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VARIETIES OF PERMUTATIVE SEMIGROUPS CLOSED UNDER DOMINIONS

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ABSTRACT. In this paper, we partially generalize a result of Isbell from the class of commutative semigroups to some generalized class of commutative semigroups by showing that dominion of such semigroups belongs to the same class by using Isbell's zigzag theorem.

1. Introduction and Preliminaries

Let U be a subsemigroup of a semigroup S. Following Isbell [5], we say that U dominates an element d of S if for every semigroup T and for all homomorphisms $\beta, \gamma : S \longrightarrow T$ and $u\beta{=}u\gamma$ for every u in U implies $d\beta{=}d\gamma$. The set of all elements of S dominated by U is called dominion of U in S and we denote it by $Dom(U, S)$. It can be easily verified that $Dom(U, S)$ is a subsemigroup of S containing U.

The following theorem provided by Isbell [5], known as Isbell's zigzag theorem, is a most useful characterization of semigroup dominions and is of basic importance to our investigations.

Theorem 1.1. *([5], Theorem 2.3) Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:*

$$d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m} \quad (1.1)$$

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where $m \geq 1$, $a_i \in U$ $(i = 0, 1, \ldots, 2m)$, $y_i, t_i \in S$ $(i = 1, 2, \ldots, m)$, *and*

$$a_0 = y_1 a_1, \qquad a_{2m-1} t_m = a_{2m},$$
$$a_{2i-1} t_i = a_{2i} t_{i+1}, \qquad y_i a_{2i} = y_{i+1} a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorization is called a zigzag in S *over* U *with value* d*, length* m *and spine* $a_0, a_1, \ldots, a_{2m}$.

The following result is from Khan [7] and is also necessary for our investigations.

Theorem 1.2. *([7], Result 3) Let* U *and* S *be semigroups with* U *as a subsemigroup of* S*. Take any* $d \in S \backslash U$ *such that* $d \in Dom(U, S)$*. Let (1) be a zigzag of shortest possible length* m *over* U *with value* d*. Then* $t_j, y_j \in S \backslash U$ *for all* $j = 1, 2, \ldots, m$.

Definition 1.3. Let

$$x_1 x_2 x_3 x_4 = x_{i_1} x_{i_2} x_{i_3} x_{i_4} \tag{1.2}$$

and

$$x_1 x_2 x_3 x_4 x_5 = x_{j_1} x_{j_2} x_{j_3} x_{j_4} x_{j_5} \tag{1.3}$$

be permutation identities, where i and j are nontrivial permutations of the sets $\{1, 2, 3, 4\}$ and $\{1, 2, 3, 4, 5\}$ respectively. Then a semigroup satisfying (1.2) is called

(i) a medial semigroup if $i_2 = 3$ and $i_3 = 2$;
(ii) a right semi-commutative semigroup if $i_3 = 4$ and $i_4 = 3$;
(iii) a left semi-commutative semigroup if $i_1 = 2$ and $i_2 = 1$;
(iv) a right cyclic commutative semigroup if $i_2 = 3$, $i_3 = 4$ and $i_4 = 2$;
(v) a left cyclic commutative semigroup if $i_1 = 2$, $i_2 = 3$ and $i_3 = 1$;
(vi) a right dual-cyclic commutative semigroup if $i_2 = 4$, $i_3 = 2$ and $i_4 = 3$;
(vii) a left dual-cyclic commutative semigroup if $i_1 = 3$, $i_2 = 1$ and $i_3 = 2$;
(viii) a right externally commutative semigroup if $i_2 = 4$ and $i_4 = 2$;
(ix) a left externally commutative semigroup if $i_1 = 3$ and $i_3 = 1$;
(x) a bi-commutative semigroup if$i_1 = 2$, $i_2 = 1$, $i_3 = 4$ and $i_4 = 3$.

while satisfying (1.3) is called

(i) a cyclic semi-normal commutative semigroup if $j_2 = 3$, $j_3 = 4$ and $j_4 = 2$;

