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THE STRUCTURE OF MODULE LIE DERIVATIONS ON TRIANGULAR BANACH ALGEBRAS

M. R. MIRI, E. NASRABADI* AND A. R. GHORCHIZADEH

Abstract. In this paper, we introduce the concept of module Lie derivations on Banach algebras and study module Lie derivations on unital triangular Banach algebras $\mathcal{T} = \begin{bmatrix} A & M \\ & B \end{bmatrix}$ to its dual. Indeed, we prove that every module (linear) Lie derivation $\delta : \mathcal{T} \to \mathcal{T}^*$ can be decomposed as $\delta = d + \tau$, where $d : \mathcal{T} \to \mathcal{T}^*$ is a module (linear) derivation and $\tau : \mathcal{T} \to Z_{\mathcal{T}}(\mathcal{T}^*)$ is a module (linear) map vanishing at commutators if and only if this happens for the corner algebras A and B.

1. Introduction

Let A and B be Banach algebras and M be a Banach A, B-module that means that M is a left Banach A-module and right Banach B-module. The Banach algebras

$$\mathcal{T} = \mathrm{Tri}(A, B, M) = \left\{ \begin{bmatrix} a & m \\ & b \end{bmatrix} : a \in A, m \in M, b \in B \right\},$$

with usual multiplication and addition actions in the space of 2×2 matrices and with the following norm

$$\left\| \begin{bmatrix} a & m \\ & b \end{bmatrix} \right\| := \|a\|_A + \|m\|_M + \|b\|_B \qquad (a \in A, m \in M, b \in B)$$

are called the triangular Banach algebra.

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*Corresponding author.

Forrest and Marcoux [4] studied (continuous) derivations on unital triangular Banach algebra. They also examined the derivations of triangular Banach algebra into its dual spaces in [5]. Amini in [1] investigated module derivations on Banach algebras and then along with Bagha [2] studied the module derivations from Banach algebra to its dual spaces. After that, Nasrabadi and Pourabbas in [7] and [6] studied the module derivations from triangular Banach algebra to its dual spaces.

On the other hand, Cheung [3] considered triangular algebras of $\mathcal{T} = \mathrm{Tri}(A, B, M)$ (without topological structure), where A and B are unital (not necessarily Banach) algebras and M is a faithful A, B-module. They obtained sufficient conditions on $\mathcal{T}$ so that every Lie derivation of $\mathcal{T}$ to $\mathcal{T}$ was a standard Lie derivation.

In this paper, we define module Lie derivation on Banach algebras and for unitanl triangular Banach algebra $\mathcal{T} = \mathrm{Tri}(A, B, M)$, we show that under what conditions these module Lie derivations from $\mathcal{T}$ to its dual (and in a special cases Lie derivations) are standard. In this way, when $\mathfrak{A}$ is a Banach algebra and A and B are Banach $\mathfrak{A}$-module with compatible actions, and M is a left Banach A-$\mathfrak{A}$-module and right Banach B-$\mathfrak{A}$-module, we show that $\mathfrak{T}$-module Lie derivation $\delta : \mathcal{T} \to \mathcal{T}^*$ can be decomposed as $\delta = d + \tau$, where $d : \mathcal{T} \to \mathcal{T}^*$ is a $\mathfrak{T}$-module derivation and $\tau : \mathcal{T} \to Z_{\mathcal{T}}(\mathcal{T}^*)$ is a $\mathfrak{T}$-module map vanishing at commutators, where $\mathfrak{T} := \left\{ \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} : \alpha \in \mathfrak{A} \right\}$. Let $\mathfrak{A}$ and A be Banach algebras such that A is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \qquad a(\alpha \cdot b) = (a \cdot \alpha)b \quad (\alpha \in \mathfrak{A},\ a, b \in A),$$

and the same is true for the right actions (for more details see [1], [6], and [7]).

Let X be a Banach A-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is, for every $\alpha \in \mathfrak{A}$, $a \in A$, $x \in X$

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \quad (a \cdot \alpha) \cdot x = a \cdot (\alpha \cdot x), \quad (a \cdot x) \cdot \alpha = a \cdot (x \cdot \alpha),$$

and the same holds for the right actions. Then we say that X is a Banach A-$\mathfrak{A}$-module.

Note also that X is an A-bimodule. The center of X on A, is as follows:

$$Z_A(X) = \{x \in X; \quad a \cdot x = x \cdot a \ \text{ for each } \ a \in A\}.$$

If X is a (commutative) Banach A-$\mathfrak{A}$-module, and so is X^*, where the actions of A and $\mathfrak{A}$ on X^* are defined by

$$(\alpha\cdot f)(x) = f(x\cdot\alpha),\ (a\cdot f)(x) = f(x\cdot a)\ (\alpha\in\mathfrak{A}, a\in A, x\in X, f\in X^*),$$

and the same holds for the actions of other side.

In particular, if A is a commutative Banach $\mathfrak{A}$-bimodule, then it is a commutative Banach A-$\mathfrak{A}$-module. In this case, the dual space $\mathfrak{A}^*$ is also a commutative Banach A-$\mathfrak{A}$-module.

