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TWO PROPERTIES OF COUSIN FUNCTORS

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Abstract. Let R be a commutative Noetherian ring with non-zero identity and $\mathcal{F}$ a filtration of $\operatorname{Spec}(R)$. We show that the Cousin functor with respect to $\mathcal{F}$, $C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \longrightarrow \operatorname{Comp}(R)$, where $\mathcal{C}_{\mathcal{F}}(R)$ is the category of R-modules which are admitted by $\mathcal{F}$ and $\operatorname{Comp}(R)$ is the category of complexes of R-modules, commutes with the formation of direct limits and is right exact. We observe that an R-module X is balanced big Cohen-Macaulay if $(R, \mathfrak{m})$ is a local ring, $\mathfrak{m}X \neq X$, and every finitely generated submodule of X is a big Cohen-Macaulay R-module with respect to some system of parameters for R.

1. Introduction

Throughout R will denote a commutative Noetherian ring with non-zero identity. For basic results, notations, and terminology not given in this paper, readers are referred to [2, 3, 6].

The notion of Cousin complex was introduced in [4] and it has a commutative algebra analogue given by Sharp in [10]. In [7], Sharp generalized this concept to the Cousin complex for an R-module X with respect to a filtration $\mathcal{F}$ of $\operatorname{Spec}(R)$ and denoted this complex by $\mathrm{C}_R(\mathcal{F}, X)$. He approved it as a powerful tool by characterizing Gorenstein rings, Cohen-Macaulay modules, local cohomology modules, and balanced big Cohen-Macaulay modules in terms of Cousin complexes (see [10, Theorem 5.4], [8, Theorem 2.4], [9, Theorem], and [7, Corollary 3.7]).

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Let $\mathcal{C}_{\mathcal{F}}(R)$ be the category of R-modules which are admitted by $\mathcal{F}$ and let $\mathrm{Comp}(R)$ be the category of complexes of R-modules. In [1], Bamdad and the first author introduced the Cousin functor with respect to $\mathcal{F}$, $C_R(\mathcal{F},-) : \mathcal{C}_{\mathcal{F}}(R) \longrightarrow \mathrm{Comp}(R)$. They used this functor to construct Cousin spectral sequences with respect to $\mathcal{F}$ and study the extension functors of Cousin cohomologies (i.e., the cohomology modules of Cousin complexes). By using this functor, they also found some equivalent conditions for vanishing of Cousin cohomologies and gave some results for modules with finite (i.e., finitely generated) Cousin cohomologies.

In this paper, we study Cousin functors and show that they commute with direct limits. We also prove that they are right exact. As a consequence, we observe that, in the case that R is local with maximal ideal $\mathfrak{m}$, an R-module X is balanced big Cohen-Macaulay if $\mathfrak{m}X \neq X$ and every finite submodule of X is a big Cohen-Macaulay R-module with respect to some system of parameters for R.

2. Cousin functors commute with direct limits

A filtration of $\mathrm{Spec}(R)$ is a descending sequence $\mathcal{F} = (F_i)_{i\geq 0}$ of subsets of $\mathrm{Spec}(R)$, so that

$$\mathrm{Spec}(R) \supseteq F_0 \supseteq F_1 \supseteq \cdots \supseteq F_i \supseteq F_{i+1} \supseteq \cdots ,$$

with the property that, for all $i \geq 0$, $F_i \setminus F_{i+1}$ is low with respect to F_i (i.e., each member of $F_i \setminus F_{i+1}$ is a minimal member of F_i with respect to inclusion). We say that $\mathcal{F}$ admits an R-module X if $\mathrm{Supp}_R(X) \subseteq F_0$.

Suppose that $\mathcal{F} = (F_i)_{i\geq 0}$ is a filtration of $\mathrm{Spec}(R)$ which admits an R-module X. The Cousin complex $C_R(\mathcal{F}, X)$ for X with respect to $\mathcal{F}$ is of the form

$$0 \xrightarrow{d_X^{-2}} X \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^{0}} \cdots \xrightarrow{d_X^{i-3}} X^{i-2} \xrightarrow{d_X^{i-2}} X^{i-1} \xrightarrow{d_X^{i-1}} X^{i} \xrightarrow{d_X^{i}} \cdots$$

