{i+1}}C_i\xrightarrow{d_i}C_{i-1}\xrightarrow{d_{i-1}}C_{i-1}\longrightarrow\ldots$$

if for every $i\in\mathbb{N}$, $Im(d_{i+1})\subseteq Ker(d_i)$. The complex $(\mathbf{C},d)$ is exact if for every $i\in\mathbb{N}$, $Im(d_{i+1})=Ker(d_i)$. By Lemma 4.15, for all n, we consider C_n as a trivial general Krasner R-hypermodule and d_n as a strong R-homomorphism. If $\mathcal{C}$ is an abelian category, then the category of all complexes in $\mathcal{C}$ is denoted by $Comp(\mathcal{C})$.

In Corollary 5.21, we will show that $_{\mathbf{R}}\mathbf{GKH}$mod is an abelian category.

Definition 5.7. Suppose $(\mathbf{C},d)$ is a complex in the $Comp({}_{\mathbf{R}}\mathbf{GKH}mod)$ and $n\in\mathbb{Z}$. Then its n-th homology of general Krasner R-hypermodule is $H_n(\mathbf{C})=Ker(d_n)/Im(d_{n+1})$. It is clear that a complex $\mathbf{C}$ is exact if and only if $H_n(\mathbf{C})=0$.

Theorem 5.8. *Assume $n\in\mathbb{Z}$. Then*

$$H_n:Comp({}_{\mathbf{R}}\mathbf{GKH}mod)\longrightarrow{}_{\mathbf{R}}\mathbf{GKH}mod$$

is an inclusion additive functor.

Proof. Let $n\in\mathbb{Z}$, $f=(f_n):(\mathbf{C},d)\to(\mathbf{C}',d')$ be a chain map, define by $H_n(f)(x+Im(d_{n+1}))=f_n(x)+Im(d'_{n+1})$. Let $x,y\in C_n$ and $x+Im(d_{n+1})=y+Im(d_{n+1})$. Then by the following commutative diagram, there exists $c\in C_{n+1}$ such that

$f_n(x) \in f_n(y)+f_n(d_{n+1})(c) \subseteq f_n(y)+d'_{n+1}(f_{n+1}(c)) \subseteq f_n(y)+Im(d'_{n+1})$. In a similar way, we have $f_n(y) + Im(d'_{n+1}) \subseteq f_n(x) + Im(d_{n+1})$ so $H_n(f)$ is well defined. It is clear that $H_n(f)$ is an inclusion additive functor.

$$\begin{array}{ccccc} C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} \\ \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ C'_{n+1} & \xrightarrow{d'_{n+1}} & C'_n & \xrightarrow{d'_n} & C'_{n-1} \end{array}$$

□

Theorem 5.9. *Assume* $(A, +_A, *)$ *and* $(B, +_B, *)$ *are general Krasner* R*-hypermodules and* $f : A \longrightarrow B$ *is a strong* R*-homomorphism.*

(*i*) *A sequence* $0 \longrightarrow A \xrightarrow{f} B$ *is exact if and only if* f *is a strong* R*-monomorphism.*

(*ii*) *A sequence* $B \xrightarrow{g} C \longrightarrow 0$ *is exact if and only if* g *is a strong* R*-epimorphism.*

(*iii*) *A sequence* $0 \longrightarrow A \xrightarrow{h} B \longrightarrow 0$ *is exact if and only if* h *is a strong* R*-isomorphism.*

Proof. It is obvious by Lemma 4.15. □

Definition 5.10. Assume $(A, +_A, *)$, $(B, +_B, *)$ and $(C, +_C, *)$ are general Krasner R-hypermodules. A short exact sequence of general Krasner R-hypermodules and strong R-homomorphisms is an exact sequence of the form $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$. We also call this short exact sequence an extension of A by C.

Example 5.11. Let $(A, +_A, *)$ be a general Krasner R-hypermodule and $S \leq A$. Then a sequence $0 \longrightarrow S \xrightarrow{i} A \xrightarrow{\pi} A/S \longrightarrow 0$ is a short exact sequence, where i is the inclusion R-homomorphism and π is the canonical R-epimorphism.

