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STRUCTURE OF ZERO-DIVISOR GRAPHS ASSOCIATED TO RING OF INTEGER MODULO n

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Abstract. For a commutative ring R with identity $1 \neq 0$, let $Z^*(R) = Z(R) \setminus \{0\}$ be the set of non-zero zero-divisors of R, where $Z(R)$ is the set of all zero-divisors of R. The zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple graph whose vertex set is $Z^*(R) = Z(R) \setminus \{0\}$ and two vertices of $Z^*(R)$ are adjacent if and only if their product is 0. In this article, we find the structure of the zero-divisor graphs $\Gamma(\mathbb{Z}_n)$, for $n = p^{N_1} q^{N_2} r$, where $2 < p < q < r$ are primes and N_1 and N_2 are positive integers.

1. Introduction

A graph is denoted by $G = G(V(G), E(G))$, where $V(G)$ is the vertex set and $E(G)$ is the edge set of G. Througout we consider simple and finite graphs. The *order* and the *size* of G are the cardinalities of $V(G)$ and $E(G)$, respectively. The *neighborhood* of a vertex v, denoted by $N(v)$, is the set of vertices of G adjacent to v. The degree of v, denoted by d_v, is the cardinality of $N(v)$. A graph G is called *r-regular*, if degree of every vertex is r.

Let R be a commutative ring with non-zero identity $1 \neq 0$. Let $Z^*(R) = Z(R) \setminus \{0\}$ be the set of non-zero zero-divisors of R, where $Z(R)$ is the set of all zero-divisors of R. An element $x \in R$, $x \neq 0$, is known as *zero-divisor* of R if we can find $y \in R$, $y \neq 0$, such that $xy = 0$. Beck [3] introduced the concept of zero-divisor

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graphs of commutative rings and included 0 in the definition. Later Anderson and Livingston [1] modified the definition of zero-divisor graphs by excluding 0 of the ring in the zero-divisor set and defined the edges between two nonzero zero-divisors if and only if their product is zero. Recent work on zero-divisor graphs can be seen in [2, 1, 7] and the references therein. In G, $x \sim y$ denotes that the vertices x and y are adjacent and xy denotes an edge. The complete graph is denoted by K_n and the complete bipartite graph by $K_{a,b}$. Other undefined notations and terminology can be seen in [5, 6].

The authors in [12] obtained the structure of the zero-divisor graphs $\Gamma(\mathbb{Z}_n)$ for $n = p^{N_1}q^{N_2}$, where $p < q$ are primes and N_1, N_2 are positive integers.

The rest of the paper is organized as follows. In Section 2, we mention some preliminaries. In Section 3, we obtain the structure of zero-divisor graphs $\Gamma(\mathbb{Z}_n)$, for $n = p^{N_1}q^{N_2}r$, where $2 < p < q < r$ are primes and N_1 and N_2 are positive integers. Moreover, the different types of spectrum of zero-divisor graphs can be seen in [8, 9, 11, 10].

2. Preliminaries

We begin with the following definition.

Definition 2.1 (Joined union)**.** Let G be a graph of order n having vertex set $\{1, 2, \dots, n\}$ and G_i be disjoint graphs of order n_i $1 \leq i \leq n$. The graph $G[G_1, G_2, \dots, G_n]$ is formed by taking the graphs $G_1, G_2, \dots, G_n$ and joining each vertex of G_i to every vertex of G_j whenever i and j are adjacent in G.

We note that G and $G[G_1, G_2, \dots, G_n]$ are of the same diameter. This graph operation is known by different names in the literature, like G-join, generalized composition, generalized join, joined union and here we follow the latter name.

Let n be a positive integer and let $\tau(n)$ denote the number of positive factors of n. Note that $d|n$ denotes d divides n. The *Euler's totient function*, or *Euler's phi function*, denoted by $\phi(n)$, is the number of positive integers less or equal to n and relatively prime to n. We say that n is in *canonical decomposition* if $n = p_1^{n_1}p_2^{n_2}\dots p_l^{n_l}$, where $l, n_1, n_2, \dots, n_l$ are positive integers and $p_1, p_2, \dots, p_l$ are distinct primes.

The following fundamental observations will be used in the sequel.

Lemma 2.2. *If n is in canonical decomposition $p_1^{n_1}p_2^{n_2}\dots p_r^{n_r}$, then*

$$\tau(n) = (n_1 + 1)(n_2 + 1)\dots(n_r + 1).$$

Theorem 2.3. *The Euler's totient function ϕ satisfies the following.*

(i) *ϕ is multiplicative, that is $\phi(pq) = \phi(p)\phi(q)$, whenever p and q are relatively prime.*
(ii) $\sum_{d|n} \phi(d) = n$.
(iii) *For prime p, $\sum_{i=1}^{l} \phi(p^l) = p^l - 1$.*

For a positive integer n, $\mathbb{Z}_n$ represents the set of congruence classes $\{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ of integer modulo n.

An integer d dividing n is a proper divisor of n if and only if $1 < d < n$. Let Υ_n be the simple graph with vertex set as the proper divisor set $\{d_1, d_2, \ldots, d_t\}$ of n, where two vertices are adjacent provided $d_i d_j$ is a multiple of n. Evidently, this graph is a connected graph [4]. If $p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$ is the canonical decomposition of n, by Lemma 2.2, it follows that the order of Υ_n is given by

$$|V(\Upsilon_n)| = (n_1 + 1)(n_2 + 1) \ldots (n_r + 1) - 2.$$

For $1 \leq i \leq t$, let $A_{d_i} = \{r \in \mathbb{Z}_n : (r, n) = d_i\}$, where (r, n) is the greatest common divisor of r and n. We observe that $A_{d_i} \cap A_{d_j} = \phi$, when $i \neq j$. So, the sets $A_{d_1}, A_{d_2}, \ldots, A_{d_t}$ are pairwise disjoint and partition the vertex set of $\Gamma(\mathbb{Z}_n)$ as $V(\Gamma(\mathbb{Z}_n)) = A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_t}$. From the definition of A_{d_i}, a vertex of A_{d_i} is adjacent to the vertex of A_{d_j} in $\Gamma(\mathbb{Z}_n)$ provided that $n | d_i d_j$, for $i, j \in \{1, 2, \ldots, t\}$ (see [4]).