where, for all $i \geq 0$,

(D1) $X^i = \bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (\mathrm{Coker}\, d_X^{i-2})_{\mathfrak{p}}$ and
(D2) $d_X^{i-1}(x) = \{(x+\mathrm{Im}\, d_X^{i-2})/1\}_{\mathfrak{p}\in F_i\setminus F_{i+1}}$ for every element x of X^{i-1};

and satisfies

(P1) $\mathrm{Supp}_R(X^i) \subseteq \mathrm{Supp}_R(X) \cap F_i$,
(P2) $\mathrm{Supp}_R(\mathrm{Coker}\, d_X^{i-2}) \subseteq \mathrm{Supp}_R(X) \cap F_i$,
(P3) $\mathrm{Supp}_R(\mathrm{H}^{i-1}(C_R(\mathcal{F}, X))) \subseteq \mathrm{Supp}_R(X) \cap F_{i+1}$, and
(P4) the natural R-homomorphism $\xi_{X^i} : X^i \longrightarrow \bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (X^i)_{\mathfrak{p}}$, where $\xi_{X^i}(x) = \{x/1\}_{\mathfrak{p}\in F_i\setminus F_{i+1}}$ for every element x of X^i, is an isomorphism

(see [10, Proposition 2.2 and Corollary 2.3], [7, Definitions 1.1, Definition 1.3, and Proposition 1.4], and [5, Proposition 1.1 and Lemma 1.2]). We adopt the convention that $X^{-1} = X$.

Bamdad and the first author proved the following lemma and used it to introduce the Cousin functor with respect to $\mathcal{F}$,

$$C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \longrightarrow \mathrm{Comp}(R),$$

which is R-linear and covariant (see [1, Theorem 2.2]).

Lemma 2.1. (see [1, Lemma 2.1]) *Let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of* $\mathrm{Spec}(R)$ *which admits R-modules X and Y. Then for every R-homomorphism $f : X \longrightarrow Y$, there exists a unique morphism of complexes*

$$C_R(\mathcal{F}, f) = (f^i)_{i\geq -2} : C_R(\mathcal{F}, X) \longrightarrow C_R(\mathcal{F}, Y)$$

such that $f^{-1} = f$.

In the above lemma, one can see that

(P5) $\xi_{Y^i} f^i = (\bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (f^i)_{\mathfrak{p}})\xi_{X^i}$ for all $i \geq 0$.

In the following theorem, we prove that the Cousin functor with respect to $\mathcal{F}$ preserves direct limits.

Theorem 2.2. *Let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of* $\mathrm{Spec}(R)$*. Then the Cousin functor with respect to $\mathcal{F}$, $C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \longrightarrow \mathrm{Comp}(R)$, commutes with the formation of direct limits.*

Proof. Assume that $(\Lambda, \preceq)$ is a (non-empty) directed partially ordered set and that $(\{X_\alpha\}_{\alpha\in\Lambda}, \{\phi^\alpha_\beta : X_\alpha \longrightarrow X_\beta\}_{\alpha\preceq\beta})$ is a direct system in $\mathcal{C}_{\mathcal{F}}(R)$ with direct limit $(\varinjlim X_\alpha, \{\phi_\alpha : X_\alpha \longrightarrow \varinjlim X_\alpha\}_{\alpha\in\Lambda})$. $(\{C_R(\mathcal{F}, X_\alpha)\}_{\alpha\in\Lambda}, \{C_R(\mathcal{F}, \phi^\alpha_\beta) : C_R(\mathcal{F}, X_\alpha) \longrightarrow C_R(\mathcal{F}, X_\beta)\}_{\alpha\preceq\beta})$ is a direct system in $\mathrm{Comp}(R)$ because $C_R(\mathcal{F}, -)$ is a covariant functor. Set $(\varinjlim C_R(\mathcal{F}, X_\alpha), \{\psi_\alpha\}_{\alpha\in\Lambda})$ be the direct limit of the later direct system where

$$\varinjlim C_R(\mathcal{F}, X_\alpha) = 0 \xrightarrow{\overrightarrow{d^{-2}}} \varinjlim X_\alpha \xrightarrow{\overrightarrow{d^{-1}}} \varinjlim (X_\alpha)^0 \xrightarrow{\overrightarrow{d^{0}}} \cdots \xrightarrow{\overrightarrow{d^{i-1}}} \varinjlim (X_\alpha)^i \xrightarrow{\overrightarrow{d^{i}}} \cdots .$$

