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ON THE STRONG DOMINATING SETS OF GRAPHS

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Abstract. Let $G = (V(G), E(G))$ be a simple graph. A set $D \subseteq V(G)$ is a strong dominating set of G, if for every vertex $x \in V(G) \setminus D$ there is a vertex $y \in D$ with $xy \in E(G)$ and $deg(x) \leq deg(y)$. The strong domination number $\gamma_{st}(G)$ is defined as the minimum cardinality of a strong dominating set. In this paper, we calculate $\gamma_{st}(G)$ for specific graphs and study the number of strong dominating sets of some graphs.

1. Introduction

Let $G = (V, E)$ be a simple graph. We use the standard graph notation ([10]). In particular, $deg(v)$, Δ and δ denote degree, maximum degree and minimum degree of G, respectively. A dominating set of a graph G is any subset S of V such that every vertex not in S is adjacent to at least one member of S. The minimum cardinality of all dominating sets of G is called the domination number of G and is denoted by $\gamma(G)$. This parameter has been extensively studied in the literature and there are hundreds of papers concerned with domination. For a detailed treatment of domination theory, the reader is referred to [10]. Also, the concept of domination and related invariants have been generalized in many ways. Most of the papers published so far deal with structural aspects of domination, trying to determine exact expressions for $\gamma(G)$ or some upper and/or lower bounds for it. The number of the dominating sets of graphs considered and studied well in the last

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decade. Regarding to enumerative side of dominating sets, Alikhani and Peng [5], have introduced the domination polynomial of a graph. The domination polynomial of graph G is the generating function for the number of dominating sets of G, i.e., $D(G,x) = \sum_{i=1}^{|V(G)|} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G with cardinality i (see [1, 5]). This polynomial and its roots has been actively studied in recent years (see for example [2]). Also some other generating functions for other kinds of dominating sets has been studied recently (see for example [8, 9]).

The corona product of two graphs G and H, $G \circ H$, is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i-th vertex of G to every vertex in the i-th copy of H. The Cartesian product of graphs G and H is a graph denoted $G\Box H$ whose vertex set is $V(G) \times V(H)$. Two vertices (g,h) and (g',h') are adjacent if either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. The join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1)$ and $v \in V(G_2)\}$.

A set $D \subseteq V(G)$ is a strong dominating set of G, if for every vertex $x \in V(G) \setminus D$ there is a vertex $y \in D$ with $xy \in E(G)$ and $deg(x) \leq deg(y)$. The strong domination number $\gamma_{st}(G)$ is defined as the minimum cardinality of a strong dominating set. A strong dominating set with cardinality $\gamma_{st}(G)$ is called a γ_{st}-set. The strong domination number was introduced in [13] and some upper bounds on this parameter presented in [12, 13]. Similar to strong domination number, a set $D \subset V$ is a weak dominating set of G if every vertex $v \in V \setminus S$ is adjacent to a vertex $u \in D$ such that $deg(v) \geq deg(u)$ (see [7]). The minimum cardinality of a weak dominating set of G is denoted by $\gamma_w(G)$. Boutrig and Chellali proved that the relation

$$\gamma_w(G) + \tfrac{3}{\Delta+1}\gamma_{st}(G) \leq n$$

holds for any connected graph of order $n \geq 3$.

In this paper, we explore the strong domination number of specific graphs. Also we study the strong domination number of some operations of two graphs. Finally, we consider the number of the strong dominating sets of graphs and investigate it for some graphs.

2. Strong domination number of some graphs

In this section, we study the strong domination number of some specific graphs. The following easy theorem shows that γ_{st}-sets of any graph does not contain vertices of degree one.

Theorem 2.1. *Suppose that G is a connected graph, $G \neq K_2$ and D is a γ_{st}-set for G. If $v \in D$, then $deg(v) \geq 2$.*

Proof. Suppose that $v \in D$ and $deg(v) \leq 1$. Since G is connected so $deg(v) = 1$ and there exist an adjacent vertex to v, say u such that $u \in V \setminus D$ and $deg(u) \geq 2$, this is a contraction. □

The following easy result is about vertices with maximum degree Δ.

