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SUMS OF UNITS IN SOME CLASSES OF NEAT RINGS

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Abstract. A ring R is said to be *clean* if every element of R is a sum of an idempotent and a unit. A ring R is a *neat ring* if every nontrivial homomorphic image is clean. In this paper, first, it is proved that every element of some classes of neat rings is *n-tuplet-good* if no factor ring of such rings isomorphic to a field of order less than $n+2$. Also by considering the structure of FGC rings, it can be proved that some clasess of FGC rings are *n-tuplet-good*.

1. Introduction

In 1953–1954, D. Zelinsky [12] proved that every linear transformation of a vector space $V(dimV \neq 1)$ over a division ring F (with three or more elements) is the sum of two automorphisms and investigation of rings generated by their units began. The *unit sum number* of a ring, denoted by, $(usn(R))$ is the least positive integer n such that every element of R can be written as a sum of exactly n units of R. These rings were named *n-good* by Vámos [11]. For additional historical background and to read more about this concept, the reader is referred to articles [1], [4], [2] and [3].
In [6] we are introduced to a concept called *n-tuplet-good*. For a natural numbers $n \geq 1$, an element $a \in R$ is n-tuplet-good if for any set $\{u_1, \ldots, u_n\}$ of central units in R, there exists a unit $u \in R$ such that $a + u_i u \in U(R)$ for each $i \geq 1$. W. K. Nicholson introduced clean rings and modules in his basic paper [8]. According to Nicholson's definition

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[8], a ring R is clean if for each $x \in R$, x can be written as $x = v + f$, where v is a unit element of R, and f is an idempotent element of R. Semi-perfect rings and linear transformations of vector spaces are exampels of clean rings. One of the primary properties of these rings is that every homomorphic image of a clean ring is clean. A ring R is a neat ring if every proper homomorphic image is clean. This definition was introduced by Warren Wm. McGovern in [7]. The ring of integers $\mathbb{Z}$ and any nonlocal PID are examples of neat rings that are not clean. Recall that a ring R is called an *FGC ring* if every finitely generated R-module decomposes into a direct sum of cyclic submodules. W. Barandal classified these rings in [5]. Integers $\mathbb{Z}$, and nonlocal PID's are examples of *FGC* rings. Let M be an R-module; M is a *linearly compact* R-module whenever $\{x_\alpha + M_\alpha\}_{\alpha \in X}$ is a family of cosets of submodules of M such that the intersection of every finite subfamily is nonempty, then $\bigcap x_\alpha + M_\alpha \neq \emptyset$. It is known that a homomorphic image of a linearly compact R-module is linearly compact (see [5]). A ring R is called a *maximal* ring if R is a linearly compact R-module. Artinian rings are maximal. For every nonzero ideal J of R if R/J is a linearly compact R-module, the ring R is said to be almost maximal and these classes of rings are neat. The ring R is a valuation ring if given any elements $x, y \in R$, $x|y$ or $y|x$. Recall that if every finitely generated ideal of a ring is cyclic, such rings were called the *Bézout ring*. Examples of Bézout domains include PIDs and valuation domains. Pm-rings was introduced as commutative rings that each prime ideal is contained in a unique maximal ideal. All the rings in this article have identity element and are commutative. Moreover, $N(R)$ denotes the nilradical of R and $J(R)$ denotes the Jacobson radical of R.

2. Main results

Ahead of discussing the main results, some basic properties are required, which are expressed in the following lemma.

Lemma 1. *Let D be a division ring. Then for any $a \in D$ and every set of n elements $\{v_1, \ldots, v_n\}$ in D, there exists an element $v \in D$ such that $a + v_i v \in D$ iff 1 is n-tuplet-good.*

Proof. For any $a \in D$, if $a = 0$ then $a + v_i v \in D$ for every $v \in D$. If $a \neq 0$, since 1 is n-tuplet-good there exists a $v \in D$ such that $1 + v_i v \in D$, thus $a + v_i a v \in D$. The converse is trivial. □

Corollary 2. *If R is a commutative local ring that has no factor ring isomorphic to a field of order less than $n + 2$, then R is n-tuplet-good.*

Lemma 3. *A ring $R \neq 0$ is local if and only if it is clean and 0 and 1 are the only idempotents in R.*

Proof. See [10, Lemma 14]. □

Theorem 4. *Let R be a neat ring with nonzero radical jacobson, then every element of R is n-tuplet-good if R has no factor isomorphic to a field of order less than $n+2$.*