The following result by Young [13] gives the cardinality of A_{d_i}.

Lemma 2.4. [13] *For a divisor d of n, the cardinality of the set A_d is equal to $\phi\left(\frac{n}{d_i}\right)$.*

We note that the induced subgraphs $\Gamma(A_{d_i})$ of $\Gamma(\mathbb{Z}_n)$ are either cliques or null graphs, as can be seen below [4].

Lemma 2.5. *For the positive integer n and its proper d_i, the following statements hold.*

(i) *If $i \in \{1, 2, \ldots, t\}$, then the subgraph $\Gamma(A_{d_i})$ of $\Gamma(\mathbb{Z}_n)$ on A_{d_i} is either the complete graph $K_{\phi\left(\frac{n}{d_i}\right)}$ or its complement $\overline{K}_{\phi\left(\frac{n}{d_i}\right)}$. Also, $\Gamma(A_{d_i})$ is $K_{\phi\left(\frac{n}{d_i}\right)}$ provided d_i^2 is a multiple of n.*
(ii) *For distinct i, j in $\{1, 2, \ldots, t\}$, a vertex of A_{d_i} is adjacent to all of A_{d_j} or none of the vertices in A_{d_j}.*
(iii) *For distinct i, j in $\{1, 2, \ldots, t\}$, a vertex of A_{d_i} is adjacent to a vertex of A_{d_j} in $\Gamma(\mathbb{Z}_n)$ provided $d_i d_j$ is a multiple of n.*

The graph formed in part (iii) of Lemma 2.5 is known as $\mathcal{G}(A(d_i))$ graph. Clearly, $\Gamma(\mathbb{Z}_n)$ can be expressed as a joined union of complete graphs and empty graphs.

Lemma 2.6. [4] *For induced subgraph $\Gamma(A_{d_i})$ of $\Gamma(\mathbb{Z}_n)$ with A_{d_i} vertices, for $1 \leq i \leq t$, the zero-divisor graph is*

$$\Gamma(\mathbb{Z}_n) = \Upsilon_n[\Gamma(A_{d_1}), \Gamma(A_{d_2}), \ldots, \Gamma(A_{d_t})].$$

3. Structure of the zero-divisor graph $\Gamma(\mathbb{Z}_{p^{N_1}q^{N_2}r})$

We begin with the following result which gives the structure of $\Gamma(\mathbb{Z}_{P^N qr})$, where N is an even number.

Theorem 3.1. *Let $\Gamma(\mathbb{Z}_n)$ be a zero-divisor graph of order $n = p^N qr$, where $2 < p < q < r$ are primes and $N = 2m$, m is any positive integer. Then*

$$\begin{aligned}\Gamma(\mathbb{Z}_n) =\Upsilon_n\Big[&\overline{K}_{\phi(p^{2m-1}qr)},\ \overline{K}_{\phi(p^{2m-2}qr)},\ \overline{K}_{\phi(p^{2m-3}qr)}, \ldots, \overline{K}_{\phi(pqr)},\\ &\overline{K}_{\phi(p^{2m}r)}, \overline{K}_{\phi(p^{2m}q)}, \overline{K}_{\phi(p^{2m-1}r)}, \overline{K}_{\phi(p^{2m-2}r)}, \ldots, \overline{K}_{\phi(r)},\\ &\overline{K}_{\phi(p^{2m-1}q)}, \overline{K}_{\phi(p^{2m-2}q)}, \ldots, \overline{K}_{\phi(q)}, \overline{K}_{\phi(p^{2m-1})}, \overline{K}_{\phi(p^{2m-2})}, \ldots,\\ &\overline{K}_{\phi(p^{m+1})}, K_{\phi(p^m)}, K_{\phi(p^{m-1})}, \ldots, K_{\phi(p^2)}, K_{\phi(p)}\Big]. \qquad (3.1)\end{aligned}$$

Proof. Let $n = p^N qr$, where $2 < p < q < r$ are primes and $N = 2m$, m is any positive integer. Then the proper divisors of n are

$$\begin{aligned}&p,\ p^2,\ p^3, \ldots,\ p^m, \ldots, p^N,\\ &q,\ r,\ pq,\ pr,\ qr,\\ &p^2q,\ p^3q, \ldots, p^mq, \ldots, p^{2m}q,\\ &p^2r,\ p^3r, \ldots, p^mr, \ldots, p^{2m}r,\\ &pqr,\ p^2qr, \ldots, p^mqr, \ldots, p^{2m-1}qr. \qquad (3.2)\end{aligned}$$

Therefore, by Lemma 2.2, the order of Υ_n is

$$(2m + 1)(1 + 1)(1 + 1) = 4(2m + 1).$$

Now, by the definition of Υ_n, we have

$$\begin{aligned}p &\sim\ p^{2m-1}qr\\ p^2 &\sim\ p^{2m-2}qr, p^{2m-1}qr\\ &\vdots\\ p^m &\sim p^mqr, p^{m+1}qr, \ldots, p^{2m-1}qr\\ &\vdots\end{aligned}$$

$$p^{2m-1} \sim p^{2m-2}qr, p^{2m-3}qr, \ldots, p^2qr, pqr.$$

The iteration of the adjacency relation is given as

$$p^i \sim \; p^jqr, \qquad i+j \geq N, \; i,j = 1,2,3,\ldots,N.$$

By the similar arguments as above, the other adjacency relations are given by

$$\begin{aligned} q \sim \; & p^N r, & r \sim \; & p^N q \\ p^i q \sim \; & p^j r, & i+j \geq N, \; & i,j = 1,2,3,\ldots,N \\ p^i r \sim \; & p^j q, & i+j \geq N, \; & i,j = 1,2,3,\ldots,N \\ p^i qr \sim \; & p^j qr, & i+j \geq N, \; & i,j = 1,2,3,\ldots,N. \end{aligned}$$