Theorem 2.2.
(i) *If v is the only vertex of G with $deg(v) = \Delta$, then for any strong dominating set S of G, $v \in S$.*
(ii) *If v is the vertex of G with $deg(v) = \Delta$ such that v is not adjacent to another vertices with maximum degree, then for any strong dominating set S of G, $v \in S$.*

Let state the following observation:

Observation 2.3. [7] If P_n and C_n are the path graph and the cycle graph of order n, respectively, then

$$\gamma_{st}(P_n) = \gamma_{st}(C_n) = \lceil \frac{n}{3} \rceil.$$

The following theorem gives the strong domination number of complete and complete bipartite graphs:

Theorem 2.4.
(i) *If $K_{m,n}$ is the complete bipartite graph and $m \leq n$, then*

$$\gamma_{st}(K_{m,n}) = \begin{cases} m & m < n \\ 2 & m = n. \end{cases}$$

(ii) *If K_n is the complete graph, then $\gamma_{st}(K_n) = 1$.*

Proof.
(i) Suppose that $m < n$ and also suppose that X and Y are two parts of $K_{m,n}$ such that $|X| = m$, $|Y| = n$. Since for every $v \in X$ and $u \in Y$, $deg(v) > deg(u)$ and all vertices in X dominates all vertices in $K_{m,n}$, so $\gamma_{st}(K_{m,n}) = m$. If $m = n$, then for every $v \in X$ and $u \in Y$, $deg(v) = deg(u)$. In this case, all vertices in Y can be dominate by one vertex of X and all vertices of X can be dominate by one vertex of Y. So $\gamma_{st}(K_{m,n}) = 2$.
(ii) Since one vertex of K_n is adjacent to all vertices of it and for every $u, v \in V(K_n)$, $deg(u) = deg(v)$, so $\gamma_{st}(K_n) = 1$.

□

The friendship (or Dutch-Windmill) graph F_n is a graph that can be constructed by coalescence n copies of the cycle graph C_3 of length 3 with a common vertex, that is $F_n = K_1 \vee nK_2$. The n-book graph B_n can be constructed by bonding n copies of the cycle graph C_4 along a common edge $\{u, v\}$, that is $B_n = K_{1,n} \Box K_2$.

Theorem 2.5. *For the friendship graph F_n and the book graph B_n, we have*

$$\gamma_{st}(F_n) = 1 \text{ and } \gamma_{st}(B_n) = 2.$$

Proof. By construction of F_n there exist one vertex such that it is adjacent to other vertices in F_n, and so this vertex have maximum degree in F_n. Therefore by Theorem 2.2, $\gamma_{st}(F_n) = 1$. In the book graph B_n, there exist two vertices that they have maximum degree and also cover all vertices in B_n. So $\gamma_{st}(B_n) = 2$. □

Two following results are about the strong domination number of corona of two graphs.

Theorem 2.6. *If G_1 and G_2 are two graphs, then*

$$\gamma_{st}(G_1 \circ G_2) = |V(G_1)|.$$

Proof. By construction of $G_1 \circ G_2$, every vertex of G_1, is adjacent to all of vertices a copy of G_2. So in $G_1 \circ G_2$, for every $v \in V(G_1)$, $u \in V(G_2)$, $deg(v) \geqslant deg(u)$, therefore $\gamma_{st}(G_1 \circ G_2) = |V(G_1)|$. □

Corollary 2.7. *If G_1 and G_2 are two graphs, then*

$$\gamma_{st}(G_1 \circ G_2) = \gamma(G_1 \circ G_2).$$

Proof. By construction of $G_1 \circ G_2$, to dominate the vertices of a copy of G_2, we need one vertex of G_1, so $\gamma(G_1 \circ G_2) = |V(G_1)|$. Since by Theorem 2.6, $\gamma_{st}(G_1 \circ G_2) = |V(G_1)|$, so $\gamma_{st}(G_1 \circ G_2) = \gamma(G_1 \circ G_2)$. □

Now we state a result about the strong domination number of Cartesian product of two graphs.