Theorem 5.12. *Assume* $(A, +_A, *)$ *and* $(B, +_B, *)$ *are general Krasner* R*-hypermodules and* $f : A \longrightarrow B$ *is a strong* R*-homomorphism.*

(*i*) *If* $(C, +_C, *)$ *is a trivial general Krasner* R*-hypermodule and a sequence* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *is a short exact sequence, then* $A \cong Im(f)$ *and* $B/Im(f) \cong C$.

(*ii*) *If* $T \subseteq S \subseteq A$ *is a tower of general Krasner* R*-subhypermodules, then there is an exact sequence*

$$0 \longrightarrow S/T \xrightarrow{f} A/T \xrightarrow{g} A/S \longrightarrow 0.$$

Proof. (i) Define $\varphi : A \to Im(f)$ by $\varphi(x) = f(x)$. Clearly φ is a strong R-isomorphism, so $A \cong Im(f)$. In addition, by Theorem 4.16, $B/Im(f) \cong C$.

(ii) It is clear by Theorem 4.18. □

Theorem 5.13. *Assume A, B, B_1 and B_2 are general Krasner R-hypermodules, $(B, +_B, *)$ is a trivial general Krasner R-hypermodule and φ and ψ are strong R-homomorphisms. Then the sequence $0 \longrightarrow B_1 \xrightarrow{\varphi} B \xrightarrow{\psi} B_2$ is exact if and only if for any general Krasner R-hypermodule A, the sequence*

$$0 \longrightarrow {}_{\mathbf{R}}\mathbf{h}om(A, B_1) \xrightarrow{\varphi^*} {}_{\mathbf{R}}\mathbf{h}om(A, B) \xrightarrow{\psi^*} {}_{\mathbf{R}}\mathbf{h}om(A, B_2)$$

is exact, where $\varphi^* = {}_{\mathbf{R}}\mathbf{h}om(A, \varphi)$ and $\psi^* = {}_{\mathbf{R}}\mathbf{h}om(A, \psi)$.

Proof. Let $f \in {}_{\mathbf{R}}\mathbf{h}\text{om}(A, B_1)$ and $\varphi^*(f) = 0_{A,B}$. Thus $\varphi \circ f = 0_{A,B}$, since φ is a strong R-monomorphism, we get that $f = 0_{A,B}$ and so φ^* is a strong R-monomorphism. Let $\beta \in Im(\varphi^*)$. Then there exists $\alpha \in {}_{\mathbf{R}}\mathbf{h}\text{om}(A, B_1)$ in such a way that $\varphi \circ \alpha = \varphi^*(\alpha) = \beta$. Since $\psi \circ \varphi = 0_{B_2}$, we get that $0_{A,B_2} = (\psi \circ \varphi) \circ \alpha = \psi \circ (\varphi \circ \alpha) = \psi \circ \beta = \psi^*(\beta)$ and so $Im(\varphi^*) \subseteq Ker(\psi^*)$.

Conversely, if $\beta \in Ker(\psi^*)$, then $0_{A,B_2} = \psi^*(\beta) = \psi \circ \beta$ and so $Im(\beta) \subseteq Ker(\psi) = Im(\varphi)$. Because $\varphi : B_1 \to Im(\varphi)$ is an R-isomorphism, we get $\varphi^{-1} \circ \beta = \alpha \in {}_{\mathbf{R}}\mathbf{h}\text{om}(A, B_1)$ and so $\varphi^*(\alpha) = \beta$. It follows that $Ker(\psi^*) \subseteq Im(\varphi^*)$. Consider $A = Ker(\varphi)$ so we have the sequence

$$0 \longrightarrow {}_{\mathbf{R}}\mathbf{h}\text{om}(Ker(\varphi), B_1) \xrightarrow{\varphi^*} {}_{\mathbf{R}}\mathbf{h}\text{om}(Ker(\varphi), B) \xrightarrow{\psi^*} {}_{\mathbf{R}}\mathbf{h}\text{om}(Ker(\varphi), B_2)$$