Now, by Lemma 2.4, cardinalities of $|A_{d_i}|$, where i is in 3.2 and $j = 1,2,3,\ldots,N$ are given by

$$|A_{d_{p^i}}| = \phi\big(p^{2m-i}qr\big), \quad |A_{d_q}| = \phi\big(p^{2m}r\big), \qquad |A_{d_r}| = \phi\big(p^{2m}q\big),$$
$$|A_{d_{p^iq}}| = \phi\big(p^{2m-i}r\big), \quad |A_{d_{p^ir}}| = \phi\big(p^{2m-i}q\big), \; |A_{d_{p^iqr}}| = \phi\big(p^{2m-i}\big).$$

Also, by Lemma 2.5, the induced subgraphs $\Gamma\big(A_{d_i}\big)$'s are

$$\Gamma\big(A_{d_{p^iqr}}\big) = \begin{cases} K_{\phi(p^{2m-i})}, & \text{for } i = m, m+1, \ldots, 2m, \\ \overline{K}_{\phi(p^{2m-i})}, & \text{for } i = 1,2,\ldots,m-1. \end{cases}$$
$$\Gamma\big(A_{d_q}\big) = \overline{K}_{\phi\left(p^{2m}r\right)}$$
$$\Gamma\big(A_{d_r}\big) = \overline{K}_{\phi\left(p^{2m}q\right)}$$
$$\Gamma\big(A_{d_{p^iq}}\big) = \overline{K}_{\phi\left(p^{2m-i}r\right)}, \; i = 1,2,3,\ldots,2m,$$
$$\Gamma\big(A_{d_{p^ir}}\big) = \overline{K}_{\phi\left(p^{2m-i}q\right)}, \; i = 1,2,3,\ldots,2m,$$
$$\Gamma\big(A_{d_{p^i}}\big) = \overline{K}_{\phi\left(p^{2m-i}qr\right)}, \; i = 1,2,3,\ldots,2m,$$

where we avoid the induced subgraph $\Gamma\big(A_{p^Nqr}\big)$ corresponding to the divisor p^Nqr. Thus, by Lemma 2.6, the structure of the zero-divisor graph $\Gamma\big(\mathbb{Z}_n\big)$ is given as in 3.1. This completes the proof. □

Now, we obtain the structure of $\Gamma(\mathbb{Z}_{p^Nqr})$, when $N = 2m+1$ is odd.

Theorem 3.2. *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph of order $n = p^Nqr$, where $2 < p < q < r$ are primes and $N = 2m+1$ is a positive integer and $m \geq 1$. Then*

$$\Gamma(\mathbb{Z}_n) = \Upsilon_n\big[\overline{K}_{\phi\left(p^{N-1}qr\right)}, \; \overline{K}_{\phi\left(p^{N-2}qr\right)}, \ldots, \overline{K}_{\phi\left(pqr\right)}, \; \overline{K}_{\phi\left(p^Nr\right)}, \; \overline{K}_{\phi\left(p^Nq\right)},$$

$$\overline{K}_{\phi\left(p^{N-1}r\right)},\ \overline{K}_{\phi\left(p^{N-2}r\right)},\dots,\overline{K}_{\phi\left(r\right)},\ \overline{K}_{\phi\left(p^{N-1}q\right)},$$
$$\overline{K}_{\phi\left(p^{N-2}q\right)},\dots,\ \overline{K}_{\phi\left(q\right)},\overline{K}_{\phi\left(p\right)},\ \overline{K}_{\phi\left(p^{2}\right)},\dots,\overline{K}_{\phi\left(p^{m}\right)},$$
$$K_{\phi\left(p^{m+1}\right)},\ K_{\phi\left(p^{m+2}\right)},\dots\ K_{\phi\left(p^{N}\right)}\big]. \tag{3.3}$$

Proof. Let $n = p^N qr$, where $2 < p < q < r$ are primes and $N = 2m+1$ is positive odd integer and $m \geq 1$. Then the proper divisors of n are given as

$$\begin{aligned}
&p,\ p^2,\ p^3,\dots,\ p^m,\ p^{m+1},\dots,p^N,\\
&q,\ r,\ pq,\ p^2q,\dots,p^mq,\ p^{m+1}q,\dots,p^Nq,\\
&pr,\ p^2r,\ p^3r,\dots,p^mr,\ p^{m+1},\dots,p^Nr,\\
&qr,\ pqr,\ p^2qr,\dots,p^mqr,\ p^{m+1}qr,\dots,p^{N-1}qr.
\end{aligned} \tag{3.4}$$

Now, by Lemma 2.2, the order of Υ_n is

$$(2m+1+1)(1+1)(1+1) = 8(m+1),$$

where $m \geq 1$.
Therefore, by definition of Υ_n, we have

$$\begin{aligned}
p &\sim\ p^{N-1}qr\\
p^2 &\sim\ p^{N-2}qr,\ p^{N-1}qr\\
&\vdots\\
p^m &\sim\ p^{m+1}qr\\
&\vdots
\end{aligned}$$

The iterations of the adjacency relations are given as

$$\begin{aligned}
p^i &\sim\ p^jqr, && i+j \geq 2m+1 \textit{ and } i,j = 1,2,3,\dots,N.\\
p^iq &\sim\ p^jr, && i+j \geq 2m+1 \textit{ and } i,j = 1,2,3,\dots,N.\\
p^ir &\sim\ p^jr, && i+j \geq 2m+1 \textit{ and } i,j = 1,2,3,\dots,N.\\
p^iqr &\sim\ p^jqr, && i+j \geq 2m+1 \textit{ and } i,j = 1,2,3,\dots,N.
\end{aligned}$$