A bounded mapping $T : A \to X$ is called an $\mathfrak{A}$-module map if

$$T(a \pm a') = T(a) \pm T(a'),\quad T(\alpha\cdot a) = \alpha\cdot T(a),\quad T(a\cdot\alpha) = T(a)\cdot\alpha,$$

where $\alpha\in\mathfrak{A},\ a, a'\in A$. Note that, τ is an additive and not necessarily linear, so it is not necessarily an $\mathfrak{A}$-module homomorphism.

Definition 1.1. An $\mathfrak{A}$-module map $d : A \to X$ is called an $\mathfrak{A}$-module derivation if

$$d(aa') = a\cdot d(a') + d(a)\cdot a' \qquad (a, a'\in A).$$

Moreover, d is called inner, if there exists $x\in X$, such that

$$d(a) = \mathbf{ad}_x(a) := a\cdot x - x\cdot a \quad (a\in A).$$

Definition 1.2. An $\mathfrak{A}$-module map $\delta : A \to X$ is called an $\mathfrak{A}$-module Lie derivation if

$$\delta([a, a']) = [\delta(a), a'] + [a, \delta(a')] \qquad (a, a'\in A),$$

where $[\ ,\ ]$ is Lie product. that is, $[a, a'] = aa' - a'a$ and

$$[x, a] = -[a, x] = xa - ax,$$

for every $a, a'\in A$ and $x\in X$.

Remark 1.3. The important point to note here is that in all the topics of this paper, if we consider $\mathfrak{A} = \mathbb{C}$, when $\mathbb{C}$-module actions are natural multiplication, then the words "$\mathfrak{A}$-module" give way to "linear", which is not usually inserted. But in general, every $\mathfrak{A}$-module (Lie) derivation is not necessarily linear, but its boundedness still implies its norm continuity (since preserved subtraction).

Definition 1.4. An ($\mathfrak{A}$-module) Lie derivation $\delta : A \to X$ is called *standard* if it can be written as the sum of an ($\mathfrak{A}$-module) derivation and an ($\mathfrak{A}$-module) mapping with the image in the center of X on A vanishing at commutators.

2. Module Lie Derivations on Triangular Banach Algebras

Let $\mathfrak{A}$, A, and B be Banach algebras such that A and B are commutative Banach $\mathfrak{A}$-bimodule with compatible actions. Furthermore, let M be a commutative Banach (A,B)-$\mathfrak{A}$-module, that is, M is a commutative Banach $\mathfrak{A}$-bimodule, left Banach A-module and right Banach B-module with compatible actions. (for more details see [6] and [7]). Let

$$\mathcal{T} = \operatorname{Tri}(A,B,M) = \left\{ \begin{bmatrix} a & m \\ & b \end{bmatrix} ; a \in A, b \in B, m \in M \right\},$$

be equipped with the usual 2×2 matrix addition and formal multiplication and with the norm $\|t\| = \|a\|_A + \|b\|_B + \|m\|_M$ for every $t = \begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$. Then it is a Banach algebra, which is called the triangular Banach algebra. We know that, as a Banach space, $\mathcal{T}$ is isomorphic to the ℓ^1-sum of A, B, and M. It is clear that $\mathcal{T}^* \simeq A^* \oplus B^* \oplus M^* = \begin{bmatrix} A^* & M^* \\ & B^* \end{bmatrix}$. Now we consider

$$\mathfrak{T} = \left\{ \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} ; \alpha \in \mathfrak{A} \right\},$$

which is a Banach algebra. $\mathcal{T}$ with the 2×2 matrix multiplication is a commutative $\mathfrak{T}$-bimodule Banach algebra with the module actions:

$$\begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \cdot \begin{bmatrix} a & m \\ & b \end{bmatrix} = \begin{bmatrix} a & m \\ & b \end{bmatrix} \cdot \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} = \begin{bmatrix} \alpha \cdot a & \alpha \cdot m \\ & \alpha \cdot b \end{bmatrix},$$

where $\begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \in \mathfrak{T}$ and $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$ (for more details see [7]).

According to [5, Section 2.5], we have the following remark.

Remark 2.1. Let $t = \begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$ and $\lambda = \begin{bmatrix} f & h \\ & g \end{bmatrix} \in \mathcal{T}^*$. Then $\mathcal{T}^*$ acts on $\mathcal{T}$ as follows: $\omega(t) = f(a) + h(m) + g(b)$. The module actions of $\mathcal{T}$ on $\mathcal{T}^*$ is give by

$$t \cdot \lambda = \begin{bmatrix} a.f + m.h & b.h \\ & b.g \end{bmatrix} \quad \text{and} \quad \lambda \cdot t = \begin{bmatrix} f.a & h.a \\ & h.m + g.b \end{bmatrix}. \tag{2.1}$$

Thus, $\mathcal{T}^*$ becomes a Banach $\mathcal{T}$-bimodule. Furthermore, since A is a commutative Banach A-$\mathfrak{A}$-module, B is a commutative Banach B-$\mathfrak{A}$-module and M is a commutative Banach (A,B)-$\mathfrak{A}$-module. That is, M is a commutative Banach $\mathfrak{A}$-bimodule left Banach A-module and right Banach B-module with compatible actions; therefore, $\mathcal{T}$ (and so $\mathcal{T}^*$) becomes a commutative Banach $\mathcal{T}$-$\mathfrak{T}$-bimodule.