is exact, where $\varphi^* = {}_{\mathbf{R}}\mathbf{h}\text{om}(Ker(\varphi), \varphi)$ and $\psi^* = {}_{\mathbf{R}}\mathbf{h}\text{om}(Ker(\varphi), \psi)$. Now, for all $x \in Ker(\varphi)$, we have

$$\varphi^*(i)(x) = (\varphi \circ i)(x) = \varphi(i(x)) = \varphi(x) = 0.$$

It concludes that $\varphi^*(i) = 0$ and so $i \in Ker(\varphi^*)$. Since φ^* is an strong R-monomorphism map, we get that $x = 0$ and so $Ker(\varphi) = \{0\}$. If consider $B_1 = A$, then we get that the sequence

$$0 \longrightarrow {}_{\mathbf{R}}\mathbf{h}\text{om}(B_1, B_1) \xrightarrow{\varphi^*} {}_{\mathbf{R}}\mathbf{h}\text{om}(B_1, B) \xrightarrow{\psi^*} {}_{\mathbf{R}}\mathbf{h}\text{om}(B_1, B_2)$$

is exact, where $\varphi^* = {}_{\mathbf{R}}\mathbf{h}\text{om}(B_1, \varphi)$ and $\psi^* = {}_{\mathbf{R}}\mathbf{h}\text{om}(B_1, \psi)$. Let $y \in Im(\varphi)$, Then there exists $x \in B_1$ such that $y = \varphi(x)$. Because $\psi^* \circ \varphi^* = 0$, we imply that $(\psi^* \circ \varphi^*)(id_{B_1}) = 0$ and so

$$0 = (\psi^* \circ \varphi^*)(id_{B_1}) = \psi^* \circ (\varphi \circ id_{B_1}) = \psi \circ (\varphi \circ id_{B_1}).$$

Thus $\psi(y) = \psi(\varphi(x)) = \psi(\varphi(id_{B_1}(x))) = 0$ and so $Im(\varphi) \subseteq Ker(\psi)$. Consider $A = Ker(\psi)$, hence the sequence

$$0 \longrightarrow {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(Ker(\psi), B_1) \xrightarrow{\varphi^*} {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(Ker(\psi), B) \xrightarrow{\psi^*} {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(Ker(\psi), B_2)$$

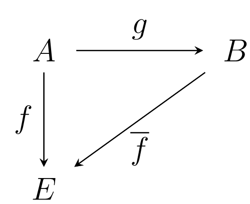
is exact, where $\varphi^* = {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(Ker(\psi), \varphi)$ and $\psi^* = {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(Ker(\psi), \psi)$. Since $i \in {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(Ker(\psi), B)$, where $i : Ker(\psi) \to B$, we get that $\psi^*(i) = 0$ and so $i \in Ker(\psi^*)$. It follows that $i \in Im(\varphi^*)$ and there exists $\alpha \in {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(Ker(\psi), B)$ in such a way that $i = \varphi^*(\alpha) = \varphi \circ \alpha$. Suppose that $y \in Ker\psi$, then $y = i(y) = \varphi(\alpha(y))$ and so $y \in Im(\varphi)$. Thus $Im(\varphi) = Ker(\psi)$. □

Corollary 5.14. *Assume A, B, B_1 and B_2 are general Krasner R-hypermodules, $(B, +_B, *)$ is a trivial general Krasner R-hypermodule and φ, ψ are strong R-homomorphisms. Then the sequence*

$$B_1 \xrightarrow{\varphi} B \xrightarrow{\psi} B_2 \longrightarrow 0$$

is exact if and only if for any general Krasner R-hypermodule A, the sequence $0 \longrightarrow {}_{\mathbf{R}}\mathbf{h}om(B_2, A) \xrightarrow{\psi^*} {}_{\mathbf{R}}\mathbf{h}om(B, A) \xrightarrow{\varphi^*} {}_{\mathbf{R}}\mathbf{h}om(B_1, A)$ is exact, where $\varphi^* = {}_{\mathbf{R}}\mathbf{h}om(\varphi, A)$ and $\psi^* = {}_{\mathbf{R}}\mathbf{h}om(\psi, A)$.