Proof. Suppose that R has no factor isomorphic to field of order less than $n+2$. We prove that each element of R is a n-tuplet-good. Let a be an arbitrary element of R and u_i central units in R for $1 \leq i \leq n$. For every ideal J of R, consider this property that:

($\star$) There exist $\overline{u} \in U(R/J)$ such that $\overline{a} + \overline{u_i u} \in U(R/J)$ for each $1 \leq i \leq n$. It suffices to show ideal 0 applies to the above property. Suppose not, let $\Omega =$\{K | K is an ideal of R such that $\star$ property faield for it\}. Clearly, Ω is non-empty and it can be easily checked that Ω is inductive. Because if $\{K_i\}$ is a chain of ideals in Ω, let $K = \bigcup_i K_i$, it can be concluded that $K \in \Omega$, Based on Zorn's Lemma, Ω has a maximal element, say, T. If $T = 0$ then, $R/J(R)$ satisfies ($\star$) then there exists $\overline{u}, \overline{v_i} \in U(R/J(R))$ such that $\overline{a} + \overline{u_i u} = \overline{v_i}$ for each $1 \leq i \leq n$. Therefore, ($\star$) is true for R, and this is a contradiction. Also R/T is an indecomposable ring and hence has no non-trivial idempotent. Since R/T is a clean ring and has no non-trivial idempotent, based on Lemma 3, R/T is a local ring. Let $T^{'} = R/T$. Then $T^{'}/J(T^{'})$ is a field. Let $x = a + T$. Since $x + J(T^{'})$ is not n-tuplet-good in $T^{'}/J(T^{'})$; So $T^{'}/J(T^{'})$ has a factor ring isomorphic to a field with less than $n+2$ elements, such that is a contradiction to the assumption. Thus, each element of R is n-tuplet-good. □

Corollary 5. *Let R be a clean ring. R is n-tuplet-good if R has no factor ring isomorphic to a field with less than $n+2$ elements.*

In general, if R is a neat ring and G is a group, then the group ring RG need not to be neat. For example, since $\omega : \mathbb{Z}G \longrightarrow \mathbb{Z}$ is a additive map and $\mathbb{Z}$ is not clean therefore, $\mathbb{Z}G$ is not neat, but in some cases we show that RG is neat. Now by applying the results for group rings, we have the following theorem.

Theorem 6. *If R is a local ring and $G = \{1, g\}$ is a group of order 2, then the group ring RG is a neat ring.*

Proof. Since R is local, $J(R)$ is the maximal ideal of R. If $2 \notin J(R)$, then 2 is invertible. Consider the map $\Psi : RG \to R \times R$ with

$\Psi(a+bg) = (a+b, a-b)$, where $a, b \in R$, and $g \in G$. Ψ is an isomorphism. Since R is a neat ring, RG is a neat ring.

If $2 \in J(R)$, since G is a 2-group and R is a local ring by [9], RG is a local ring. Therefore, RG is a neat ring. □

Corollary 7. *If R is a local ring and $G = \{1, g\}$ is a group of order 2, then group ring RG is n-tuplet-good if has no factor isomorphic to a field with less than $n+2$ elements.*

3. Some clasess of FGC rings

Recall that a ring R is called an FGC ring if every finitely generated module is isomorphic to a direct sum of cyclics. In this section, the main purpose is to show that some classes of FGC rings are n-tuplet-good. By [5, Theorem 9.1], we know that any FGC ring R is a finite direct product of almost maximal valuation rings, almost maximal Bézout domains, and torch rings. The first and second classes are neat, but the third class is never neat. To show that an FGC ring is n-tuplet-good, first it is attempted show that each direct summand is n-tuplet-good.

Theorem 8. [13, Zelinsky] *Let R be a maximal ring, then R is a finite direct product of local rings.*

Based on the Zelinsky theorem, it is obvious that every maximal ring and almost maximal ring is neat. So we have next corollary.

Corollary 9. *A maximal ring is clean. Moreover, an almost maximal ring is neat.*

A domain R is h-local if every nonzero element is contained in only finitely many maximal ideals and every nonzero prime ideal of R is a subset of just one maximal ideal. For example, every local domain is h-local. It is well known that h-local domains are neat. Recall that if every finitely generated ideal of a ring is cyclic, such rings are called *Bézout rings.* PIDs and valuation domains are examples of *Bézout ring.*

Definition 10. *Let R be a nonlocal ring satisfying the following conditions:*
(1) R has a anly one nonzero minimal prime ideal P.
(2) $R/N(R)$ is an h-local domain.
(3) R is a locally almost maximal Bézout ring.
Such rings were called torch rings.