Now, by Lemma 2.4, the cardinalities of $|A_{d_i}|$, where i is given by 3.4 and $j = 1,2,3,\dots,N$, are given by

$$|A_{d_{p^j}}| = \phi\big(p^{N-j}qr\big),\quad |A_{d_{p^jq}}| = \phi\big(p^{N-j}r\big),$$
$$|A_{d_{p^jr}}| = \phi\big(p^{N-j}q\big),\quad |A_{d_{p^jqr}}| = \phi\big(p^{N-j}\big).$$

Thus, by Lemma 2.5, the induced subgraphs $\Gamma\big(A_{d_i}\big)$ are given by

$$\Gamma\big(A_{d_{p^j qr}}\big) = \begin{cases} K_{\phi\left(p^{N-j}\right)}, & j = 1,2,3,\dots,m. \\ \overline{K}_{\phi\left(p^{N-j}\right)}, & j = m+1, m+2,\dots,N, \end{cases}$$
$$\Gamma\big(A_{d_{p^j}}\big) = \overline{K}_{\phi\left(p^{N-j}qr\right)}, \ j = 1,2,3,\dots,N,$$
$$\Gamma\big(A_{d_{p^j q}}\big) = \overline{K}_{\phi\left(p^{N-j}r\right)}, \ j = 1,2,3,\dots,N,$$
$$\Gamma\big(A_{d_{p^j r}}\big) = \overline{K}_{\phi\left(p^{N-j}q\right)}, \ j = 1,2,3,\dots,N,$$
$$\Gamma\big(A_{d_q}\big) = \overline{K}_{\phi\left(p^{N}r\right)},$$
$$\Gamma\big(A_{d_r}\big) = \overline{K}_{\phi\left(p^{N}q\right)}$$

where we avoid the induced subgraph $\Gamma\big(A_{p^N qr}\big)$ corresponding to the divisor $p^N qr$. Thus, by Lemma 2.6, the structure of the zero-divisor graph $\Gamma\big(\mathbb{Z}_n\big)$ is given as in 3.3, which proves the result. □

The next result gives the structure of $\Gamma(\mathbb{Z}_{p^{N_1}q^{N_2}r})$, where $N_1 = 2m_1+1$ is odd and $N_2 = 2m_2$ is even.

Theorem 3.3. *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph of order $n = p^{N_1}q^{N_2}r$, where $2 < p < q < r$ are primes, $N_1 = 2m_1+1$ and $N_2 = 2m_2$ are positive integers and $m_1, m_2 \geq 1$. Then*

$$\begin{aligned}\Gamma\big(\mathbb{Z}_n\big) = \Upsilon_n\big[&\overline{K}_{\phi\left(p^{N_1-1}q^{N_2}r\right)}, \ \overline{K}_{\phi\left(p^{N_1-2}q^{N_2}r\right)}, \dots, \overline{K}_{\phi\left(pq^{N_2}r\right)}, \\ &\overline{K}_{\phi\left(q^{N_2}r\right)}, \ \overline{K}_{\phi\left(p^{N_1}q^{N_2-1}r\right)}, \overline{K}_{\phi\left(p^{N_1}q^{N_2-2}r\right)}, \dots, \overline{K}_{\phi\left(p^{N_1}qr\right)}, \\ &\overline{K}_{\phi\left(p^{N_1}r\right)}, \ K_{\phi\left(q\right)}, \ K_{\phi\left(q^2\right)}, \dots, K_{\phi\left(q^{m_2}\right)}, \overline{K}_{\phi\left(q^{m_2+1}\right)}, \\ &\overline{K}_{\phi\left(q^{m_2+2}\right)}, \dots, \overline{K}_{\phi\left(q^{2m_2}\right)}, \\ &K_{\phi\left(p\right)}, \ K_{\phi\left(p^2\right)}, \dots, K_{\phi\left(p^{m_1}\right)}, \dots, \\ &K_{\phi\left(pq^{m_2}\right)}, \dots, K_{\phi\left(p^{m_1}q\right)}, \ \dots, K_{\phi\left(p^{m_1}q^{m_2}\right)}, \ \overline{K}_{\phi\left(p^{m_1+1}q^{m_2}\right)}, \\ &\overline{K}_{\phi\left(p^{m_1+1}q^{m_2+1}\right)}, \dots, \overline{K}_{\phi\left(p^{2m_1+1}q^{2m_2}\right)}, \ \overline{K}_{\phi\left(r\right)}, \\ &\overline{K}_{\phi\left(pqr\right)}, \overline{K}_{\phi\left(p^2qr\right)}, \ \overline{K}_{\phi\left(pq^2r\right)}, \dots, \overline{K}_{\phi\left(p^{m_1}q^{m_2}r\right)}, \dots, \\ &\overline{K}_{\phi\left(p^{2m_1+1}q^{2m_2-1}r\right)}, \dots, \overline{K}_{\phi\left(p^{2m_1}q^{2m_2}r\right)}\big]. \end{aligned} \tag{3.5}$$

Proof. Let $n = p^{N_1}q^{N_2}r$, where $2 < p < q < r$ are primes, $N_1 = 2m_1+1$ and $N_2 = 2m_2$ are positive integers with $m_1, \ m_2 \geq 1$. Then the proper divisors of n are

$$\begin{aligned}
&p,\ p^2,\ldots,p^{m_1},\ p^{m_1+1},\ldots,p^{2m_1+1},\\
&q,q^2,\ldots,q^{m_2},\ q^{m_2+1},\ldots,q^{2m_2},\\
&r,pq,\ p^2q,\ldots,p^{m_1}q,\ldots,p^{2m_1+1}q,\\
&pq^2,\ldots,pq^{2m_2},\ldots,p^{2m_1+1}q^{2m_2},\\
&pr,\ldots,p^{2m_1+1}r,\\
&qr,\ldots,q^{2m_2}r,\ pqr,\ldots,p^{m_1}q^{m_2}r,\ldots,p^{2m_1+1}q^{2m_2-1}r,\\
&p^{2m_1}q^{2m_2}r=p^{N_1-1}q^{N_2}r. \qquad (3.6)
\end{aligned}$$