Definition 5.15. Assume $(A, +_A, *)$, $(B, +_B, *)$ and $(E, +_E, *)$ are general Krasner R-hypermodules. Then $(E, +_E, *)$ is called a normal injective general Krasner R-hypermodule if for strong R-monomorphism $g \in {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(A, B)$ and $f \in {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(A, E)$, there exists $\overline{f} \in {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(B, E)$ such that $\overline{f} \circ g = f$ or the following diagram is commutative.



Theorem 5.16. *Assume $(E, +_E, *)$ is a general Krasner R-hypermodule. Then $(E, +_E, *)$ is a normal injective if and only if ${}_{\mathbf{R}}\mathbf{h}om(-, E)$ is an exact functor.*

Proof. Let $0 \longrightarrow B_1 \xrightarrow{\varphi} B \xrightarrow{\psi} B_2 \longrightarrow 0$ be an exact sequence and $(B, +_B, *)$ be a trivial general Krasner R-hypermodule. By Corollary 5.14, the sequence

$$0 \longrightarrow {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(B_2, E) \xrightarrow{\psi^*} {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(B, E) \xrightarrow{\varphi^*} {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(B_1, E)$$

is exact. Let $f : B_1 \to E$. Thus there exists $\overline{f} : B \to E$ in such a way that $\overline{f}\varphi = f$. Hence for all $f \in {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(B_1, E)$ there exists

$\overline{f} \in {}_{\mathbf{R}}\mathbf{h}\mathrm{om}(B, E)$ such that $\varphi^*(\overline{f}) = f$ so φ^* is a strong R-epimorphism and so ${}_{\mathbf{R}}\mathbf{h}\mathrm{om}(-, E)$ is an exact functor. Conversely, is clear. □

Given a collection $\{A_i\}_{i\in I}$ of general Krasner R-hypermodules, the direct product $\prod\limits_{i\in I} A_i$ is just the product of the underlying sets A_i with general Krasner R-hypermodule hyperstructure given by componentwise hyperaddition and left external multiplication, i.e., for all $(a_i)_{i\in I}, (a'_i)_{i\in I} \in \prod\limits_{i\in I} A_i$ and $r \in R$,

$$(a_i)_{i\in I} +' (a'_i)_{i\in I} = \{(a''_i)_{i\in I} \mid a''_i \in a_i +_{A_i} a'_i, i \in I\}$$

and $r *' (a_i)_{i\in I} = \{(t_i)_{i\in I} \mid t_i \in r * a_i, i \in I\}$. The direct sum $\bigoplus\limits_{i\in I} A_i$ is a R-subhypermodule of the direct product $\prod\limits_{i\in I} A_i$ consisting of elements $(a_i)_{i\in I}$ such that all but a finitely many a_i are zero.
Now by the hyperoperations "$+'$" and "$*'$" in above notation, we have the following results.

Proposition 5.17. *Let $(A, +_R, *)$ be a general Krasner R-hypermodule. Then*

- (i) *If A_1, A_2 are general Krasner R-subhypermodules of A, then $A_1 +_R A_2$ is a general Krasner R-subhypermodule of A.*
- (ii) *If $\{A_j\}_{j\in J}$ is a family of general Krasner R-subhypermodules of A, then $\bigcap\limits_{j\in J} A_j$ is a general Krasner R-subhypermodule of A.*
- (iii) *If $\{A_j\}_{j\in J}$ is a family of general Krasner R-subhypermodules of A, then $(\prod\limits_{j\in I} A_j, +', *')$ is a general Krasner R-subhypermodule of A.*
- (iv) *If $\{A_j\}_{j\in J}$ is a family of general Krasner R-subhypermodules of A, then $(\bigoplus\limits_{j\in I} A_j, +', *')$ is a general Krasner R-subhypermodule of A.*