Proposition 2.2. *The center of $\mathcal{T}^*$ on $\mathcal{T}$ is given by*

$$Z_{\mathcal{T}}(\mathcal{T}^*) = \left\{ \begin{bmatrix} f & 0 \\ & g \end{bmatrix}; \quad f \in Z_A(A^*), \quad g \in Z_B(B^*) \right\}.$$

Proof. Suppose that $f \in Z_A(A^*)$ and $g \in Z_B(B^*)$. It is easy to verify $\begin{bmatrix} f & 0 \\ & g \end{bmatrix} \in Z_{\mathcal{T}}(\mathcal{T}^*)$.

Conversely, if $\begin{bmatrix} f & h \\ & g \end{bmatrix} \in Z_{\mathcal{T}}(\mathcal{T}^*)$, by (2.1), we have

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ & 0 \end{bmatrix} &= \begin{bmatrix} f & h \\ & g \end{bmatrix} \begin{bmatrix} 1_A & 0 \\ & 0 \end{bmatrix} - \begin{bmatrix} 1_A & 0 \\ & 0 \end{bmatrix} \begin{bmatrix} f & h \\ & g \end{bmatrix} \\ &= \begin{bmatrix} f - f & h \\ & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & h \\ & 0 \end{bmatrix}. \end{aligned}$$

Therefore, $h = 0$. So if $\begin{bmatrix} f & 0 \\ & g \end{bmatrix} \in Z_{\mathcal{T}}(\mathcal{T}^*)$, for every arbitrary $a \in A$ and $b \in B$, we have

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ & 0 \end{bmatrix} &= \begin{bmatrix} f & 0 \\ & g \end{bmatrix} \begin{bmatrix} a & 0 \\ & b \end{bmatrix} - \begin{bmatrix} a & 0 \\ & b \end{bmatrix} \begin{bmatrix} f & 0 \\ & g \end{bmatrix} \\ &= \begin{bmatrix} fa & 0 \\ & gb \end{bmatrix} - \begin{bmatrix} af & 0 \\ & bg \end{bmatrix} \\ &= \begin{bmatrix} fa - af & 0 \\ & gb - bg \end{bmatrix}. \end{aligned}$$

Thus, $fa = af$ and $gb = bg$, that means $f \in Z_A(A^*)$ and $g \in Z_B(B^*)$. □

3. Main results

All over this section, A is an unital commutative Banach A-$\mathfrak{A}$-module, B is an unital commutative Banach B-$\mathfrak{A}$-module, M is a commutative Banach (A, B)-$\mathfrak{A}$-module (M is a commutative Banach $\mathfrak{A}$-bimodule, left Banach A-module and right Banach B-module) and $\mathcal{T} = \begin{bmatrix} A & M \\ & B \end{bmatrix}$ is the triangular Banach algebra associated with A, M, and B, which becomes an unital commutative Banach $\mathcal{T}$-$\mathfrak{T}$-module.

Proposition 3.1. *The map $\delta : \mathcal{T} \longrightarrow \mathcal{T}^*$ is a ($\mathfrak{T}$-module) Lie derivation if and only if δ is of the form*

$$\delta\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) = \begin{bmatrix} l_A(a) + h_B(b) - mm_0 & m_0a - bm_0 \\ & l_B(b) + h_A(a) + m_0m \end{bmatrix}, \quad (3.1)$$

where $m_0 \in M^*$, $l_A : A \longrightarrow A^*$ and $l_B : B \longrightarrow B^*$ are ($\mathfrak{A}$-module) Lie derivations, $h_A : A \longrightarrow Z_B(B^*)$ and $h_B : B \longrightarrow Z_A(A^*)$ are ($\mathfrak{A}$-module) maps satisfying $h_A([a, a']) = 0$ and $h_B([b, b']) = 0$.

Proof. Due to remark 1.3, we provide the proof in the general state (module state). For convenience, for every $a \in A$, $b \in B$, and $m \in M$, we symbolize $\boldsymbol{p} = \begin{bmatrix} 1_A & 0 \\ & 0 \end{bmatrix}$, $\boldsymbol{q} = \begin{bmatrix} 0 & 0 \\ & 1_B \end{bmatrix}$, $\boldsymbol{a} = \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}$, $\boldsymbol{m} = \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}$. Also

$$\delta(\boldsymbol{p}) = \begin{bmatrix} \phi_{\boldsymbol{p}} & \varphi_{\boldsymbol{p}} \\ & \psi_{\boldsymbol{p}} \end{bmatrix}, \quad \delta(\boldsymbol{q}) = \begin{bmatrix} \phi_{\boldsymbol{q}} & \varphi_{\boldsymbol{q}} \\ & \psi_{\boldsymbol{q}} \end{bmatrix},$$

$$\delta(\boldsymbol{a}) = \begin{bmatrix} \phi_{\boldsymbol{a}} & \varphi_{\boldsymbol{a}} \\ & \psi_{\boldsymbol{a}} \end{bmatrix}, \quad \delta(\boldsymbol{m}) = \begin{bmatrix} \phi_{\boldsymbol{m}} & \varphi_{\boldsymbol{m}} \\ & \psi_{\boldsymbol{m}} \end{bmatrix}, \quad \delta(\boldsymbol{b}) = \begin{bmatrix} \phi_{\boldsymbol{b}} & \varphi_{\boldsymbol{b}} \\ & \psi_{\boldsymbol{b}} \end{bmatrix}.$$

The proof begins with the following six claims.