Since $(\varinjlim X_\alpha, \{\phi_\alpha : X_\alpha \longrightarrow \varinjlim X_\alpha\}_{\alpha\in\Lambda})$ is the direct limit of $(\{X_\alpha\}_{\alpha\in\Lambda}, \{\phi^\alpha_\beta : X_\alpha \longrightarrow X_\beta\}_{\alpha\preceq\beta})$,

- $C_R(\mathcal{F}, \varinjlim X_\alpha)$ is an object in $\mathrm{Comp}(R)$,
- $C_R(\mathcal{F}, \phi_\alpha) : C_R(\mathcal{F}, X_\alpha) \longrightarrow C_R(\mathcal{F}, \varinjlim X_\alpha)$ is a morphism in $\mathrm{Comp}(R)$ for all $\alpha \in \Lambda$, and
- $C_R(\mathcal{F}, \phi_\beta)C_R(\mathcal{F}, \phi^\alpha_\beta) = C_R(\mathcal{F}, \phi_\alpha)$ for all $\alpha \preceq \beta$.

Thus there exists a unique morphism

$$f = (f^i)_{i\geq -2} : \varinjlim C_R(\mathcal{F}, X_\alpha) \longrightarrow C_R(\mathcal{F}, \varinjlim X_\alpha)$$

such that $f\psi_\alpha = C_R(\mathcal{F}, \phi_\alpha)$ for all $\alpha \in \Lambda$. Therefore we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \xrightarrow{\overrightarrow{d^{-2}}} & \varinjlim X_\alpha & \xrightarrow{\overrightarrow{d^{-1}}} & \varinjlim(X_\alpha)^0 & \xrightarrow{\overrightarrow{d^{0}}} \cdots \xrightarrow{\overrightarrow{d^{i-1}}} & \varinjlim(X_\alpha)^i & \xrightarrow{\overrightarrow{d^{i}}} & \cdots \\
 & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{0}} & & \downarrow{\scriptstyle f^{i}} & & \\
0 & \xrightarrow{d^{-2}_{\varinjlim X_\alpha}} & \varinjlim X_\alpha & \xrightarrow{d^{-1}_{\varinjlim X_\alpha}} & (\varinjlim X_\alpha)^0 & \xrightarrow{d^{0}_{\varinjlim X_\alpha}} \cdots \xrightarrow{d^{i-1}_{\varinjlim X_\alpha}} & (\varinjlim X_\alpha)^i & \xrightarrow{d^{i}_{\varinjlim X_\alpha}} & \cdots .
\end{array}$$

By using induction on $i \geq -1$, we prove that f^i is an isomorphism. The case $i = -1$ is clear. Suppose that $i \geq 0$ and that f^j is an isomorphism for all $-1 \leq j \leq i-1$. Let $\mathfrak{p} \in F_i \setminus F_{i+1}$. By (P1) and (P3),

$$\begin{array}{ccccccc}
(\varinjlim(X_\alpha)^{i-2})_\mathfrak{p} & \xrightarrow{(\overrightarrow{d^{i-2}})_\mathfrak{p}} & (\varinjlim(X_\alpha)^{i-1})_\mathfrak{p} & \xrightarrow{(\overrightarrow{d^{i-1}})_\mathfrak{p}} & (\varinjlim(X_\alpha)^{i})_\mathfrak{p} & \xrightarrow{(\overrightarrow{d^{i}})_\mathfrak{p}} & 0 \\
\downarrow{\scriptstyle (f^{i-2})_\mathfrak{p}} & & \downarrow{\scriptstyle (f^{i-1})_\mathfrak{p}} & & \downarrow{\scriptstyle (f^{i})_\mathfrak{p}} & & \\
((\varinjlim X_\alpha)^{i-2})_\mathfrak{p} & \xrightarrow{(d^{i-2}_{\varinjlim X_\alpha})_\mathfrak{p}} & ((\varinjlim X_\alpha)^{i-1})_\mathfrak{p} & \xrightarrow{(d^{i-1}_{\varinjlim X_\alpha})_\mathfrak{p}} & ((\varinjlim X_\alpha)^{i})_\mathfrak{p} & \xrightarrow{(d^{i}_{\varinjlim X_\alpha})_\mathfrak{p}} & 0
\end{array}$$

is a commutative diagram with exact rows. Hence $(f^i)_\mathfrak{p}$ is an isomorphism from the Five Lemma [6, Proposition 2.72]. Thus