Therefore, by Lemma 2.2, the order of

$$\Upsilon_n=(N_1+1)(N_2+1)(1+1)=2(N_1+1)(N_2+1).$$

Now, by the definition of Υ_n, we have

$$\begin{aligned}
p &\sim p^{N_1-1}q^{N_2}r\\
p^2 &\sim p^{N_1-2}q^{N_2}r,\ p^{N_1-1}q^{N_2}r,\\
&\vdots\\
p^{m_1} &\sim p^{m_1+1}q^{N_2}r\\
&\vdots
\end{aligned}$$

The iterations of the adjacency relations are given as

$$\begin{aligned}
p^i &\sim\ p^jq^{N_2}r,\ \ i+j\geq 2m_1+1,\ \ i,j=1,2,3,\ldots,2m_1+1,\\
q^i &\sim\ p^Nq^jr,\ \ \ i+j\geq 2m_2,\ \ i,j=1,2,3,\ldots,2m_2,\\
pq^i &\sim\ p^kq^jr,\ \ \ i+j\geq 2m_2,\ \ i,j=1,2,3,\ldots,2m_2,\ \ k\geq 2m_1,\\
&\vdots\\
p^{m_1}q^i &\sim\ p^kq^jr,\ \ \ i+j\geq 2m_2,\ \ k\geq m_1+1,\ \ i,j=1,2,3,\ldots,2m_2,\\
&\vdots\\
p^{2m_1+1}q^i &\sim\ p^kq^jr,\ \ \ i+j\geq 2m_2,\ \ k\geq 0,\ \ i,j=1,2,3,\ldots,2m_2,\\
&\vdots\\
p^tq^sr &\sim\ p^{t'}q^{s'}r,\ \ \ t+t'\geq 2m_1+1,\ \ s+s'\geq\ 2m_2.
\end{aligned}$$

Thus, by Lemma 2.4, the cardinalities of $|A_{d_i}|$, where

$$i=1,2,\ldots,2m_1+1=N_1,\ \ j=1,2,\ldots,2m_2=N_2,$$

are given by

$$\begin{aligned} |A_{p^iq^jr}| &= \phi\big(p^{N_1-i}q^{N_2-j}\big), & |A_{p^iq^j}| &= \phi\big(p^{N_1-i}q^{N_2-j}r\big),\\ |A_{p^i}| &= \phi\big(p^{N_1-i}q^{N_2}r\big), & |A_{q^j}| &= \phi\big(p^{N_1}q^{N_2-j}r\big),\\ |A_r| &= \phi\big(p^{N_1}q^{N_2}\big), & |A_{p^ir}| &= \phi\big(p^{N_1-i}q^{N_2}\big),\\ |A_{q^jr}| &= \phi\big(p^{N_1}q^{N_2-j}\big). \end{aligned}$$

Therefore, by Lemma 2.6, the induced subgraphs $\Gamma\big(A_{d_i}\big)$, where d_i is from Equation 3.6, are given by

$$\begin{aligned} \Gamma(A_{d_{p^i}}) &= \overline{K}_{\phi(p^{N_1-i}q^{N_2}r)},\ 1 \le i \le 2m_1+1,\\ \Gamma(A_{d_{q^j}}) &= \overline{K}_{\phi(p^{N_1}q^{N_2-j}r)},\ 1 \le j \le 2m_2,\\ \Gamma(A_{d_{p^{N_1}r}}) &= \begin{cases} K_{\phi(q^j)},\ 1 \le j \le m_2,\\ \overline{K}_{\phi(q^j)},\ m_2+1 \le j \le 2m_2, \end{cases}\\ \Gamma(A_{d_{q^{N_2}r}}) &= \begin{cases} \overline{K}_{\phi(p^i)},\ m_1+1 \le i \le 2m_1+1,\\ K_{\phi(p^i)},\ 1 \le i \le m_1, \end{cases}\\ \Gamma(A_{d_r}) &= \begin{cases} \overline{K}_{\phi(p^iq^j)},\ m_1+1 \le i \le 2m_1+1,\ \textit{and}\\ \qquad\qquad m_2+1 \le j \le 2m_2,\\ K_{\phi(p^iq^j)},\ 1 \le i \le m_1,\ \textit{and}\ 1 \le j \le m_2, \end{cases}\\ \Gamma(A_{d_{p^{N_1}q^{N_2}}}) &= \overline{K}_{\phi(r)}, \end{aligned}$$

where we avoid the induced subgraph $\Gamma\big(A_{p^{N_1}q^{N_2}r}\big)$ corresponding to the divisor $p^{N_1}q^{N_2}r$. Thus, by Lemma 2.6, the structure of the zero-divisor graph $\Gamma\big(\mathbb{Z}_n\big)$ is given by 3.5. □

The following result gives the structure of $\Gamma(\mathbb{Z}_{p^{N_1}q^{N_2}r})$, where $N_1 = 2m_1$ is even and $N_2 = 2m_2+1$ is odd. The proof is similar to the arguments as in the above theorems.