Proof. It is straightforward by definition. □

The direct product $\prod\limits_{i\in I} A_i$ is equipped with a collection of strong R-homomorphisms $\{\pi_j : \prod\limits_{i\in I} A_i \to A_j\}_{j\in I}$ given by $\pi_j((a_i)_{i\in I}) = a_j$, for all $(a_i)_{i\in I} \in \prod\limits_{i\in I} A_i$. Similarly, the direct sum $\bigoplus\limits_{i\in I} A_i$ is equipped with a collection of strong R-homomorphisms $\{p_j : A_j \to \prod_{i\in I} A_i\}_{j\in I}$ given by $p_j(a_j) = (a_i)_{i\in I}$, for all $j \in I$, where for each $i \neq j, a_i = 0$ and $a_i = 0$, for all $a_j \in A_j$.

Theorem 5.18. ${}_{\mathbf{R}}\mathbf{GKH}$*mod is equipped with products and coproducts.*

Proof. Given a collection $\{A_i\}_{i\in I}$ of general Krasner R-hypermodules. By Proposition 5.17, the direct product $\prod_{i\in I} A_i$ with the collection of strong R-homomorphism $\{\pi_j : \prod_{i\in I} A_i \to A_j\}_{j\in I}$ is a product of $\{A_i\}_{i\in I}$. Similarly, the direct sum $\bigoplus_{i\in I} A_i$ with the collection of strong R-homomor phism$\{p_j : A_j \to \prod_{i\in I} A_i\}_{j\in I}$ is a coproduct of $\{A_i\}_{j\in I}$. □

Theorem 5.19. *The category* ${}_{\mathbf{R}}\mathbf{GKH}$*mod has kernels and cokernels.*

Proof. Let A, A' be general Krasner R-hypermodules, where $(A', +_{A'}, *)$ is trivial and $\varphi : A \to A'$ be a strong R-homomorphism. It can be shown that $Ker(\varphi)$ with the inclusion R-homomorphism

$$i : Ker(\varphi) \to A$$

is the kernel of φ in the categorical sense. Similarly, $A'/Im(\varphi)$ with the natural projection R-homomorphism $\pi : A' \to A'/Im(\varphi)$ mapping $\pi(a) = \bigcup_{t\in a+_{A'}Im(\varphi)} t$ is the cokernel of φ. □

Theorem 5.20. *In* ${}_{\mathbf{R}}\mathbf{GKH}$*mod:*

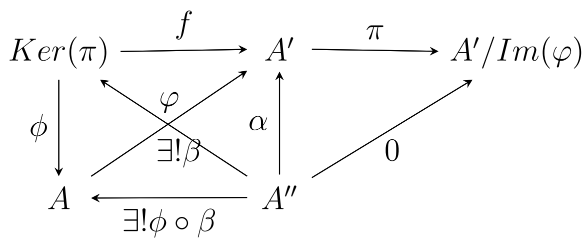
(*i*) *every strong R-monomorphism is the kernel of its cokernel.*
(*ii*) *every strong R-epimorphism is the cokernel of its kernel.*

Proof. (*i*) Let A, A' be general Krasner R-hypermodules, where $(A', +_{A'}, *)$ is trivial and $\varphi : A \to A'$ be a strong R-monomorphism. We know that the cokernel of φ is $A'/Im(\varphi)$ with the strong R-homomorphism $\pi : A' \to A'/Im(\varphi)$. Now, the

$$Ker(\pi) = Im(\varphi) \cong A,$$

together with the inclusion R-homomorphism $i : Ker(\pi) \to A'$. Hence, there exists a strong R-isomorphism $\phi : Ker(\pi) \to A$. If A'' is any general Krasner R-hypermodule with a strong R-homomorphism $\alpha : A'' \to A'$ such that $\pi \circ \alpha = 0$, then by the universal property of the kernel, there exists a unique strong R-homomorphism $\beta : A'' \to Ker(\pi)$