$$\bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (f^i)_\mathfrak{p} : \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (\varinjlim(X_\alpha)^i)_\mathfrak{p} \longrightarrow \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} ((\varinjlim X_\alpha)^i)_\mathfrak{p}$$

is an isomorphism. On the other hand, since

$$\xi_{(X_\alpha)^i} : (X_\alpha)^i \longrightarrow \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} ((X_\alpha)^i)_\mathfrak{p}$$

is an isomorphism for all $\alpha \in \Lambda$ by (P4),

$$\overrightarrow{\xi_i} : \varinjlim(X_\alpha)^i \longrightarrow \varinjlim(\bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} ((X_\alpha)^i)_\mathfrak{p})$$

is an isomorphism. Let

$$\nu : \varinjlim(\bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} ((X_\alpha)^i)_\mathfrak{p}) \longrightarrow \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} \varinjlim((X_\alpha)^i)_\mathfrak{p}$$

and

$$\mu : \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} \varinjlim((X_\alpha)^i)_\mathfrak{p} \longrightarrow \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (\varinjlim(X_\alpha)^i)_\mathfrak{p}$$

be the natural isomorphisms. Thus

$$\xi^{-1}_{(\varinjlim X_\alpha)^i}(\bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (f^i)_\mathfrak{p})\mu\nu\overrightarrow{\xi_i} : \varinjlim(X_\alpha)^i \longrightarrow (\varinjlim X_\alpha)^i$$

is an isomorphism. Since $\mu\nu\overrightarrow{\xi_i} = \xi_{\varinjlim(X_\alpha)^i}$ and

$$\xi_{(\varinjlim X_\alpha)^i} f^i = (\bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (f^i)_{\mathfrak{p}})\xi_{\varinjlim (X_\alpha)^i},$$

f^i is an isomorphism as we desired. □

The following result is an immediate application of the above theorem.

Corollary 2.3. *Let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of* $\operatorname{Spec}(R)$*. Then the Cousin functor with respect to $\mathcal{F}$, $C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \longrightarrow \operatorname{Comp}(R)$, commutes with direct sums.*

Proof. This follows from Theorem 2.2. □

Since each R-module can be viewed as the direct limit of its finite submodules and homology functors commute with the formation of direct limits, we have the following corollaries.

Corollary 2.4. *Let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of* $\operatorname{Spec}(R)$ *which admits an R-module X. Then*

$$C_R(\mathcal{F}, X) \cong \varinjlim_{\alpha\in\Lambda} C_R(\mathcal{F}, X_\alpha),$$

where X_α is a finite submodule of X for all $\alpha \in \Lambda$.

Proof. It follows from Theorem 2.2. □

Corollary 2.5. *Let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of* $\operatorname{Spec}(R)$ *which admits an R-module X. Then, for all $i \geq -1$,*

$$\mathrm{H}^i(C_R(\mathcal{F}, X)) \cong \varinjlim_{\alpha\in\Lambda} \mathrm{H}^i(C_R(\mathcal{F}, X_\alpha)),$$

where X_α is a finite submodule of X for all $\alpha \in \Lambda$.

Proof. The assertion follows from Corollary 2.4. □

Corollary 2.6. *Suppose that $\mathcal{F} = (F_i)_{i\geq 0}$ is a filtration of* $\operatorname{Spec}(R)$ *which admits an R-module X. Assume also that $X \cong \varinjlim_{\alpha\in\Lambda} X_\alpha$ where, for all $\alpha \in \Lambda$, X_α is a finite submodule of X such that $C_R(\mathcal{F}, X_\alpha)$ is exact. Then $C_R(\mathcal{F}, X)$ is exact. In particular, $C_R(\mathcal{F}, X)$ is exact if $C_R(\mathcal{F}, Y)$ is exact for every finite submodule Y of X.*

Proof. This follows from Theorem 2.2. □

Let R be a local ring, X an arbitrary R-module, and $r_1, \ldots, r_n$ a system of parameters for R. Recall that, X is said to be big Cohen-Macaulay with respect to $r_1, \ldots, r_n$ if $r_1, \ldots, r_n$ is a regular sequence on X. Also, X is said to be balanced big Cohen-Macaulay if X is big Cohen-Macaulay with respect to every system of parameters for R. It is well known that if X is a finite big Cohen-Macaulay with respect to some system of parameters for R, then X is a balanced big Cohen-Macaulay R-module.