Theorem 3.4. *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph of order $n = p^{N_1}q^{N_2}r$, where $2 < p < q < r$ are primes, $N_1 = 2m_1$ and $N_2 = 2m_2+1$ are positive integers and $m_1, m_2 \ge 1$. Then*

$$\begin{aligned} \Gamma\big(\mathbb{Z}_n\big) = \Upsilon_n\big[&\overline{K}_{\phi\left(p^{N_1-1}q^{N_2}r\right)},\ \overline{K}_{\phi\left(p^{N_1-2}q^{N_2}r\right)}, \dots, \overline{K}_{\phi\left(pq^{N_2}r\right)},\ \overline{K}_{\phi\left(q^{N_2}r\right)},\\ &\overline{K}_{\phi\left(p^{N_1}q^{N_2-1}r\right)},\ \overline{K}_{\phi\left(p^{N_1}q^{N_2-2}r\right)}, \dots, \overline{K}_{\phi\left(p^{N_1}qr\right)},\ \overline{K}_{\phi\left(p^{N_1}r\right)},\\ &K_{\phi\left(q\right)},\ K_{\phi\left(q^2\right)}, \dots, K_{\phi\left(q^{m_2-1}\right)},\ \overline{K}_{\phi\left(q^{m_2}\right)},\\ &\overline{K}_{\phi\left(q^{m_2+1}\right)}, \overline{K}_{\phi\left(q^{m_2+2}\right)}, \dots, \overline{K}_{\phi\left(q^{2m_2}\right)}, \end{aligned}$$

$$K_{\phi(p)}, K_{\phi(p^2)}, \dots, K_{\phi(p^{m_1})}, \dots,$$
$$K_{\phi(pq^{m_2})}, \dots, K_{\phi(p^{m_1}q)}, \dots, K_{\phi(p^{m_1}q^{m_2})},$$
$$\overline{K}_{\phi(p^{m_1+1}q^{m_2})}, \ \overline{K}_{\phi(p^{m_1+1}q^{m_2+1})}, \dots, \overline{K}_{\phi(p^{2m_1}q^{2m_2+1})},$$
$$\overline{K}_{\phi(r)}, \ \overline{K}_{\phi(pqr)}, \ \overline{K}_{\phi(p^2qr)}, \ \overline{K}_{\phi(pq^2r)}, \dots,$$
$$\overline{K}_{\phi(p^{m_1}q^{m_2}r)}, \dots, \overline{K}_{\phi(p^{2m_1-1}q^{2m_2+1}r)}, \dots, \overline{K}_{\phi(p^{2m_1}q^{2m_2}r)}\big].$$

Now, we obtain the structure of $\Gamma(\mathbb{Z}_{p^{N_1}q^{N_2}r})$, where both $N_1 = 2m_1$ and $N_2 = 2m_2$ are even.

Theorem 3.5. *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph of order $n = p^{N_1}q^{N_2}r$, where $2 < p < q < r$ are primes, $N_1 = 2m_1$ and $N_2 = 2m_2 > 2$ with $N_2 < N_1$ are positive integers and $m_1, m_2 > 1$. Then*

$$\begin{aligned}\Gamma(\mathbb{Z}_n) = \Upsilon_n \big[&\overline{K}_{\phi(p^{N-1}q^{N_2}r)}, \ \overline{K}_{\phi(p^{N_1-2}q^{N_2}r)}, \dots, \overline{K}_{\phi(q^{N_2}r)},\\ &\overline{K}_{\phi(p^{N_1}q^{N_2-1}r)}, \overline{K}_{\phi(p^{N_1}q^{N_2-2}r)}, \dots, \overline{K}_{\phi(p^{N_1}r)},\\ &K_{\phi(p)}, \ \overline{K}_{\phi(p^2)}, \dots, \overline{K}_{\phi(p^{N_1})},\\ &K_{\phi(q)}, K_{\phi(q^2)}, \dots, K_{\phi(q^{m_2})},\\ &\overline{K}_{\phi(p^{m_2+1})}, \ \overline{K}_{\phi(p^{m_2+2})}, \dots, \overline{K}_{\phi(p^{2m_2})},\\ &K_{\phi(pq)}, K_{\phi(p^2q)}, \dots, K_{\phi(p^{m_1}q^{m_2})}, \ \overline{K}_{\phi(p^{m_1+1}q)},\\ &\overline{K}_{\phi(p^{m_1+1}q^{m_2+1})}, \dots, \overline{K}_{\phi(p^{2m_1}q^{2m_2})}, \ \overline{K}_{\phi(r)}\big].\end{aligned} \tag{3.7}$$

Proof. Let $n = p^{N_1}q^{N_2}r$, where $2 < p < q < r$ are primes, $N_1 = 2m_1$ and $N_2 = 2m_2 > 2$, $N_2 < N_1$ are positive integers with $m_1, \ m_2 > 1$. Then the proper divisors of n are

$$\begin{aligned}&p, \ p^2, \dots, p^{m_1}, p^{m_1+1}, \dots, p^{2m_1},\\ &q, \ q^2, \dots, q^{m_2}, q^{m_2+1}, \dots, q^{2m_2}, \ r,\\ &pq, pq^2, \dots, pq^{2m_2}, \ p^2q, \ p^2q^2, \dots, p^{2m_1}q^{2m_2},\\ &pr, \dots, p^{2m_1}r, qr, \dots, q^{2m_2}r,\\ &pqr, \ p^2qr, \dots, p^{2m_1}qr, \dots, p^{2m_1}q^{2m_2-1}r, p^{2m_1-1}q^{2m_2}r.\end{aligned}$$

Therefore, by Lemma 2.2, the order of Υ_n is

$$(N_1 + 1)(N_2 + 1)(1 + 1) = 2(N_1 + 1)(N_2 + 1).$$

Also, by the definition of Υ_n, we have

$$\begin{aligned}
p &\sim p^{N_1-1}q^{N_2}r\\
p^2 &\sim p^{N_1-2}q^{N_2}r,\ p^{N_1-1}q^{N_2}r,\\
&\vdots\\
p^{m_1} &\sim p^{m_1}q^{N_2}r\\
&\vdots
\end{aligned}$$