such that $i \circ \beta = \alpha$. Then from $\phi \circ \beta : A'' \to A$ is a strong R-homomorphism and moreover, $\varphi \circ (\phi \circ \beta) = (\varphi \circ \phi) \circ \beta = i \circ \beta = \alpha$. Moreover, if $\gamma : A'' \to A$ is any other strong R-monomorphism such that $\varphi \circ \gamma = \alpha$, then $i \circ (\phi^{-1} \circ \gamma) = (i \circ \phi^{-1}) \circ \gamma = \varphi \circ \gamma = \alpha$. Thus, by the uniqueness of $\beta, \beta = \phi^{-1} \circ \gamma$. Thus, $\gamma = \phi \circ \beta$, and so there exists a unique strong R-monomorphism $\phi \circ \beta : A'' \to A$ such that $\varphi \circ (\phi \circ \beta) = \alpha$. By the universal property of the kernel, A with the strong R-monomorphism $\varphi : A \to A'$ is the kernel of $\pi : A' \to A'/Im(\varphi)$, which is the cokernel of $\varphi : A \to A'$. Thus, every strong R- monomorphism is the kernel of its cokernel.



(ii) It is similar to (i). □

Corollary 5.21. $_{\mathbf{R}}$**GKH***mod is an abelian category.*

5.3. **Normal injective resolution in $_{\mathbf{R}}$GKHmod.** In this subsection, we define the concept of normal injective resolutions and prove that any trivial general Krasner R-hypermodule, where R is commutative, has a normal injective resolution.

Definition 5.22. A normal injective resolution of

$$A \in obj(_{\mathbf{R}}\mathbf{GKH}mod),$$

is an exact sequence

$$\mathbf{E} : 0 \longrightarrow A \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \dots$$

in which for all n, E^n is a normal injective trivial general Krasner R-hypermodule. If $\mathbf{E}$ is an injective resolution of A, then its deleted normal injective resolution is the complex

$$E^A : 0 \longrightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \dots.$$

Indeed, by deleting of A in sequence $\mathbf{E}$, we do not lose any information, for $A \cong Kerd^0$.

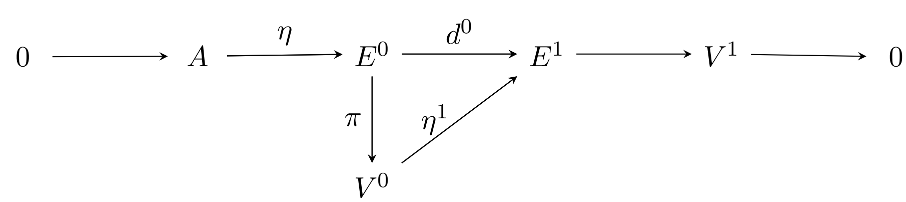
It is easy to see that any abelian group is a canonical hypergroup. So in the following theorem, consider A as trivial canonical hypergroup.

Theorem 5.23. *Assume A is a trivial general Krasner R-hypermodule, where R is commutative. Then A can be embedded as a general Krasner R-subhypermodule of a normal injective general Krasner R-hypermodule.*

Proof. Since $\mathbb{Z}$ is a general Krasner hyperring and A a trivial general Krasner R-hypermodule, by Theorem 5.3, ${}_{\mathbb{Z}}\mathbf{h}\mathrm{om}(R, A)$ is a general Krasner Z-hypermodule. Define $\varphi : A \to {}_{\mathbb{Z}}\mathbf{h}\mathrm{om}(R, A)$ by $\varphi(a) = \varphi_a$, where for $r \in R$, $\varphi_a(r) = r * a$. It is easy to see that φ is a strong R-monomorphism. In addition, there exist an injective abelian group D and a strong $\mathbb{Z}$-monomorphism $i : A \to D$. Left exactness of ${}_{\mathbf{R}}\mathbf{h}\mathrm{om}(R, -)$ gives an injection $i^* : {}_{\mathbb{Z}}\mathbf{h}\mathrm{om}(R, A) \to {}_{\mathbb{Z}}\mathbf{h}\mathrm{om}(R, D)$, and so the composite $i^* \circ \varphi$ is a strong $\mathbb{Z}$-monomorphism. By Theorem 4.7, $i^* \circ \varphi$ is a strong R-monomorphism and the proof is finished. □