Corollary 2.7. *Suppose that R is a local ring with maximal ideal $\mathfrak{m}$ and that X is an arbitrary R-module such that $\mathfrak{m}X \neq X$. Assume also that $X \cong \varinjlim_{\alpha\in\Lambda} X_\alpha$ where, for all $\alpha \in \Lambda$, X_α is a finite submodule of X and also is a big Cohen-Macaulay R-module with respect to some system of parameters for R. Then X is balanced big Cohen-Macaulay. In particular, X is balanced big Cohen-Macaulay if every finite submodule of X is a big Cohen-Macaulay R-module with respect to some system of parameters for R.*

Proof. By assumptions, for all $\alpha \in \Lambda$, X_α is a balanced big Cohen-Macaulay R-module. Thus, from [7, Corollary 3.7] (or [1, Corollary 4.6]), for all $\alpha \in \Lambda$, the Cousin complex for X_α with respect to the dimension filtration (i.e., $\mathcal{D}(R) = (D_i(R))_{i\geq 0}$ where

$$D_i(R) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \dim(R) - \dim_R(R/\mathfrak{p}) \geq i\}$$

for all $i \geq 0$) is exact. Hence the Cousin complex for X with respect to the dimension filtration is exact by Corollary 2.6. Therefore, again from [7, Corollary 3.7] (or [1, Corollary 4.6]), X is a balanced big Cohen-Macaulay R-module. □

3. Cousin functors are right exact

In this section, we show that the Cousin functor with respect to $\mathcal{F}$ is right exact. We need the following lemma for this purpose.

Lemma 3.1. *Let*

$$\begin{array}{ccccccc}
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \longrightarrow & 0 \\
\downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \gamma'} & & \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
X'' & \xrightarrow{f''} & Y'' & \xrightarrow{g''} & Z'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

be a commutative diagram of R-modules with exact columns. If the top two rows are exact, then the bottom row is exact.

Proof. This is easy and left to the reader. □

Now, we can state and prove the main result of this section.

Theorem 3.2. *Let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of* $\operatorname{Spec}(R)$. *Then the Cousin functor with respect to $\mathcal{F}$, $C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \longrightarrow \operatorname{Comp}(R)$, is right exact.*

Proof. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an exact sequence in $\mathcal{C}_{\mathcal{F}}(R)$. By Lemma 2.1, there exist morphisms of complexes

$$C_R(\mathcal{F}, f) = (f^i)_{i\geq -2} : C_R(\mathcal{F}, X) \longrightarrow C_R(\mathcal{F}, Y)$$

and

$$C_R(\mathcal{F}, g) = (g^i)_{i\geq -2} : C_R(\mathcal{F}, Y) \longrightarrow C_R(\mathcal{F}, Z)$$

such that $f^{-1} = f$ and $g^{-1} = g$. Thus

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow{\scriptstyle d_X^{-2}} & & \downarrow{\scriptstyle d_Y^{-2}} & & \downarrow{\scriptstyle d_Z^{-2}} & & \\
X & \xrightarrow{f^{-1}} & Y & \xrightarrow{g^{-1}} & Z & \longrightarrow & 0 \\
\downarrow{\scriptstyle d_X^{-1}} & & \downarrow{\scriptstyle d_Y^{-1}} & & \downarrow{\scriptstyle d_Z^{-1}} & & \\
X^0 & \xrightarrow{f^0} & Y^0 & \xrightarrow{g^0} & Z^0 & \longrightarrow & 0 \\
\downarrow{\scriptstyle d_X^{0}} & & \downarrow{\scriptstyle d_Y^{0}} & & \downarrow{\scriptstyle d_Z^{0}} & & \\
\vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle d_X^{i-1}} & & \downarrow{\scriptstyle d_Y^{i-1}} & & \downarrow{\scriptstyle d_Z^{i-1}} & & \\
X^i & \xrightarrow{f^i} & Y^i & \xrightarrow{g^i} & Z^i & \longrightarrow & 0 \\
\downarrow{\scriptstyle d_X^{i}} & & \downarrow{\scriptstyle d_Y^{i}} & & \downarrow{\scriptstyle d_Z^{i}} & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