Theorem 2.8. *If G and H are two graphs, then*

$$\gamma_{st}(G \Box H) \leqslant min\{|V(H)|\gamma_{st}(G), |V(G)|\gamma_{st}(H)\}.$$

Proof. Suppose that D is the $\gamma_{st}(G)$-set of G. If in every copy of G in $G \Box H$, we choose the same γ_{st}-set, then we have a strong dominating set for $G \Box H$. Similarly, we can have a strong dominating set for Cartesian product of G and H with cardinality $|V(G)|\gamma_{st}(H)$. Therefore, we have the result. □

As an immediate consequence, we have the following results for the ladder graph $L_n = P_n \Box K_2$ and tour $P_m \Box P_n$:

Corollary 2.9.

$$\gamma_{st}(L_n) \leq \begin{cases} 2\lceil \frac{n}{3} \rceil & \textit{if } n \equiv 0 \ (mod\ 3) \textit{ or } n \equiv 2 \ (mod\ 3) \\ 2\lceil \frac{n}{3} \rceil - 1 & \textit{if } n \equiv 1 \ (mod\ 3) \end{cases}$$

Corollary 2.10. *If $m, n > 2$, then*

$$\gamma_{st}(P_n \Box P_m) \leq m \cdot \gamma_{st}(P_n)$$

We close this section with the following theorem:

Theorem 2.11. *If Δ is a maximum degree of G, then*

$$\frac{n}{\Delta + 1} \leq \gamma_{st}(G) \leq n - \Delta.$$

Moreover, this inequality is sharp for the complete graph.

Proof. Suppose that $v \in V(G)$ and $deg(v) = \Delta$, vertex v dominates $\Delta + 1$ vertices of G, so the maximum number of vertices that we need for dominating other vertices are $n - (\Delta + 1)$. Therefore

$$\gamma_{st}(G) \leq n - (\Delta + 1) + 1 = n - \Delta.$$

Suppose that D is a strong dominating set with minimum cardinality, since for every $v \in D$, $deg(v) \leq \Delta$, so the maximum number of vertices which dominate by v are $\Delta + 1$, and so $n \leq \gamma_{st}(G)(\Delta + 1)$. Therefore $\frac{n}{\Delta+1} \leq \gamma_{st}(G)$. □

3. The number of strong dominating sets

In this section, we consider the number of the strong dominating sets of graphs and investigate it for some specific graphs. Let $\mathcal{D}_{st}(G, i)$ be the family of the strong dominating sets of a graph G with cardinality i and let $d_{st}(G, i) = |\mathcal{D}_{st}(G, i)|$. We denote the generating function for the number of strong dominating sets of G by $D_{st}(G, x)$ and is the polynomial

$$D_{st}(G, x) = \sum_{i=1}^{|V(G)|} d_{st}(G, i) x^i.$$

Note that it is possible that for a graph G, $\gamma(G) = \gamma_{st}(G)$, but for some k, $d(G, k) \neq d_{st}(G, k)$. For example in the tree T in Figure 1, $\gamma_{st}(T) = \gamma(T) = 4$, $d_{st}(T, 4) = 5$ and $d(T, 4) > 5$.

Theorem 3.1. *If G is a k-regular graph, then $\gamma_{st}(G) = \gamma(G)$ and $D_{st}(G, x) = D(G, x)$.*

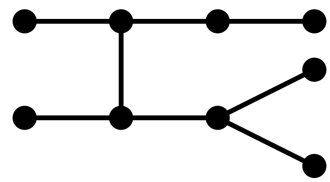

FIGURE 1. Tree T with $d(T,4) > d_{st}(T,4)$.

Proof. Since for every $u,v \in V(G)$, $deg(v) = deg(u)$, so if D is a dominating set then it is a strong dominating set, therefore $\gamma_{st}(G) = \gamma(G)$. Since every dominating sets of regular graph G is the strong dominating set of G, so $d_{st}(G,i) = d(G,i)$ and we have the result. □

Here we consider the number of strong dominating sets of cycles. Let $C_n, n \geq 3$, be the cycle with n vertices $V(C_n) = [n]$ and

$$E(C_n) = \big\{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\big\}.$$

We investigate the number of the strong dominating sets of cycles. We need the following theorem which is a recurrence relation for the number of dominating sets of a cycle:

Theorem 3.2. [3] *The number of dominating sets of cycle C_n satisfies the following recursive relation:*

$$d(C_n,i) = d(C_{n-1},i-1) + d(C_{n-2},i-1) + d(C_{n-3},i-1).$$

The following theorem gives the explicit formula for the number of dominating sets of C_n ([6]):

Theorem 3.3. [6] *For every $n \geq 3$,*

$$d(C_n,k) = \sum_{m=0}^{\lfloor\frac{n-k}{2}\rfloor+1} \binom{k-1}{n-k-m}\left(\binom{n-k-m+2}{m+2}\binom{n-k-m}{m-2}\right).$$

Using Theorems 3.1, 3.2 and 3.3 we have the following corollary which gives the number of the strong dominating sets of cycles:

Corollary 3.4. *For every $n \geq 3$,*

$$d_{st}(C_n,k) = \sum_{m=0}^{\lfloor\frac{n-k}{2}\rfloor+1} \binom{k-1}{n-k-m}\left(\binom{n-k-m+2}{m+2}\binom{n-k-m}{m-2}\right).$$

In the following theorem, we obtain the number of the strong dominating sets of complete bipartite graph.

Theorem 3.5. (i) *If $m < n$, then*

$$d_{st}(K_{m,n}, i) = \begin{cases} 0 & i < m \\ 1 & i = m \\ \binom{n}{i-m} & i > m \end{cases}$$

(ii) $D_{st}(K_{n,n}, x) = ((1+x)^n - 1)^2 + 2x^n$.

Proof. (i) By Theorem 2.4, if $m < n$ then $\gamma_{st}(K_{m,n}) = m$, so obviously for $i < m$, $d_{st}(K_{m,n}, i) = 0$ and also for $i = m$, $d_{st}(K_{m,n}, i) = 1$. Suppose that X, Y are parts of $K_{m,n}$ such that $|X| = m, |Y| = n$. Since for every $v \in X, u \in Y$,

$$deg(v) = n > deg(u) = m$$

so we should select m vertex of X and $i - m$ vertex of Y, therefore in this case $d_{st}(K_{m,n}, i) = \binom{n}{i-m}$

(ii) Since $K_{n,n}$ is n-regular graph, so by Theorem 3.1,

$$D_{st}(K_{n,n}, x) = D(K_{n,n}, x).$$

We know that $D(K_{n,n}, x) = ((1+x)^n - 1)^2 + 2x^n$ (see [5]), therefore we have the result. □

□

Here we compute the number of the strong dominating sets of the complete graph, the friendship and the book graph:

Theorem 3.6. (i) *If K_n is complete graph, then*

$$d_{st}(K_n, i) = \binom{n}{i}$$

(ii) *If F_n is the friendship graph, then*

$$d_{st}(F_n, i) = \binom{2n}{i-1}$$

(iii) *If B_n is the book graph, then*

$$d_{st}(B_n, i) = \binom{2n}{i-2}.$$

Proof. (i) Since $\gamma_{st}(K_n) = 1$ and for every $v \in V(K_n)$, $deg(v) = n - 1$ so $d_{st}(K_n, i) = \binom{n}{i}$.

(ii) To have a strong dominating set of F_n of size i, we have to choose a center vertex and $i - 1$ vertices from another $2n$ vertices. So $d_{st}(F_n, i) = \binom{2n}{i-1}$.

(iii) To have a strong dominating set of the book graph B_n we have to choose two vertices of degree two and then we choose $i - 2$ vertices from another $2n$ vertices of B_n. Therefore

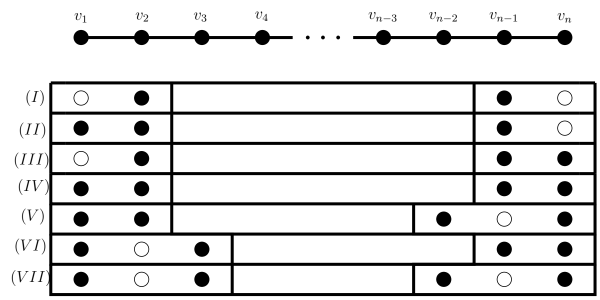

FIGURE 2. Making strong dominating sets of P_n related to the proof of Theorem 3.8

$$d_{st}(B_n, i) = \binom{2n}{i-2}.$$

□

Now, we consider the number of strong dominating sets of paths. First we recall the following theorem which is a recurrence relation for the number of dominating sets of paths.