Claim 1: $\phi_{\boldsymbol{m}} = -m\varphi_{\boldsymbol{p}},\ \psi_{\boldsymbol{m}} = \varphi_{\boldsymbol{p}}m$ and $\varphi_{\boldsymbol{m}} = 0$.

$$\begin{aligned}
\begin{bmatrix} \phi_{\boldsymbol{m}} & \varphi_{\boldsymbol{m}} \\ & \psi_{\boldsymbol{m}} \end{bmatrix} &= \delta(\boldsymbol{m}) = \delta([\boldsymbol{p}, \boldsymbol{m}]) \\
&= [\delta(\boldsymbol{p}), \boldsymbol{m}] + [\boldsymbol{p}, \delta(\boldsymbol{m})] \\
&= \delta(\boldsymbol{p})\boldsymbol{m} - \boldsymbol{m}\delta(\boldsymbol{p}) + \boldsymbol{p}\delta(\boldsymbol{m}) - \delta(\boldsymbol{m})\boldsymbol{p} \\
&= \begin{bmatrix} \phi_{\boldsymbol{p}} & \varphi_{\boldsymbol{p}} \\ & \psi_{\boldsymbol{p}} \end{bmatrix} \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix} - \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix} \begin{bmatrix} \phi_{\boldsymbol{p}} & \varphi_{\boldsymbol{p}} \\ & \psi_{\boldsymbol{p}} \end{bmatrix} \\
&+ \begin{bmatrix} 1_A & 0 \\ & 0 \end{bmatrix} \begin{bmatrix} \phi_{\boldsymbol{m}} & \varphi_{\boldsymbol{m}} \\ & \psi_{\boldsymbol{m}} \end{bmatrix} - \begin{bmatrix} \phi_{\boldsymbol{m}} & \varphi_{\boldsymbol{m}} \\ & \psi_{\boldsymbol{m}} \end{bmatrix} \begin{bmatrix} 1_A & 0 \\ & 0 \end{bmatrix} \\
&= \begin{bmatrix} -m\varphi_{\boldsymbol{p}} & 0 \\ & \varphi_{\boldsymbol{p}}m \end{bmatrix} + \begin{bmatrix} 0 & -\varphi_{\boldsymbol{m}} \\ & 0 \end{bmatrix} = \begin{bmatrix} -m\varphi_{\boldsymbol{p}} & -\varphi_{\boldsymbol{m}} \\ & \varphi_{\boldsymbol{p}}m \end{bmatrix},
\end{aligned}$$

therefore, $\phi_{\boldsymbol{m}} = -m\varphi_{\boldsymbol{p}},\ \psi_{\boldsymbol{m}} = \varphi_{\boldsymbol{p}}m$ and $\varphi_{\boldsymbol{m}} = 0$.

Claim 2: $a\phi_{\boldsymbol{p}} = \phi_{\boldsymbol{p}}a,\ \varphi_{\boldsymbol{a}} = \varphi_{\boldsymbol{p}}a$.

$$\begin{aligned}
\begin{bmatrix} 0 & 0 \\ & 0 \end{bmatrix} &= \delta([\boldsymbol{a}, \boldsymbol{p}]) = [\delta(\boldsymbol{a}), \boldsymbol{p}] + [\boldsymbol{a}, \delta(\boldsymbol{p})] \\
&= \begin{bmatrix} \phi_{\boldsymbol{a}} & \varphi_{\boldsymbol{a}} \\ & \psi_{\boldsymbol{a}} \end{bmatrix} \begin{bmatrix} 1_A & 0 \\ & 0 \end{bmatrix} - \begin{bmatrix} 1_A & 0 \\ & 0 \end{bmatrix} \begin{bmatrix} \phi_{\boldsymbol{a}} & \varphi_{\boldsymbol{a}} \\ & \psi_{\boldsymbol{a}} \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&+\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} \phi_{\boldsymbol{p}} & \varphi_{\boldsymbol{p}} \\ & \psi_{\boldsymbol{p}} \end{bmatrix}-\begin{bmatrix} \phi_{\boldsymbol{p}} & \varphi_{\boldsymbol{p}} \\ & \psi_{\boldsymbol{p}} \end{bmatrix}\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\\
&=\begin{bmatrix} \phi_{\boldsymbol{a}} & \varphi_{\boldsymbol{a}} \\ & 0 \end{bmatrix}-\begin{bmatrix} \phi_{\boldsymbol{a}} & 0 \\ & 0 \end{bmatrix}+\begin{bmatrix} a\phi_{\boldsymbol{p}} & 0 \\ & 0 \end{bmatrix}-\begin{bmatrix} \phi_{\boldsymbol{p}}a & \varphi_{\boldsymbol{p}}a \\ & 0 \end{bmatrix}\\
&=\begin{bmatrix} a\phi_{\boldsymbol{p}}-\phi_{\boldsymbol{p}}a & \varphi_{\boldsymbol{a}}-\varphi_{\boldsymbol{p}}a \\ & 0 \end{bmatrix},
\end{aligned}
$$