(ii) a middle right semi-commutative semigroup if $j_4 = 5$ and $j_5 = 4$;
(iii) middle left cyclic commutative semigroup if $j_1 = 2$, $J_2 = 3$ and $j_3 = 1$;
(iv) a double right semi-commutative semigroup if $j_2 = 4$, $j_3 = 5$, $j_4 = 2$ and $j_5 = 3$;
(v) a double left semi-commutative semigroup if $j_1 = 3$, $j_2 = 4$, $j_3 = 1$ and $j_4 = 2$;
(vi) a middle right dual-cyclic commutative semigroup if $j_3 = 5$, $j_4 = 3$ and $j_5 = 4$;
(vii) a middle left dual-cyclic commutative semigroup if $j_1 = 3$, $j_2 = 1$ and $j_3 = 2$;
(viii) a dual right semi-commutative semigroup if $j_2 = 5$, $j_3 = 4$, $j_4 = 2$ and $j_5 = 3$;
(ix) a middle left externally commutative semigroup if $j_1 = 3$ and $j_3 = 1$;
(x) a left dual-cyclic right semi-commutative if $j_1 = 3$, $j_2 = 1$, $j_3 = 2$, $j_4 = 5$ and $j_5 = 4$.

The semigroup theoretic notations and conventions of Clifford and Preston [3] and Howie [4] will be used throughout without explicit mention.

2. Dominions and some generalized classes of commutative semigroups

Isbell [5], Corollary 2.5, showed that the dominion of a commutative semigroup is commutative. But in [6], Khan gave a counter-example to show that this stronger result is false for each (nontrivial) permutation identity other than commutativity. Recently Alam, Higgins and Khan [2] generalized Isbell's result from commutative semigroups to $\mathcal{H}$-commutative semigroups. Also, Abbas and Ashraf in [1], found some generalized classes of commutative semigroups for which this stronger result is true in some weaker form. Further, in the same direction, I found some more permutative semigroups for which $Dom(U, S)$ satisfies the identity of U.

Theorem 2.1. *Let U be a medial sub-semigroup of a cyclic semi-normal commutative semigroup S. Then $Dom(U, S)$ is medial semigroup.*

Proof. Let U be a medial sub-semigroup of a cyclic semi-normal commutative semigroup S. Then we have to show that $Dom(U, S)$

is also medial semigroup.

Case (i): If $d_1, d_2, d_3, d_4 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U,S)\backslash U$ and $d_2, d_3, d_4 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= y_m(a_{2m}d_2d_3d_4) \text{ (by zigzag equations)} \\ &= y_m a_{2m} d_3 d_2 d_4 \text{ (since } U \text{ is medial semigroup)} \\ &= d_1d_3d_2d_4 \text{ (by zigzag equations),} \end{aligned}$$

as required.
Case (iii): $d_1, d_2 \in Dom(U,S)\backslash U$ and $d_3, d_4 \in U$.
Then, by Theorem 1.1, d_2 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= d_1a_ot_1d_3d_4 \text{ (by zigzag equations)} \\ &= d_1t_1d_3a_od_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_3x_4x_2x_5) \\ &= d_1t_1d_3(y_1a_1)d_4 \text{ (by zigzag equations)} \\ &= d_1d_3y_1a_1t_1d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_3x_4x_2x_5) \\ &= d_1d_3y_1a_2t_2d_4 \text{ (by zigzag equations)} \\ &= d_1d_3y_2a_3t_2d_4 \text{ (by zigzag equations)} \\ &\vdots \\ &= d_1d_3y_ma_{2m-1}t_md_4 \\ &= d_1d_3y_ma_{2m}d_4 \text{ (by zigzag equations)} \\ &= d_1d_3d_2d_4 \text{ (by zigzag equations),} \end{aligned}$$