is a commutative diagram of R-modules. By using induction on i, we prove that

$$X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i \longrightarrow 0$$

is an exact sequence of R-modules. The case $i = -1$ is clear. Suppose that $i \geq 0$ and

$$X^j \xrightarrow{f^j} Y^j \xrightarrow{g^j} Z^j \longrightarrow 0$$

is exact for all $-1 \le j \le i-1$. Let $\mathfrak{p} \in F_i \setminus F_{i+1}$. By (P1) and (P3),

$$\begin{array}{ccccccc}
(X^{i-2})_{\mathfrak{p}} & \xrightarrow{(f^{i-2})_{\mathfrak{p}}} & (Y^{i-2})_{\mathfrak{p}} & \xrightarrow{(g^{i-2})_{\mathfrak{p}}} & (Z^{i-2})_{\mathfrak{p}} & \longrightarrow & 0 \\
\downarrow{\scriptstyle (d_X^{i-2})_{\mathfrak{p}}} & & \downarrow{\scriptstyle (d_Y^{i-2})_{\mathfrak{p}}} & & \downarrow{\scriptstyle (d_Z^{i-2})_{\mathfrak{p}}} & & \\
(X^{i-1})_{\mathfrak{p}} & \xrightarrow{(f^{i-1})_{\mathfrak{p}}} & (Y^{i-1})_{\mathfrak{p}} & \xrightarrow{(g^{i-1})_{\mathfrak{p}}} & (Z^{i-1})_{\mathfrak{p}} & \longrightarrow & 0 \\
\downarrow{\scriptstyle (d_X^{i-1})_{\mathfrak{p}}} & & \downarrow{\scriptstyle (d_Y^{i-1})_{\mathfrak{p}}} & & \downarrow{\scriptstyle (d_Z^{i-1})_{\mathfrak{p}}} & & \\
(X^{i})_{\mathfrak{p}} & \xrightarrow{(f^{i})_{\mathfrak{p}}} & (Y^{i})_{\mathfrak{p}} & \xrightarrow{(g^{i})_{\mathfrak{p}}} & (Z^{i})_{\mathfrak{p}} & \longrightarrow & 0 \\
\downarrow{\scriptstyle (d_X^{i})_{\mathfrak{p}}} & & \downarrow{\scriptstyle (d_Y^{i})_{\mathfrak{p}}} & & \downarrow{\scriptstyle (d_Z^{i})_{\mathfrak{p}}} & & \\
0 & & 0 & & 0 & &
\end{array}$$

is a commutative diagram with exact columns. On the other hand, by the induction hypothesis, the top two rows are also exact. Therefore

$$(X^i)_{\mathfrak{p}} \xrightarrow{(f^i)_{\mathfrak{p}}} (Y^i)_{\mathfrak{p}} \xrightarrow{(g^i)_{\mathfrak{p}}} (Z^i)_{\mathfrak{p}} \longrightarrow 0$$

is exact from Lemma 3.1. Thus

$$\bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (X^i)_{\mathfrak{p}} \xrightarrow{\bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (f^i)_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (Y^i)_{\mathfrak{p}} \xrightarrow{\bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (g^i)_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (Z^i)_{\mathfrak{p}} \longrightarrow 0$$

is exact. Since, by (P5),

$$\begin{array}{ccccccc}
X^i & \xrightarrow{f^i} & Y^i & \xrightarrow{g^i} & Z^i & \longrightarrow & 0 \\
\downarrow{\scriptstyle \xi_{X^i}} & & \downarrow{\scriptstyle \xi_{Y^i}} & & \downarrow{\scriptstyle \xi_{Z^i}} & & \\
\bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (X^i)_{\mathfrak{p}} & \xrightarrow{\bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (f^i)_{\mathfrak{p}}} & \bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (Y^i)_{\mathfrak{p}} & \xrightarrow{\bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (g^i)_{\mathfrak{p}}} & \bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (Z^i)_{\mathfrak{p}} & \longrightarrow & 0.
\end{array}$$

is a commutative diagram and, by (P4), ξ_{X^i}, ξ_{Y^i}, and ξ_{Z^i} are isomorphisms,

$$X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i \longrightarrow 0$$

is exact. Thus

$$C_R(\mathcal{F}, X) \xrightarrow{C_R(\mathcal{F},f)} C_R(\mathcal{F}, Y) \xrightarrow{C_R(\mathcal{F},g)} C_R(\mathcal{F}, Z) \longrightarrow 0$$