that shows, $a\phi_{\boldsymbol{p}} = \phi_{\boldsymbol{p}}a$ and $\varphi_{\boldsymbol{a}} = \varphi_{\boldsymbol{p}}a$.

Claim 3: $b\psi_{\boldsymbol{p}} = \psi_{\boldsymbol{p}}b,\ \varphi_{\boldsymbol{b}} = -b\varphi_{\boldsymbol{p}}$.

$$
\begin{aligned}
\begin{bmatrix} 0 & 0 \\ & 0 \end{bmatrix} &= \delta([\boldsymbol{b},\boldsymbol{p}]) = [\delta(\boldsymbol{b}),\boldsymbol{p}]+[\boldsymbol{b},\delta(\boldsymbol{p})]\\
&=\begin{bmatrix} \phi_{\boldsymbol{b}} & \varphi_{\boldsymbol{b}} \\ & \psi_{\boldsymbol{b}} \end{bmatrix}\begin{bmatrix} 1_A & 0 \\ & 0 \end{bmatrix}-\begin{bmatrix} 1_A & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} \phi_{\boldsymbol{b}} & \varphi_{\boldsymbol{b}} \\ & \psi_{\boldsymbol{b}} \end{bmatrix}\\
&+\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\begin{bmatrix} \phi_{\boldsymbol{p}} & \varphi_{\boldsymbol{p}} \\ & \psi_{\boldsymbol{p}} \end{bmatrix}-\begin{bmatrix} \phi_{\boldsymbol{p}} & \varphi_{\boldsymbol{p}} \\ & \psi_{\boldsymbol{p}} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\\
&=\begin{bmatrix} \phi_{\boldsymbol{b}} & \varphi_{\boldsymbol{b}} \\ & 0 \end{bmatrix}-\begin{bmatrix} \phi_{\boldsymbol{b}} & 0 \\ & 0 \end{bmatrix}+\begin{bmatrix} 0 & b\varphi_{\boldsymbol{b}} \\ & b\psi_{\boldsymbol{p}} \end{bmatrix}-\begin{bmatrix} 0 & 0 \\ & b\psi_{\boldsymbol{p}} \end{bmatrix}\\
&=\begin{bmatrix} 0 & \varphi_{\boldsymbol{b}}+b\varphi_{\boldsymbol{p}} \\ & b\psi_{\boldsymbol{p}}-\psi_{\boldsymbol{p}}b \end{bmatrix},
\end{aligned}
$$

so $b\psi_{\boldsymbol{p}} = \psi_{\boldsymbol{p}}b$ and $\varphi_{\boldsymbol{b}} = -b\varphi_{\boldsymbol{p}}$.

Claim 4: $\phi_{\boldsymbol{b}} \in Z_A(A^*)$ and $\psi_{\boldsymbol{a}} \in Z_B(B^*)$.

$$
\begin{aligned}
\begin{bmatrix} 0 & 0 \\ & 0 \end{bmatrix} &= \delta([\boldsymbol{a},\boldsymbol{b}]) = [\delta(\boldsymbol{a}),\boldsymbol{b}]+[\boldsymbol{a},\delta(\boldsymbol{b})]\\
&=\left[\begin{bmatrix} \phi_{\boldsymbol{a}} & \varphi_{\boldsymbol{a}} \\ & \psi_{\boldsymbol{a}} \end{bmatrix},\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\right]+\left[\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix},\begin{bmatrix} \phi_{\boldsymbol{b}} & \varphi_{\boldsymbol{b}} \\ & \psi_{\boldsymbol{b}} \end{bmatrix}\right]\\
&=\begin{bmatrix} 0 & -b\varphi_{\boldsymbol{a}} \\ & \psi_{\boldsymbol{a}}b-b\psi_{\boldsymbol{a}} \end{bmatrix}+\begin{bmatrix} a\phi_{\boldsymbol{b}}-\phi_{\boldsymbol{b}}a & -\varphi_{\boldsymbol{b}}a \\ & 0 \end{bmatrix}\\
&=\begin{bmatrix} [a,\phi_{\boldsymbol{b}}] & -b\varphi_{\boldsymbol{a}}-\varphi_{\boldsymbol{b}}a \\ & [\psi_{\boldsymbol{a}},b] \end{bmatrix},
\end{aligned}
$$

thus, $[a,\phi_{\boldsymbol{b}}] = 0$ and $[\psi_{\boldsymbol{a}}, b] = 0$. Since $a \in A$ and $b \in B$ are arbitrary, we show that, $\phi_{\boldsymbol{b}} \in Z_A(A^*)$ and $\psi_{\boldsymbol{a}} \in Z_B(B^*)$. Note that, equation $-b\varphi_{\boldsymbol{a}} - \varphi_{\boldsymbol{b}}a = 0$ confirms the second part of claims 2 and 3.