as required.
Case (iv): $d_1, d_2, d_3 \in Dom(U,S)\backslash U$ and $d_4 \in U$.
Then, by Theorem 1.1, d_3 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= d_1d_2a_ot_1d_4 \text{ (by zigzag equations)} \\ &= d_1d_2(y_1a_1)t_1d_4 \text{ (by zigzag equations)} \\ &= d_1y_1a_1t_1d_2d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_3x_4x_2x_5) \\ &= d_1y_1a_2t_2d_2d_4 \text{ (by zigzag equations)} \\ &= d_1y_2a_3t_2d_2d_4 \text{ (by zigzag equations)} \\ &\vdots \end{aligned}$$

$$= d_1 y_m a_{2m-1} t_m d_2 d_4$$
$$= d_1 y_m a_{2m} d_2 d_4 \text{ (by zigzag equations)}$$
$$= d_1 d_3 d_2 d_4 \text{ (by zigzag equations),}$$

as required.

Case (v): $d_1, d_2, d_3, d_4 \in Dom(U, S) \backslash U$.

Then, by Theorem 1.1, d_4 has zigzag equations of type (1.1) in S over U of length m. Now

$$d_1 d_2 d_3 d_4 = (d_1 d_2 d_3 a_o) t_1 \text{ (by zigzag equations)}$$
$$= d_1 d_3 d_2 a_o t_1 \text{ (by case (iv))}$$
$$= d_1 d_3 d_2 d_4 \text{ (by zigzag equations),}$$

as required. Thus the proof of the theorem is completed. □

Theorem 2.2. *Let U be a right semi-commutative sub-semigroup of a middle right semi-commutative semigroup S. Then $Dom(U, S)$ is right semi-commutative semigroup.*

Proof. Let U be a right semi-commutative sub-semigroup of a middle right semi-commutative semigroup S. Then we have to show that $Dom(U, S)$ is also right semi-commutative semigroup.

Case (i): If $d_1, d_2, d_3, d_4 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U, S) \backslash U$ and $d_2, d_3, d_4 \in U$.

Then, by Theorem 1.1, d_1 has zigzag equations of type (1.1) in S over U of length m. Now

$$d_1 d_2 d_3 d_4 = y_m a_{2m} d_2 d_3 d_4 \text{ (by zigzag equations)}$$
$$= y_m a_{2m} d_2 d_4 d_3 \text{ (as } S \text{ satisfies } x_1 x_2 x_3 x_4 x_5 = x_1 x_2 x_3 x_5 x_4)$$
$$= d_1 d_2 d_4 d_3 \text{(by zigzag equations),}$$

as required.

Case (iii): $d_1, d_2 \in Dom(U, S) \backslash U$ and $d_3, d_4 \in U$.

Then, by Theorem 1.1, d_2 has zigzag equations of type (1.1) in S over U of length m. Now

$$d_1 d_2 d_3 d_4 = d_1 a_o t_1 d_3 d_4 \text{ (by zigzag equations)}$$
$$= d_1 a_o t_1 d_4 d_3 \text{ (as } S \text{ satisfies } x_1 x_2 x_3 x_4 x_5 = x_1 x_2 x_3 x_5 x_4)$$
$$= d_1 d_2 d_4 d_3 \text{ (by zigzag equations),}$$

as required.

Case (iv): $d_1, d_2, d_3 \in Dom(U, S) \backslash U$ and $d_4 \in U$.