The iterations of the adjacency relations are given as

$$\begin{aligned}
p^i &\sim\ p^jq^{N_2}r,\ i+j\geq 2m_1,\ i,j=1,2,3,\dots,2m_1,\\
q^i &\sim\ p^{N_1}q^jr,\ i+j\geq 2m_2,\ i,j=1,2,3,\dots,2m_2,\\
pq^i &\sim\ p^kq^jr,\ i+j\geq 2m_2,\ k\geq 2m_1-1,\\
&\vdots\\
p^{m_1}q^i &\sim\ p^kq^jr,\ i+j\geq 2m_2,\ k\geq m_1,\ i,j=1,2,3,\dots,2m_2,\\
&\vdots\\
p^{2m_1}q^i &\sim\ p^kq^jr,\ i+j\geq 2m_2,\ k\geq 0,\ i,j=1,2,3,\dots,2m_2,\\
&\vdots\\
p^tq^sr &\sim\ p^{t'}q^{s'}r,\ t+t'\geq 2m_1,\ s+s'\geq 2m_2.
\end{aligned}$$

For $i=1,2,3,\dots,2m_1$, $j=1,2,3,\dots,2m_2$, by Lemma 2.4, the cardinalities of A_{d_i} are given by

$$\begin{aligned}
|A_{p^iqjr}| &= \phi\big(p^{N_1-i}q^{N_2-j}\big),\ |A_{p^iqj}|=\phi\big(p^{N_1-i}q^{N_2-j}r\big),\\
|A_{p^i}| &= \phi\big(p^{N_1-i}q^{N_2}r\big),\dots,|A_{q^jr}|=\phi\big(p^{N_1}q^{N_2-j}r\big),\\
|A_{p^ir}| &= \phi\big(p^{N_1-i}q^{N_2}\big),\dots,|A_{q^jr}|=\phi\big(p^{N_1}q^{N_2-j}\big),\dots,\\
|A_r| &= \phi\big(p^{N_1}q^{N_2}\big),\ |A_{p^{N_1}q^{N_2}}|=\phi\big(r\big),
\end{aligned}$$

Thus, by Lemma 2.5, the induced subgraphs $\Gamma\big(A_{d_{p^i}}\big)$ are given by

$$\begin{aligned}
\Gamma\big(A_{d_{p^i}}\big) &= \overline{K}_{\phi\left(p^{N_1-i}q^{N_2}r\right)},\ i=1,2,3,\dots,2m_1,\\
\Gamma\big(A_{d_{q^j}}\big) &= \overline{K}_{\phi\left(p^{N_1}q^{N_2-j}r\right)},\ j=1,2,3,\dots,2m_2,\\
\Gamma\big(A_{d_{p^iq^{N_2}r}}\big) &= \overline{K}_{\phi\left(p^k\right)},\ i=1,2,3,\dots,2m_1,\ \textit{and}\ 2\leq k\leq 2m_1,\\
\Gamma\big(A_{d_{p^{N_1-1}q^{N_2}r}}\big) &= K_{\phi(p)},
\end{aligned}$$

$$\Gamma\big(A_{d_{p^{N_1}q^jr}}\big)=\begin{cases}K_{\phi(q^k)}, & j=1,2,3,\ldots,2m_2 \textit{ and}\\ & 1\le k\le m_2,\\ \overline{K}_{\phi(q^s)}, & j=1,2,3,\ldots,2m_2 \textit{ and}\\ & m_2+1\le s\le 2m_2,\end{cases}$$

$$\Gamma\big(A_{d_{p^iq^jr}}\big)=\begin{cases}K_{\phi(p^kq^s)}, & 1\le i\le 2m_1, 1\le j\le 2m_2,\\ & 1\le k\le m_1 \textit{ and } 1\le s\le m_2,\\ \overline{K}_{\phi(p^kq^s)}, & 1\le i\le 2m_1,\ 1\le j\le 2m_2,\\ & m_1+1\le k\le 2m_1 \textit{ and}\\ & m_2+1\le s\le 2m_2,\end{cases}$$

$$\Gamma\big(A_{d_{p^{N_1}q^{N_2}}}\big)=\overline{K}_{\phi(r)}.$$

where we avoid the induced subgraph $\Gamma\big(A_{p^{N_1}q^{N_2}r}\big)$ corresponding to the divisor $p^{N_1}q^{N_2}r$. Thus, by Lemma 2.6, the structure of zero-divisor graph $\Gamma\big(\mathbb{Z}_n\big)$ is given as in 3.7. □

We have the following observations.

Corollary 3.6. *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph of order $n=p^{N_1}q^{N_2}r$, where $2<p<q<r$ are primes, $N_1=2m_1$ and $N_2=2m_2$ are positive integers. If $N_1=N_2$, then the structure of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ is given as*

$$\begin{aligned}\Gamma(\mathbb{Z}_n)=\Upsilon_n\big[&\overline{K}_{\phi\left(p^{N_1-1}q^{N_2}r\right)},\ \overline{K}_{\phi\left(p^{N_1-2}q^{N_2}r\right)},\ldots,\overline{K}_{\phi\left(pq^{N_2}r\right)},\ \overline{K}_{\phi\left(q^{N_2}r\right)},\\ &\overline{K}_{\phi\left(p^{N_1}q^{N_2-1}r\right)},\ \overline{K}_{\phi\left(p^{N_1}q^{N_2-2}r\right)},\ldots,\overline{K}_{\phi\left(p^{N_1}qr\right)},\\ &K_{\phi\left(p^{m_1}\right)},\overline{K}_{\phi\left(p\right)},\ldots,\overline{K}_{\phi\left(p^{m_1-1}\right)},\ \overline{K}_{\phi\left(p^{m_1+1}\right)},\ldots,\\ &\overline{K}_{\phi\left(p^{N_1}\right)},\ K_{\phi\left(q^{m_2}\right)},\overline{K}_{\phi\left(q\right)},\ldots,\overline{K}_{\phi\left(q^{m_2-1}\right)},\overline{K}_{\phi\left(q^{m_2+1}\right)},\ldots,\\ &\overline{K}_{\phi\left(q^{M}\right)},K_{\phi\left(pq\right)},K_{\phi\left(p^2q\right)},\ldots,K_{\phi\left(p^{m_1}q^{m_2}\right)},\overline{K}_{\phi\left(p^{m_1+1}q\right)},\\ &\overline{K}_{\phi\left(p^{m_1+1}q^{m_2+1}\right)},\ldots,\overline{K}_{\phi\left(p^{2m_1}q^{2m_2}\right)},\ \overline{K}_{\phi\left(r\right)}\big].\end{aligned}$$