Proposition 11. *Let R be a torch ring which has no factor ring isomorphic to a field F with $\mid F \mid < n+2$. Then R is n-tuplet-good.*

Proof. If R is a torch ring, then R has a unique minimal prime ideal $N(R)$, and $R/N(R)$ is an h-local domain. Put $T := R/N(R)$, therefore $T/J(T)$ is a pm ring and $V(J(T))$ is finite. By Lemma 2.4 of [5], $T/J(T)$ is direct sum of indecomposible modules of the form T/J_i such that $J(R) \leq J_i$ for $i \in I$. We claim that T/J_i is local ring for every $i \in I$. Suppose towards contradiction, by Proposition 2.5 of [5], for each i, and every nontrivial partitions V_1, V_2 of $V(J_i)$, there are two maximal ideals M_1, M_2 such that, there is a prime ideal P belong to them but this is a contradiction. Thus $T/J(T)$ is direct sum of local rings, so by Remark 2, the result is obtained. □

Proposition 12. *Let R be a maximal ring. If R has no factor ring isomorphic to a field F with $\mid F \mid< n+2$, then R is n-tuplet-good.*

Proof. Clearly since R is maximal, hence $R/J(R)$ is a maximal ring, and based on Zelinsky's Theorem, $R/J(R)$ is a finite direct product of local rings. Hence, based on Remark 2, R is n-tuplet-good. □

Corollary 13. [7, Corollary 3.9] *FGC domains are neat.*

Theorem 14. [5, Theorem 9.1] *A ring is an FGC ring if and only if it is a finite direct product of the following types of rings:*
1. Maximal valuation rings.
2. Almost maximal Bézout domains.
3. Torch rings.

Now we have enough information to prove the main results for two classes of FGC rings.

Theorem 15. *Let R be a clean FGC ring. If R has no factor ring isomorphic to a field F with $\mid F \mid< n+2$, then R is n-tuplet-good.*

Proof. Since R is FGC ring by Theorem 14, we can write

$$R = R_1 \times \cdots \times R_n$$

such that none of the R_i can be torch because otherwise for only minimal prime ideal P, R/P is local ring. This contradicts the definition of the torch ring. Therefore, every R_i is either maxmal valuation ring or clean domain, in both cases every R_i is local ring and by Remark 2, R is n-tuplet-good. □

Theorem 16. *Let R be an FGC ring with not J-semisimple homomorphic image, then every element of R is n-tuplet-good if no factor ring of R is isomorphic to a field F with $\mid F \mid< n+2$.*

Proof. Suppose R is an FGC ring. Based on Theorem 14, R is a finite direct product of maximal valuation rings, almost maximal Bézout domains, and torch rings. Based on ([5], Theorem 5.2), every almost maximal Bézout domain is an FGC domain, and therefore; by Corollary 13 is neat. Therefore based on Theorem 4, Proposition 11, and Proposition 12, the result follows. □

In the end, we ask two open problems:

Problem 17. *The integer ring* $\mathbb{Z}$*, and polynomial ring over field* F *are examples of neat ring which are not* n*-tuplet-good for any* n*, can we say that is true for every PID but not a field?*

Problem 18. *For every* $n \geq 2$*, can we contruct a ring with the property that every element is* n*-tuplet-good?*

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مجموع عناصر وارون‌پذیر در برخی از کلاس‌های حلقه‌های آراسته

ندا پویان

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حلقه R خوش‌ترکیب گفته می‌شود، اگر هر عضو آن‌را بتوان به‌صورت مجموع یک عنصر وارون‌پذیر و یک عنصر خودتوان نوشت. حلقه R یک حلقه‌ی آراسته است اگر هر تصویر همریخت غیربدیهی آن خوش‌ترکیب باشد. در این مقاله ابتدا ثابت شده است که هر عنصر از برخی از کلاس‌های حلقه‌های آراسته، n-توپلت-گود است اگر هیچ فاکتوری از چنین حلقه‌هایی، با میدانی از مرتبه ی کمتر از $n+2$ یکریخت نباشد. همچنین با در نظر گرفتن ساختار حلقه‌های FGC ثابت شده است که برخی از کلاس‌های حلقه‌های FGC، n-توپلت-گود هستند.

کلمات کلیدی: عدد جمعی یکالی، n-توپلت-گود، حلقه آراسته، حلقه FGC.