Then, by Theorem 1.1, d_3 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned}
d_1d_2d_3d_4 &= d_1d_2y_ma_{2m}d_4 \text{ (by zigzag equations)}\\
&= d_1d_2y_md_4a_{2m} \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_3x_5x_4)\\
&= (d_1d_2y_md_4a_{2m-1})t_m \text{ (by zigzag equations)}\\
&= d_1d_2y_ma_{2m-1}d_4t_m\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_3x_5x_4)\\
&= (d_1d_2y_{m-1}a_{2m-2}d_4)t_m \text{ (by zigzag equations)}\\
&= d_1d_2y_{m-1}d_4a_{2m-2}t_m\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_3x_5x_4)\\
&= (d_1d_2y_{m-1}d_4a_{2m-3})t_{m-1} \text{ (by zigzag equations)}\\
&= d_1d_2y_{m-1}a_{2m-3}d_4t_{m-1}\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_3x_5x_4)\\
&\;\;\vdots\\
&= d_1d_2y_1a_1d_4t_1\\
&= (d_1d_2a_od_4)t_1 \text{ (by zigzag equations)}\\
&= d_1d_2d_4a_ot_1 \text{ (by case (iii))}\\
&= d_1d_2d_4d_3 \text{ (by zigzag equations),}
\end{aligned}$$

as required.

Case (v): $d_1, d_2, d_3, d_4 \in Dom(U, S)\backslash U$.

Then, by Theorem 1.1, d_4 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned}
d_1d_2d_3d_4 &= (d_1d_2d_3a_0)t_1 \text{ (by zigzag equations)}\\
&= d_1d_2a_od_3t_1 \text{ (by case (iv))}\\
&= d_1d_2a_ot_1d_3 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_3x_5x_4)\\
&= d_1d_2d_4d_3 \text{ (by zigzag equations),}
\end{aligned}$$

as required. Thus the proof of the theorem is completed. □

Theorem 2.3. *Let U be a left semi-commutative sub-semigroup of a middle left cyclic commutative semigroup S. Then $Dom(U, S)$ is left semi-commutative semigroup.*

Proof. Let U be a left semi-commutative sub-semigroup of a middle left cylic commutative semigroup S. Then we have to show that $Dom(U, S)$ is also left semi-commutative semigroup.

Case (i): $d_1, d_2, d_3, d_4 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U, S)\backslash U$ and $d_2, d_3, d_4 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1.1) in S over U of length m. Now

$$
\begin{aligned}
d_1d_2d_3d_4 &= y_m a_{2m} d_2d_3d_4 \text{ (by zigzag equations)}\\
&= a_{2m} d_2 y_m d_3 d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_2x_3x_1x_4x_5)\\
&= a_{2m-1} t_m (d_2 y_m) d_3 d_4 \text{ (by zigzag equations)}\\
&= t_m d_2 y_m a_{2m-1} d_3 d_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_2x_3x_1x_4x_5)\\
&= t_m d_2 (y_{m-1} a_{2m-2}) d_3 d_4 \text{ (by zigzag equations)}\\
&= d_2 y_{m-1} a_{2m-2} t_m d_3 d_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_2x_3x_1x_4x_5)\\
&= d_2 y_{m-1} a_{2m-3} t_{m-1} d_3 d_4 \text{ (by zigzag equations)}\\
&\ \vdots\\
&= d_2 y_1 a_1 t_1 d_3 d_4\\
&= d_2 a_o t_1 d_3 d_4 \text{ (by zigzag equations)}\\
&= d_2 d_1 d_3 d_4 \text{ (by zigzag equations),}
\end{aligned}
$$

as required.
Case (iii): $d_1, d_2 \in Dom(U, S)\backslash U$ and $d_3, d_4 \in U$.
Then, by Theorem 1.1, d_2 has zigzag equations of type (1.1) in S over U of length m. Now

$$
\begin{aligned}
d_1d_2d_3d_4 &= d_1 a_o t_1 d_3 d_4 \text{ (by zigzag equations)}\\
&= a_o t_1 d_1 d_3 d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_2x_3x_1x_4x_5)\\
&= d_2 d_1 d_3 d_4 \text{ (by zigzag equations),}
\end{aligned}
$$

as required.
Case (iv): $d_1, d_2, d_3 \in Dom(U, S)\backslash U$ and $d_4 \in U$.
Then, by Theorem 1.1, d_3 has zigzag equations of type (1.1) in S over U of length m. Now