Claim 5: $\phi_{[a,a']} = [\phi_{\boldsymbol{a}}, a'] + [a, \phi_{\boldsymbol{a'}}]$ and $\psi_{[\boldsymbol{a},\boldsymbol{a'}]} = 0$.

$$\begin{bmatrix} \phi_{[a,a']} & \varphi_{[a,a']} \\ & \psi_{[a,a']} \end{bmatrix} = \delta\left(\begin{bmatrix} [a,a'] & 0 \\ & 0 \end{bmatrix}\right) = \delta([\boldsymbol{a}, \boldsymbol{a'}])$$
$$= \begin{bmatrix} \phi_{\boldsymbol{a}} & \varphi_{\boldsymbol{a}} \\ & \psi_{\boldsymbol{a}} \end{bmatrix}\begin{bmatrix} a' & 0 \\ & 0 \end{bmatrix} - \begin{bmatrix} a' & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} \phi_{\boldsymbol{a}} & \varphi_{\boldsymbol{a}} \\ & \psi_{\boldsymbol{a}} \end{bmatrix}$$
$$+ \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} \phi_{\boldsymbol{a'}} & \varphi_{\boldsymbol{a'}} \\ & \psi_{\boldsymbol{a'}} \end{bmatrix} - \begin{bmatrix} \phi_{\boldsymbol{a'}} & \varphi_{\boldsymbol{a'}} \\ & \psi_{\boldsymbol{a'}} \end{bmatrix}\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \phi_{\boldsymbol{a}}a' - a'\phi_{\boldsymbol{a}} + a\phi_{\boldsymbol{a'}} - \phi_{\boldsymbol{a'}}a & \varphi_{\boldsymbol{a}}a' - \varphi_{\boldsymbol{a'}}a \\ & 0 \end{bmatrix}$$
$$= \begin{bmatrix} [\phi_{\boldsymbol{a}}, a'] + [a, \phi_{\boldsymbol{a'}}] & \varphi_{\boldsymbol{a}}a' - \varphi_{\boldsymbol{a'}}a \\ & 0 \end{bmatrix},$$

this shows that, $\phi_{[a,a']} = [\phi_{\boldsymbol{a}}, a'] + [a, \phi_{\boldsymbol{a'}}]$ and $\psi_{[a,a']} = 0$.

Claim 6: $\psi_{[b,b']} = [\psi_{\boldsymbol{b}}, b'] + [b, \psi_{\boldsymbol{b'}}]$ and $\phi_{[\boldsymbol{b},\boldsymbol{b'}]} = 0$.
Proof is similar to claim 5.

We now begin the main body of proof. Define

$$\begin{aligned} l_A : A \to A \quad & \text{by} \quad \delta_A(a) := \phi_{\boldsymbol{a}}, \\ l_B : B \to B \quad & \text{by} \quad l_B(b) := \psi_{\boldsymbol{b}}, \\ h_A : A \to Z_B(B^*) \quad & \text{by} \quad h_A(a) := \psi_{\boldsymbol{a}}, \\ h_B : B \to Z_A(A^*) \quad & \text{by} \quad h_B(b) := \phi_{\boldsymbol{b}}, \\ & \text{and} \\ m_0 \in M^* \quad & \text{by} \quad m_0 := \varphi_{\boldsymbol{p}}. \end{aligned}$$

Claims **1** to **6**, show that (3.1) is valid. Let δ is a $\mathfrak{T}$-module map. For every $\begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \in \mathfrak{T}$ and $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$, we have

$$\begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix}\delta\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) = \delta\left(\begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix}\begin{bmatrix} a & m \\ & b \end{bmatrix}\right). \tag{3.2}$$

Now by (3.1) and replacing 0 instead of b and m in (3.2), we get

$$\begin{bmatrix} \alpha l_A(a) & (\alpha m_0)(a) \\ & \alpha h_A(a) \end{bmatrix} = \begin{bmatrix} l_A(\alpha a) & m_0(\alpha a) \\ & h_A(\alpha a) \end{bmatrix}.$$

that shows, l_A and h_A are $\mathfrak{A}$-module map. Similarly, by (3.1) and replacing 0 instead of a and m in (3.2), we can show that l_B and h_B are $\mathfrak{A}$-module maps.