Corollary 3.7. *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph of order $n=p^{N_1}q^{N_2}r$, where $2<p<q<r$ are primes, $N_1=2m_1$ and $N_2=2$ are positive integers. Then the structure of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ is given as*

$$\begin{aligned}\Gamma(\mathbb{Z}_n)=\Upsilon_n\big[&\overline{K}_{\phi\left(p^{N_1-1}q^2r\right)},\ \overline{K}_{\phi\left(p^{N_1-2}q^2r\right)},\ldots,\overline{K}_{\phi\left(pq^2r\right)},\ \overline{K}_{\phi\left(p^{N_1}qr\right)},\\ &\overline{K}_{\phi\left(p^{N_1}r\right)},\ K_{\phi\left(p\right)},\ K_{\phi\left(p^2\right)},\ldots,\ K_{\phi\left(p^{m_1}\right)},\end{aligned}$$

$$\overline{K}_{\phi\left(p^{m_1+1}\right)}, \dots, \overline{K}_{\phi\left(p^{2m_1}\right)}, K_{\phi\left(q\right)}, \overline{K}_{\phi\left(q^2\right)},$$
$$K_{\phi\left(pq\right)}, K_{\phi\left(p^2q\right)}, \dots, K_{\phi\left(p^{m_1}q^2\right)}, \overline{K}_{\phi\left(p^{m_1+1}q\right)},$$
$$\overline{K}_{\phi\left(p^{m_1+1}q^{m_2+1}\right)}, \dots, \overline{K}_{\phi\left(p^{2m_1}q^2\right)}, \overline{K}_{\phi\left(r\right)}\Big].$$

The structure of $\Gamma(\mathbb{Z}_{p^{N_1}q^{N_2}r})$, where both $N_1 = 2m_1 + 1$ and $N_2 = 2m_2 + 1$ are odd, is as follows. The proof is similar to Theorems 3.1 and 3.2.

Theorem 3.8. *Let* $\Gamma(\mathbb{Z}_n)$ *be the zero-divisor graph of order* $n = p^{N_1}q^{N_2}r$, *where* $2 < p < q < r$ *are primes,* $N_1 = 2m_1 + 1$ *and* $N_2 = 2m_2 + 1$ *are positive integers, and* $2m_1 + 1 \leq 2m_2 + 1$. *Then*

$$\Gamma(\mathbb{Z}_n) = \Upsilon_n\Big[\overline{K}_{\phi\left(p^{N_1-1}q^{N_2}r\right)}, \overline{K}_{\phi\left(p^{N_1-2}q^{N_2}r\right)}, \dots, \overline{K}_{\phi\left(pq^{N_2}r\right)},$$
$$\overline{K}_{\phi\left(p^{N_1}q^{N_2-1}r\right)}, \overline{K}_{\phi\left(p^{N_1}q^{N_2-2}r\right)}, \dots, \overline{K}_{\phi\left(p^{N_1}qr\right)}, \overline{K}_{\phi\left(p^{N_1}r\right)},$$
$$K_{\phi\left(p\right)}, K_{\phi\left(p^2\right)}, \dots, K_{\phi\left(p^{m_1}\right)}, \overline{K}_{\phi\left(p^{m_1+1}\right)}, \dots, \overline{K}_{\phi\left(p^{2m_1+1}\right)},$$
$$K_{\phi\left(q\right)}, K_{\phi\left(q^2\right)}, \dots, K_{\phi\left(q^{m_2}\right)},$$
$$\overline{K}_{\phi\left(p^{m_2+1}\right)}, \overline{K}_{\phi\left(p^{m_2+2}\right)}, \dots, \overline{K}_{\phi\left(p^{2m_2+1}\right)},$$
$$K_{\phi\left(pq\right)}, K_{\phi\left(p^2q\right)}, \dots, K_{\phi\left(p^{m_1}q^{m_2}\right)}, \overline{K}_{\phi\left(p^{m_1+1}q\right)},$$
$$\overline{K}_{\phi\left(p^{m_1+1}q^{m_2+1}\right)}, \dots, \overline{K}_{\phi\left(p^{2m_1+1}q^{2m_2+1}\right)}, \overline{K}_{\phi\left(r\right)}\Big].$$

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Journal of Algebraic Systems

STRUCTURE OF ZERO-DIVISOR GRAGHS ASSOCIATED TO RING OF INTEGER MODULO n

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بررسی ساختار گراف‌های مقسوم‌علیه صفر وابسته به حلقه‌ی اعداد صحیح به پیمانه n

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برای حلقه‌ی جابه‌جایی یکدار R با $۰ \neq ۱$، فرض می‌کنیم $Z(R)$ مجموعه‌ی همه‌ی مقسوم‌علیه‌های صفر R باشد و $Z^*(R) = Z(R) \setminus \{۰\}$. گراف مقسوم‌علیه صفر R که آن را با نماد $\Gamma(R)$ نشان می‌دهیم، گراف ساده‌ای است که مجموعه‌ی رئوس آن برابر است با $Z^*(R) = Z(R) \setminus \{۰\}$ و دو رأس از $Z^*(R)$ با هم مجاورند اگر و تنها اگر حاصلضرب آن‌ها صفر باشد. ما در این مقاله، ساختار گراف‌های مقسوم علیه صفر $\Gamma(\mathbb{Z}_n)$ را برای $n = p^{N_۱}q^{N_۲}r$، زمانی که $۲ < p < q < r$ اعداد اول و $N_۱$ و $N_۲$ اعداد صحیح مثبت هستند، را مورد بررسی قرار می‌دهیم.

کلمات کلیدی: گراف مقسوم‌علیه صفر، حلقه‌ی جابه‌جایی، حلقه به پیمانه اعداد صحیح، اجتماع الحاقی.