$$
\begin{aligned}
d_1d_2d_3d_4 &= d_1 d_2 a_o t_1 d_4 \text{ (by zigzag equations)}\\
&= d_2 a_o d_1 t_1 d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_2x_3x_1x_4x_5)\\
&= (d_2 y_1) a_1 d_1 t_1 d_4 \text{ (by zigzag equations)}\\
&= a_1 d_1 (d_2 y_1) t_1 d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_2x_3x_1x_4x_5)\\
&= d_1 d_2 y_1 a_1 t_1 d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_2x_3x_1x_4x_5)
\end{aligned}
$$

$$= d_1d_2(y_1a_2)t_2d_4 \text{ (by zigzag equations)}$$
$$= d_2y_1a_2d_1t_2d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_2x_3x_1x_4x_5)$$
$$= d_2y_2a_3(d_1t_2)d_4 \text{ (by zigzag equations)}$$
$$= (y_2a_3)d_2d_1t_2d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_2x_3x_1x_4x_5)$$
$$= d_2d_1y_2a_3t_2d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_2x_3x_1x_4x_5)$$
$$\vdots$$
$$= d_2d_1y_ma_{2m-1}t_md_4$$
$$= d_2d_1y_ma_{2m}d_4 \text{ (by zigzag equations)}$$
$$= d_2d_1d_3d_4 \text{ (by zigzag equations),}$$

as required.
Case (v): $d_1, d_2, d_3, d_4 \in Dom(U, S)\backslash U$.
Then, by Theorem 1.1, d_4 has zigzag equations of type (1.1) in S over U of length m. Now

$$d_1d_2d_3d_4 = (d_1d_2d_3a_0)t_1 \text{ (by zigzag equations)}$$
$$= d_2d_1d_3a_ot_1 \text{ (by Case (iv))}$$
$$= d_2d_1d_3d_4 \text{ (by zigzag equations),}$$

as required. Thus the proof of the theorem is completed. □

Theorem 2.4. *Let U be a right cyclic commutative sub-semigroup of a double right semi-commutative semigroup S. Then $Dom(U, S)$ is right cyclic commutative semigroup.*

Proof. Let U be a right cyclic commutative sub-semigroup of a double right semi-commutative semigroup S. Then we have to show that $Dom(U, S)$ is also right cyclic commutative semigroup.

Case (i): If $d_1, d_2, d_3, d_4 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U, S)\backslash U$ and $d_2, d_3, d_4 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1.1) in S over U of length m. Now

$$d_1d_2d_3d_4 = y_m(a_{2m}d_2d_3d_4) \text{ (by zigzag equations)}$$
$$= y_ma_{2m}d_3d_4d_2 \text{ (by Case (i))}$$
$$= d_1d_3d_4d_2 \text{ (by zigzag equations),}$$

as required.
Case (iii): $d_1, d_2 \in Dom(U, S)\backslash U$ and $d_3, d_4 \in U$.

Then, by Theorem 1.1, d_2 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= d_1a_ot_1d_3d_4 \text{ (by zigzag equations)} \\ &= d_1d_3d_4a_ot_1 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_4x_5x_2x_3) \\ &= d_1d_3d_4d_2 \text{ (by zigzag equations),} \end{aligned}$$

as required.
Case (iv): $d_1, d_2, d_3 \in Dom(U, S)\backslash U$ and $d_4 \in U$.
Then, by Theorem 1.1, d_3 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= d_1d_2y_ma_{2m}d_4 \text{ (by zigzag equations)} \\ &= d_1a_{2m}d_4d_2y_m \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_4x_5x_2x_3) \\ &= d_1(a_{2m-1}t_m)d_4d_2y_m \text{ (by zigzag equations)} \\ &= d_1d_2y_ma_{2m-1}t_md_4 \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_4x_5x_2x_3) \\ &= d_1d_2y_{m-1}(a_{2m-2}t_m)d_4 \text{ (by zigzag equations)} \\ &= d_1a_{2m-2}t_md_4d_2y_{m-1} \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_4x_5x_2x_3) \\ &= d_1a_{2m-3}t_{m-1}(d_4d_2)y_{m-1} \text{ (by zigzag equations)} \\ &= d_1d_4d_2y_{m-1}a_{2m-3}t_{m-1} \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_4x_5x_2x_3) \\ &\vdots \\ &= d_1d_4d_2y_1a_1t_1 \\ &= d_1d_4d_2a_ot_1 \text{ (by zigzag equations)} \\ &= d_1a_ot_1d_4d_2 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_4x_5x_2x_3) \\ &= d_1d_3d_4d_2 \text{ (by zigzag equations),} \end{aligned}$$