Conversely, let l_A, l_B, h_A and h_B are $\mathfrak{A}$-module maps. Let $\omega = \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \in \mathfrak{T}$ and $t = \begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$, since M is a commutative $\mathfrak{A}$-bimodule, by reusing (3.1) we have

$$\begin{aligned} \delta(\omega t) &= \begin{bmatrix} l_A(\alpha a) + h_B(\alpha b) - \alpha m m_0 & m_0(\alpha a) - \alpha b m_0 \\ & l_B(\alpha b) + h_A(\alpha a) + m_0(\alpha m) \end{bmatrix} \\ &= \begin{bmatrix} \alpha l_A(a) + \alpha h_B(b) - \alpha m m_0 & (\alpha m_0)(a) - \alpha b m_0 \\ & \alpha l_B(b) + \alpha h_A(a) + (\alpha m_0)(m) \end{bmatrix} \\ &= \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \begin{bmatrix} l_A(a) + h_B(b) - m m_0 & m_0 a - b m_0 \\ & l_B(b) + h_A(a) + m_0 m \end{bmatrix} \\ &= \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \delta\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) \\ &= \omega\delta(t), \end{aligned}$$

that shows, δ is a $\mathfrak{T}$-module map and the proof is complete. □

By remark 1.3, a special form of the previous proposition is as follows, which we omit to prove

Proposition 3.2. *A map $\delta : \mathcal{T} \longrightarrow \mathcal{T}^*$ is a Lie derivation if and only if δ is of the form*

$$\delta\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) = \begin{bmatrix} l_A(a) + h_B(b) - m m_0 & m_0 a - b m_0 \\ & l_B(b) + h_A(a) + m_0 m \end{bmatrix},$$

where $m_0 \in M^*$, $l_A : A \longrightarrow A^*$and $l_B : B \longrightarrow B^*$ are Lie derivations, $h_A : A \longrightarrow Z_B(B^*)$ and $h_B : B \longrightarrow Z_A(A^*)$ are linear maps vanishing on each commutator.

Theorem 3.3. *Let $\delta : \mathcal{T} \longrightarrow \mathcal{T}^*$ be a ($\mathfrak{T}$-module) Lie derivation as above. Then, δ is standard if and only if both $l_A : A \longrightarrow A^*$ and $l_B : B \longrightarrow B^*$ are standard.*

Proof. We provide the proof in the general state (module state). Suppose $\mathfrak{T}$-module Lie derivation $\delta : \mathcal{T} \to \mathcal{T}^*$ is standard, written as $d+\tau$, where $d : \mathcal{T} \to \mathcal{T}^*$ is an $\mathfrak{T}$-module derivation and $\tau : \mathcal{T} \to Z_{\mathcal{T}}(\mathcal{T}^*)$ is an $\mathfrak{T}$-module map vanishing on each commutator. According to [7, Lemma 1.1], there exist $\mathfrak{A}$-module derivations $l'_A : A \to A^*$ and $l'_B : B \to B^*$ and an element $\gamma \in M^*$ such that

$$d\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix}\Big) = \begin{bmatrix} l'_A(a) - m\gamma & \gamma a - b\gamma \\ & l'_B(b) + \gamma m \end{bmatrix}.$$

It is easy to show that $\gamma = m_0$. Now we have,

$$\tau\left(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\right) = \delta\left(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\right) - d\left(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} (l_A - l'_A)(a) & 0 \\ & h_A(a) \end{bmatrix}.$$

So we observe that,

$$\begin{bmatrix} (l_A - l'_A)(a) & 0 \\ & h_A(a) \end{bmatrix} \in Z_{\mathcal{T}}(\mathcal{T}^*) = \begin{bmatrix} Z_A(A^*) & \\ & Z_B(B^*) \end{bmatrix}.$$

This means that, $(l_A - l'_A)(a) \in Z_A(A^*)$. We now define maps $\tau_A : A \to Z_A(A^*)$ by $\tau_A(a) = (l_A - l'_A)(a)$. Since l_A and l'_A are $\mathfrak{A}$-module Lie derivations,τ_A is an $\mathfrak{A}$-module (Lie derivation) map such that

$$\begin{aligned}\tau_A([a, a']) &= [\tau_A(a), a'] + [a, \tau_A(a')] \\ &= \tau_A(a)a' - a'\tau_A(a) + a\tau_A(a') - \tau_A(a')a \\ &= \tau_A(a)a' - \tau_A(a)a' + \tau_A(a')a - \tau_A(a')a \\ &= 0,\end{aligned}$$

where the third equation holds because of $\tau_A(A) \subseteq Z_A(A^*)$. This means that τ_A is vanishing on each commutator. Therefore, the decomposition of $l_A = l'_A + \tau_A$ requires all the conditions to be standard. Similarly we can show that, l_B is standard.

Conversely, suppose $\delta : \mathcal{T} \to \mathcal{T}^*$ is a $\mathfrak{T}$-module Lie derivation of the form (3.1) and l_A and l_B are standard, that is, $l_A = l'_A + \tau_A$ and $l_B = l'_B + \tau_B$, which $l'_A : A \to A^*$ and $l'_B : B \to B^*$ are $\mathfrak{A}$-module Lie derivations and $\tau_A : A \to Z_A(A^*)$ and $\tau_B : B \to Z_B(B^*)$ are $\mathfrak{A}$-module maps vanishing at commutators. According to [7, Lemma 1.1], the mapping $d : \mathcal{T} \to \mathcal{T}^*$ defined by

$$d\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) := \begin{bmatrix} l'_A(a) - mm_0 & m_0a - bm_0 \\ & l'_B(b) + m_0m \end{bmatrix},$$

is $\mathfrak{T}$-module derivation. Now define the map $\tau : \mathcal{T} \to Z_{\mathcal{T}}(\mathcal{T}^*)$ by

$$\tau\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) := \begin{bmatrix} h_B(b) + \tau_A(a) & 0 \\ & h_A(a) + \tau_B(b) \end{bmatrix}.$$