Theorem 3.7. [4] *The number of dominating sets of path graph P_n, satisfies the following recursive relation:*

$$d(P_n, i) = d(P_{n-1}, i-1) + d(P_{n-1}, i-2) + d(P_{n-1}, i-3).$$

Now we find a lower bound for the number of strong dominating sets of paths.

Theorem 3.8. *For every $n \geq 7$, the number of strong dominating sets of path P_n with cardinality i, $d_{st}(P_n, i)$, satisfies:*

$$d_{st}(P_n, i) \geq d(P_{n-4}, i-2) + 2d(P_{n-4}, i-3) + d(P_{n-4}, i-4) + 2d(P_{n-5}, i-4) + d(P_{n-6}, i-4).$$

Proof. Since we have two vertices of degree one, then their neighbours need to be in our strong dominating set, say S. In case we do not have v_2 or v_{n-1} in S, then it is necessary to have $v_1, v_3 \in S$ or $v_{n-2}, v_n \in S$, respectively (See figure 2). We consider the following cases:

(I) $v_2, v_{n-1} \in S$ and $v_1, v_n \notin S$. We find a dominating set with cardinality $i-2$ for the rest of vertices which is a path of order $n-4$. So, in this case, we have $d(P_{n-4}, i-2)$ strong dominating sets.

(II) $v_1, v_2, v_{n-1} \in S$ and $v_n \notin S$. In this case, we find a dominating set with cardinality $i-3$ for the rest of vertices which is a path of order $n-4$. So, in this case, we have $d(P_{n-4}, i-3)$ strong dominating sets.

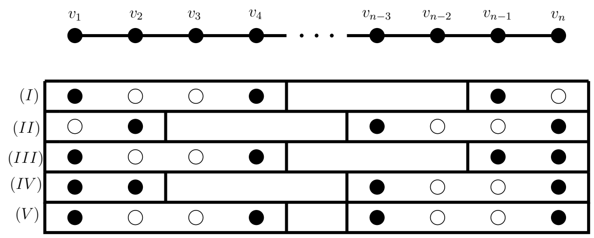

FIGURE 3. Some cases which are not strong dominating sets of P_n related to the proof of Theorem 3.9

(III) $v_2, v_{n-1}, v_n \in S$ and $v_1 \notin S$. In this case, we have similar argument as case (II).

(IV) $v_1, v_2, v_{n-1}, v_n \in S$. In this case, we find a dominating set with cardinality $i-4$ for the rest of vertices which is a path of order $n-4$. So, in this case, we have $d(P_{n-4}, i-4)$ strong dominating sets.

(V) $v_1, v_2, v_{n-2}, v_n \in S$ and $v_{n-1} \notin S$. In this case, we find a dominating set with cardinality $i-4$ for the rest of vertices which is a path of order $n-5$. So, in this case, we have $d(P_{n-5}, i-4)$ strong dominating sets.

(VI) $v_1, v_3, v_{n-1}, v_n \in S$ and $v_2 \notin S$. In this case, we have similar argument as case (V).

(VII) $v_1, v_3, v_{n-2}, v_n \in S$ and $v_2, v_{n-1} \notin S$. In this case, we find a dominating set with cardinality $i-4$ for the rest of vertices which is a path of order $n-6$. So, in this case, we have $d(P_{n-6}, i-4)$ strong dominating sets.

Hence, the number of strong dominating sets of path graph of order n with cardinality i should be greater than or equal to summation of these cases. Therefore, we have the result. □

Here we obtain an upper bound for the number of strong dominating sets of paths.