as required.
Case (v): $d_1, d_2, d_3, d_4 \in Dom(U, S)\backslash U$.
Then, by Theorem 1.1, d_4 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= d_1d_2d_3y_ma_{2m} \text{ (by zigzag equations)} \\ &= d_1y_ma_{2m}d_2d_3 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_4x_5x_2x_3) \\ &= d_1(y_ma_{2m-1})t_md_2d_3 \text{ (by zigzag equations)} \\ &= d_1d_2d_3y_ma_{2m-1}t_m \end{aligned}$$

$$
\begin{aligned}
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_4x_5x_2x_3)\\
&= d_1(d_2d_3)y_{m-1}a_{2m-2}t_m \text{ (by zigzag equations)}\\
&= d_1a_{2m-2}t_md_2d_3y_{m-1}\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_4x_5x_2x_3)\\
&= d_1(a_{2m-3}t_{m-1})d_2d_3y_{m-1} \text{ (by zigzag equations)}\\
&= d_1d_3y_{m-1}a_{2m-3}t_{m-1}d_2\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_4x_5x_2x_3)\\
&\vdots\\
&= d_1d_3y_1a_1t_1d_2\\
&= d_1d_3a_ot_1d_2 \text{ (by zigzag equations)}\\
&= d_1d_3d_4d_2 \text{ (by zigzag equations)},
\end{aligned}
$$

as required. Thus the proof of the theorem is completed. □

Theorem 2.5. *Let U be a left cyclic commutative sub-semigroup of a double left semi-commutative semigroup S. Then $Dom(U,S)$ is left cyclic commutative semigroup.*

Proof. Let U be a left cyclic commutative sub-semigroup of a double left semi-commutative semigroup S. Then we have to show that $Dom(U,S)$ is also left cyclic commutative semigroup.

Case (i): If $d_1, d_2, d_3, d_4 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U,S)\backslash U$ and $d_2, d_3, d_4 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1.1) in S over U of length m. Now

$$
\begin{aligned}
d_1d_2d_3d_4 &= a_ot_1d_2d_3d_4 \text{ (by zigzag equations)}\\
&= d_2d_3a_ot_1d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_4x_1x_2x_5)\\
&= d_2d_3d_1d_4 \text{ (by zigzag equations)},
\end{aligned}
$$

as required.
Case (iii): $d_1, d_2 \in Dom(U,S)\backslash U$ and $d_3, d_4 \in U$.
Then, by Theorem 1.1, d_2 has zigzag equations of type (1.1) in S over U of length m. Now

$$
\begin{aligned}
d_1d_2d_3d_4 &= d_1a_ot_1d_3d_4 \text{ (by zigzag equations)}\\
&= t_1d_3d_1a_od_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_4x_1x_2x_5)\\
&= t_1d_3d_1(y_1a_1)d_4 \text{ (by zigzag equations)}
\end{aligned}
$$

$$
\begin{aligned}
&= d_1y_1a_1t_1d_3d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_4x_1x_2x_5)\\
&= d_1(y_1a_2)t_2d_3d_4 \text{ (by zigzag equations)}\\
&= t_2d_3d_1y_1a_2d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_4x_1x_2x_5)\\
&= t_2(d_3d_1)y_2a_3d_4 \text{ (by zigzag equations)}\\
&= y_2a_3t_2d_3d_1d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_4x_1x_2x_5)\\
&\ \vdots\\
&= y_ma_{2m-1}t_md_3d_1d_4\\
&= y_ma_{2m}d_3d_1d_4 \text{ (by zigzag equations)}\\
&= d_2d_3d_1d_4 \text{ (by zigzag equations)},
\end{aligned}
$$

as required.