Clearly, $\delta = d + \tau$ and τ is a $\mathfrak{T}$-module map, because h_A, h_B, τ_A, and τ_B are $\mathfrak{A}$-module maps. Now to complete the proof it suffices to show that τ is vanishing at commutators. Assuming

$$t = \begin{bmatrix} a & m \\ & b \end{bmatrix}, t = \begin{bmatrix} a' & m' \\ & b' \end{bmatrix} \in \mathcal{T},$$

we have

$$\begin{aligned}\tau([t,t']) &= \tau\left(\begin{bmatrix}[a,a'] & am'+mb'-a'm-m'b\\ & [b,b']\end{bmatrix}\right)\\ &= \begin{bmatrix}h_B([b,b'])+\tau_A([a,a']) & 0\\ & h_A([a,a'])+\tau([b,b'])\end{bmatrix}\\ &= \begin{bmatrix}0 & 0\\ & 0\end{bmatrix}.\end{aligned}$$

Therefore, δ is standard. □

Finally, as a direct consequence of Proposition 3.1 and Theorem 3.3, the following theorem is obtained

Theorem 3.4. *Every ($\mathfrak{T}$-module) Lie derivation on $\mathcal{T}$ is standard if and only if every ($\mathfrak{A}$-module) Lie derivation on corner algebras A and B is standard.*

Remark 3.5. The authors of this paper speculate that the results of this paper are also correct for the case where A and B has a bounded approximate identity.

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References

1. M. Amini, Module amenability for semigroup algebras, *Semigroup Forum*, **69** (2004), 243–254.
2. M. Amini and D. E. Bagha, Weak module amenability for semigroup algebras, *Semigroup Forum*, **71** (2005), 18–26.
3. W. Cheung, Lie derivations of triangular algebras, *Linear Multilinear Algebra*, **51**(3) (2003), 299–310.
4. B. E. Forrest and L. W. Marcoux, Derivation of triangular Banach algebras, *Indiana University Mathematics Society*, **45** (1996), 441–462.
5. B. E. Forrest and L. W. Marcoux, Weak amenability of triangular Banach algebras, *Trans. Amer. Math. Soc.*, **354** (2002), 1435–1452.
6. E. Nasrabadi, Weak module amenability of triangular Banach algebras II, *Math. Slovaca*, **69**(2) (2019), 425–432.
7. A. Pourabbas and E. Nasrabadi, Weak module amenability of triangular Banach algebras, *Math. Slovaca*, **61** (2011), 949–958.

Mohammad Reza Miri
Department of Mathematics, University of Birjand, P.O. Box 9717434765, Birjand, Iran.
Email: mrmiri@birjand.ac.ir

Ebrahim Nasrabadi
Department of Mathematics, University of Birjand, P.O. Box 9717434765, Birjand, Iran.
Email: nasrabadi@birjand.ac.ir

Ali Reza Ghorchizadeh
Department of Mathematics, University of Birjand, P.O. Box 9717434765, Birjand, Iran.
Email: alireza.ghorchizadeh@birjand.ac.ir

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THE STRUCTURE OF MODULE LIE DERIVATIONS ON TRIANGULAR BANACH ALGEBRAS

M. R. MIRI, E. NASRABADI AND A. R. GHORCHIZADEH

ساختار اشتقاق‌های لی مدولی روی جبرهای مثلثی باناخ

محمد رضا میری[۱] ، ابراهیم نصرآبادی[۲] و علیرضا قورچی زاده[۳]

[۱,۲,۳]گروه ریاضی، دانشگاه بیرجند، بیرجند، ایران

در این مقاله، ما مفهوم اشتقاق‌های لی مدولی روی جبرهای باناخ را معرفی می‌کنیم. همچنین، اشتقاق‌های لی مدولی از جبر مثلثی باناخ یکانیِ $\mathcal{T} = \begin{bmatrix} A & M \\ & B \end{bmatrix}$ دوگان آن را مطالعه می‌کنیم. در واقع، نشان می‌دهیم که هر لی مدولی (خطی) $\delta : \mathcal{T} \to \mathcal{T}^*$ می‌تواند به صورت $\delta = d + \tau$ تجزیه شود، وقتی $d : \mathcal{T} \to \mathcal{T}^*$ یک اشتقاق مدولی (خطی) و $\tau : \mathcal{T} \to Z_{\mathcal{T}}(\mathcal{T}^*)$ یک نگاشت مدولی (خطی) است که روی جابجاگرها صفر می‌شود اگر و تنها اگر این اتفاق برای هرکدام از جبرهای گوشه‌ای A و B رخ دهد.

کلمات کلیدی: جبر مثلثی باناخ، اشتقاق لی مدولی، اشتقاق لی استاندارد.