is an exact sequence in $\mathrm{Comp}(R)$ as desired. □

Definition 3.3. Let n be a non-negative integer and let $\mathcal{F} = (F_i)_{i\ge 0}$ be a filtration of $\mathrm{Spec}(R)$ such that $F_0 = \mathrm{Spec}(R)$. We define the nth Cousin functor with respect to $\mathcal{F}$, denoted by $C_{n,R}(\mathcal{F}, -)$, to be the nth left derived functor of $C_R(\mathcal{F}, -)$. For an arbitrary R-module X, we call $C_{n,R}(\mathcal{F}, X)$ the nth Cousin complex for X with respect to $\mathcal{F}$.

Corollary 3.4. *Let n be a non-negative integer and let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of* $\operatorname{Spec}(R)$ *such that* $F_0 = \operatorname{Spec}(R)$*. Then* $C_{0,R}(\mathcal{F}, -)$ *is naturally equivalent to* $C_R(\mathcal{F}, -)$*.*

Proof. The assertion follows because $C_R(\mathcal{F}, -)$ is right exact by Theorem 3.2. □

Corollary 3.5. *Let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of* $\operatorname{Spec}(R)$ *such that* $F_0 = \operatorname{Spec}(R)$ *and let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence of R-modules. Then there is a long exact sequence

$$\cdots \longrightarrow$$

$$C_{n,R}(\mathcal{F}, X) \xrightarrow{C_{n,R}(\mathcal{F},f)} C_{n,R}(\mathcal{F}, Y) \xrightarrow{C_{n,R}(\mathcal{F},g)} C_{n,R}(\mathcal{F}, Z) \longrightarrow$$

$$\cdots \longrightarrow$$

$$C_{1,R}(\mathcal{F}, X) \xrightarrow{C_{1,R}(\mathcal{F},f)} C_{1,R}(\mathcal{F}, Y) \xrightarrow{C_{1,R}(\mathcal{F},g)} C_{1,R}(\mathcal{F}, Z) \longrightarrow$$

$$C_R(\mathcal{F}, X) \xrightarrow{C_R(\mathcal{F},f)} C_R(\mathcal{F}, Y) \xrightarrow{C_R(\mathcal{F},g)} C_R(\mathcal{F}, Z) \longrightarrow 0$$

in $\operatorname{Comp}(R)$*.*

Proof. This follows from Corollary 3.4. □

The Cousin complex is a powerful tool in commutative and homological algebra which characterizes some other commutative and homological concepts such as Gorenstein rings, Cohen-Macaulay modules, local cohomology modules, and balanced big Cohen-Macaulay modules (see [10, Theorem 5.4], [8, Theorem 2.4], [9, Theorem], and [7, Corollary 3.7]). Since the Cousin complex is isomorphic to the 0th Cousin complex by Corollary 3.4, it is natural to raise the following question.

Question 3.6. Let n be a positive integer. Does the nth Cousin complex characterize any commutative and homological concepts?

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TWO PROPERTIES OF COUSIN FUNCTORS

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دو خاصیت از تابعگون‌های کوزین

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فرض کنیم R یک حلقه نوتری جابه‌جایی با واحد ناصفر و $\mathcal{F}$ یک صافی از $\mathrm{Spec}(R)$ باشد. نشان می‌دهیم که تابعگون کوزین نسبت به $\mathcal{F}$، $C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \longrightarrow \mathrm{Comp}(R)$، که $\mathcal{C}_{\mathcal{F}}(R)$ رسته R-مدول‌هایی است که توسط $\mathcal{F}$ پذیرفته می‌شوند و $\mathrm{Comp}(R)$ رسته هم‌بافت‌های R-مدولی است، با حد مستقیم جابه‌جا شده و دقیق راست می‌باشد. مشاهده می‌کنیم که یک R-مدول مانند X کوهن-مکالی بزرگ متوازن است اگر $(R, \mathfrak{m})$ یک حلقه موضعی، $\mathfrak{m}X \neq X$ و هر زیرمدول متناهی‌مولد از X یک R-مدول کوهن-مکالی بزرگ نسبت به دستگاهی از پارامترها برای R باشد.

کلمات کلیدی: تابعگون‌های دقیق راست، تابعگون‌های کوزین، حدهای مستقیم، هم‌بافت‌های کوزین.