Case (iv): $d_1, d_2, d_3 \in Dom(U, S)\backslash U$ and $d_4 \in U$.

Then, by Theorem 1.1, d_3 has zigzag equations of type (1.1) in S over U of length m. Now

$$
\begin{aligned}
d_1d_2d_3d_4 &= d_1d_2a_ot_1d_4 \text{ (by zigzag equations)}\\
&= a_ot_1d_1d_2d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_4x_1x_2x_5)\\
&= y_1a_1t_1(d_1d_2)d_4 \text{ (by zigzag equations)}\\
&= t_1d_1d_2(y_1a_1)d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_4x_1x_2x_5)\\
&= d_2y_1a_1t_1d_1d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_4x_1x_2x_5)\\
&= d_2y_1a_2t_2d_1d_4 \text{ (by zigzag equations)}\\
&= d_2y_2a_3t_2d_1d_4 \text{ (by zigzag equations)}\\
&\ \vdots\\
&= d_2y_ma_{2m-1}t_md_1d_4\\
&= d_2y_ma_{2m}d_1d_4 \text{ (by zigzag equations)}\\
&= d_2d_3d_1d_4 \text{ (by zigzag equations)},
\end{aligned}
$$

as required.

Case (v): $d_1, d_2, d_3, d_4 \in Dom(U, S)\backslash U$.

Then, by Theorem 1.1, d_4 has zigzag equations of type (1.1) in S over U of length m. Now

$$
\begin{aligned}
d_1d_2d_3d_4 &= (d_1d_2d_3a_o)t_1 \text{ (by zigzag equations)}\\
&= d_2d_3d_1a_ot_1 \text{ (by Case (iv))}\\
&= d_2d_3d_1d_4 \text{ (by zigzag equations)},
\end{aligned}
$$

as required. Thus the proof of the theorem is completed. □

Theorem 2.6. *Let U be a right dual-cyclic commutative sub-semigroup of a middle right dual-cyclic commutative semigroup S. Then $Dom(U, S)$ is right dual-cyclic commutative semigroup.*

Proof. Let U be a right dual-cyclic commutative sub-semigroup of a middle right dual-cyclic commutative semigroup S. Then we have to show that $Dom(U, S)$ is also right dual-cyclic commutative semigroup.

Case (i): If $d_1, d_2, d_3, d_4 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U, S)\backslash U$ and $d_2, d_3, d_4 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= y_m a_{2m} d_2d_3d_4 \text{ (by zigzag equations)} \\ &= y_m a_{2m} d_4d_2d_3 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4) \\ &= d_1d_4d_2d_3 \text{ (by zigzag equations)}, \end{aligned}$$

as required.
Case (iii): $d_1, d_2 \in Dom(U, S)\backslash U$ and $d_3, d_4 \in U$.
Then, by Theorem 1.1, d_2 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= d_1 y_m a_{2m} d_3d_4 \text{ (by zigzag equations)} \\ &= d_1 y_m d_4 a_{2m} d_3 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4) \\ &= d_1(y_m d_4 a_{2m-1} t_m d_3) \text{ (by zigzag equations)} \\ &= (d_1 y_m d_4 d_3 a_{2m-1}) t_m \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4) \\ &= d_1 y_m a_{2m-1} d_4 d_3 t_m \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4) \\ &= d_1 y_{m-1} a_{2m-2} d_4 (d_3 t_m) \text{ (by zigzag equations)} \\ &= d_1(y_{m-1} d_3 t_m a_{2m-2} d_4) \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4) \\ &= d_1(y_{m-1} d_3 d_4 t_m a_{2m-2}) \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4) \\ &= (d_1 y_{m-1} d_3 a_{2m-2} d_4) t_m \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4) \\ &= d_1 y_{m-1} d_4 d_3 a_{2m-2} t_m \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4) \end{aligned}$$