Theorem 5.24. *Suppose A is a trivial general Krasner R-hypermodule, where R is commutative. Then A has a normal injective resolution.*

Proof. By Theorem 5.23, A can be embedded as a general Krasner R-subhypermodule of a normal injective general Krasner R-hypermodule. Thus, there are a normal injective general Krasner R-hypermodule E^0, a strong R-monomorphism $\eta : A \to E^0$ and an exact sequence $0 \longrightarrow A \xrightarrow{\eta} E^0 \xrightarrow{\pi} V^0 = coker(\eta) \longrightarrow 0$, where π is the natural R-epimorphism. By continuous of this process, there are an injective module E^1 and an embedding $\eta^1 : V^0 \to E^1$, as follows:



,

where d^0 is the composite $d^0 = \eta^1 \circ \pi$. □

6. Conclusion

The current paper has defined and considered the notion of trivial general Krasner R-hypermodule, and has investigated the categorical properties of general Krasner R-hypermodules. We construct the general Krasner R-hypermodules based on any non-empty set and generalize R-modules to general Krasner R-hypermodules. The strong R-isomorphism theorems on trivial general Krasner R-hypermodules, are proved and so are constructed the quotient of general Krasner

R-hypermodules. We try to define a hyperoperation and a left general Krasner hypermodule on the set of all strong homomorphisms of general Krasner R-hypermodules and construct a new class of general Krasner R-hypermodules. Besides, we show that the category of trivial general Krasner R-hypermodules is an abelian category and so has an injection resolution. In this work, we define some functors on trivial general Krasner R-hypermodules and prove that these functors are exact functors.

We hope that these results are helpful for further studies in module theory. In our future studies, we hope to obtain more results regarding homology on general Krasner R-hypermodules and fuzzy general Krasner R-hypermodules.

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NORMAL INJECTIVE RESOLUTION OF GENERAL KRASNER HYPERMODULES

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تحلیل انژکتیو نرمال از ابرمدول‌های کراسنر عمومی

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در این مقاله مفهوم ابرحلقه‌های کراسنر عمومی را بر اساس ساختار حلقه و ابرمدول‌های کراسنر عمومی چپ را روی ساختار مدول می‌سازیم. ابرمدول‌های کراسنر عمومی چپ بدیهی را معرفی کرده و ثابت می‌کنیم که ابرمدول‌های کراسنر عمومی چپ بدیهی با ابرمدول‌های کراسنر چپ متفاوت هستند. نشان می‌دهیم که برای هر ابرحلقه کراسنر عمومی R و هر دو تا ابرمدول کراسنر عمومی چپ بدیهی A, B، ساختار $_{\mathbf{R}}\mathrm{hom}(A, B)$ یک ابرمدول کراسنر عمومی چپ است و $_{\mathbf{R}}\mathrm{hom}(-, B)$، یک تابعگون هموردای دقیق و $_{\mathbf{R}}\mathrm{hom}(A, -)$ یک تابعگون ناوردای دقیق است. در پایان نشان می‌دهیم که رسته $_{\mathbf{R}}\mathbf{GKH}\mathrm{mod}$ (رسته تمام ابرمدول‌های کراسنر عمومی چپ بدیهی و تمام همریختی‌ها) یک رسته آبلی است و ابرمدول‌های کراسنر عمومی چپ بدیهی یک تحلیل انژکتیو نرمال دارند.

کلمات کلیدی: ابرحلقه کراسنر عمومی، ابرمدول‌های کراسنر عمومی (نرمال انژکتیو)، تحلیل انژکتیو نرمال، رسته آبلی.