Theorem 3.9. *For every $n \geq 9$, the number of strong dominating sets of path P_n with cardinality i, $d_{st}(P_n, i)$, satisfies:*

$$d_{st}(P_n, i) \leq d(P_n, i) - 2d(P_{n-6}, i-3) - 2d(P_{n-6}, i-4) - d(P_{n-8}, i-4).$$

Proof. To find an upper bound, first we find all dominating sets of a path. Then we remove cases which are not strong dominating sets from the dominating sets. We consider the following cases (See figure 3):

(I) $v_1, v_4, v_{n-1} \in S$ and $v_2, v_3, v_n \notin S$. We find a dominating set with cardinality $i-3$ for the rest of vertices which is a path of

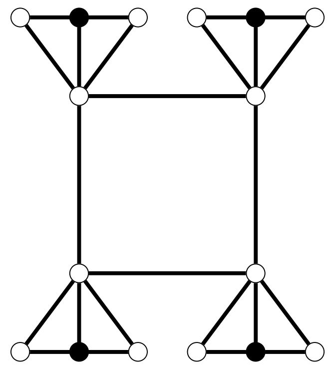

FIGURE 4. Black vertices of $C_4 \circ P_3$ is a dominating set but is not a strong dominating.

order $n-6$. So, in this case, we have $d(P_{n-6}, i-3)$ dominating sets which are not strong dominating sets.

(II) $v_2, v_{n-3}, v_n \in S$ and $v_1, v_{n-2}, v_{n-1} \notin S$. In this case, we have similar argument as case (I).

(III) $v_1, v_4, v_{n-1}, v_n \in S$ and $v_2, v_3 \notin S$. We find a dominating set with cardinality $i-4$ for the rest of vertices which is a path of order $n-6$. So, in this case, we have $d(P_{n-6}, i-4)$ dominating sets which are not strong dominating sets.

(IV) $v_1, v_2, v_{n-3}, v_n \in S$ and $v_{n-2}, v_{n-1} \notin S$. In this case, we have similar argument as case (III).

(V) $v_1, v_4, v_{n-3}, v_n \in S$ and $v_2, v_3, v_{n-2}, v_{n-1} \notin S$. We find a dominating set with cardinality $i-4$ for the rest of vertices which is a path of order $n-8$. So, in this case, we have $d(P_{n-8}, i-4)$ dominating sets which are not strong dominating sets.

Now by removing these cases from all dominating sets with cardinality i of a path, we have the result. □

We saw that $\gamma_{st}(G \circ H) = \gamma(G \circ H)$ but each dominating sets of $G \circ H$ is not necessarily a strong dominating set of $G \circ H$. As an example, the four black vertices of the graph $C_4 \circ P_3$ in Figure 4 is a dominating set but is not a strong dominating set. The following theorem gives a lower bound for the number of the strong dominating sets of the corona product of two graphs:

Theorem 3.10. *If G_1 and G_2 are two graphs of order m and n, respectively, then for $i \geq m$*

$$d_{st}(G_1 \circ G_2, i) \geq \binom{mn}{i-m}.$$

Proof. Since $\gamma_{st}(G_1 \circ G_2) = |V(G_1)| = m$, so to have a strong dominating set with i vertices, we can select m vertices of G_1 and $i-m$ vertices of other vertices of $G_1 \circ G_2$. Therefore there exist at least $\binom{mn}{i-m}$ sets that they are strong dominating sets. □

Corollary 3.11. (i) *If G is connected graph of order m, then for every $i \geq m$,*

$$d_{st}(G \circ K_1, i) \geq \binom{m}{i-m}.$$

(ii) *If P_n is the path graph, then*

$$d_{st}(P_n \circ K_1, i) \geq \binom{n}{i-n}.$$

4. Conclusions

We studied the number of the strong dominating sets for certain graphs. For some graphs we found the exact formula for the number of strong dominating sets of cardinality i, but for paths the problem looks difficult. We presented some inequalities for $d_{st}(P_n, i)$, but until now all attempts to find a formula for $d_{st}(P_n, i)$ failed. Also, we think that the study of the number of the strong dominating sets of operation of two graphs, such as join, corona, lexicographic and Cartesian is a good subject. There are recurrence relations for the number of the dominating sets of arbitrary graph G with cardinality i ([11]) and it is interesting problem to find recurrence relations for the number of strong dominating sets, too. The paper leaves some open problems, among them:

Open Problem 1. Find the explicit formulas for the number of the strong dominating sets of operation of two graphs, such as join, corona, lexicographic and Cartesian.

Open Problem 2. Find the explicit formulas for $d_{st}(P_n, i)$.

Open Problem 3. Find the recurrence relations and splitting formulas for the $d_{st}(G, i)$ using simple graph operations.

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