$$
\begin{aligned}
&= (d_1 y_{m-1} d_4 d_3 a_{2m-3}) t_{m-1} \text{ (by zigzag equations)}\\
&= d_1 y_{m-1} a_{2m-3} d_4 d_3 t_{m-1}\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4)\\
&\vdots\\
&= d_1 y_1 a_1 d_4 d_3 t_1\\
&= (d_1 a_0 d_4 d_3) t_1 \text{ (by zigzag equations)}\\
&= (d_1 d_3 a_0 d_4) t_1 \text{ (by case (ii))}\\
&= d_1 d_4 d_3 a_0 t_1 \text{ (by case (ii))}\\
&= d_1 d_4 t_1 d_3 a_0 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4)\\
&= d_1 d_4 a_0 t_1 d_3 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4)\\
&= d_1 d_4 d_2 d_3 \text{ (by zigzag equations)},
\end{aligned}
$$

as required.
Case (iv): $d_1, d_2, d_3 \in Dom(U,S)\backslash U$ and $d_4 \in U$.
Then, by Theorem 1.1, d_3 has zigzag equations of type (1.1) in S over U of length m. Now

$$
\begin{aligned}
d_1d_2d_3d_4 &= d_1 d_2 y_m a_{2m} d_4 \text{ (by zigzag equations)}\\
&= d_1 d_2 d_4 y_m a_{2m} \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4)\\
&= (d_1 d_2 d_4 y_m a_{2m-1}) t_m \text{ (by zigzag equations)}\\
&= (d_1 d_2 a_{2m-1} d_4 y_m) t_m\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4)\\
&= d_1 d_2 y_m a_{2m-1} d_4 t_m \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4)\\
&= (d_1 d_2 y_{m-1} a_{2m-2} d_4) t_m \text{ (by zigzag equations)}\\
&= d_1 d_2 d_4 y_{m-1} a_{2m-2} t_m\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4)\\
&= (d_1 d_2 d_4 y_{m-1} a_{2m-3}) t_{m-1} \text{ (by zigzag equations)}\\
&= (d_1 d_2 a_{2m-3} d_4 y_{m-1}) t_{m-1}\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4)\\
&= d_1 d_2 y_{m-1} a_{2m-3} d_4 t_{m-1}\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4)\\
&\vdots\\
&= d_1 d_2 y_1 a_1 d_4 t_1\\
&= (d_1 d_2 a_0 d_4) t_1 \text{ (by zigzag equations)}
\end{aligned}
$$

$$= d_1 d_4 d_2 a_0 t_1 \text{ (by case (iii))}$$
$$= d_1 d_4 d_2 d_3 \text{ (by zigzag equations),}$$

as required.

Case (v): $d_1, d_2, d_3, d_4 \in Dom(U,S)\backslash U$.

Then, by Theorem 1.1, d_4 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= (d_1d_2d_3a_o)t_1 \text{ (by zigzag equations)} \\ &= d_1a_0d_2d_3t_1 \text{ (by Case (iv))} \\ &= d_1a_0t_1d_2d_3 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_2x_5x_3x_4) \\ &= d_1d_4d_2d_3 \text{ (by zigzag equations),} \end{aligned}$$

as required. Thus the proof of the theorem is completed. □

Theorem 2.7. *Let U be a left dual-cyclic commutative sub-semigroup of a middle left dual-cyclic commutative semigroup S. Then $Dom(U,S)$ is left dual-cyclic commutative semigroup.*

Proof. Let U be a left dual-cyclic commutative sub-semigroup of a middle left dual-cyclic commutative semigroup S. Then, we have to show that $Dom(U,S)$ is also left dual-cyclic commutative semigroup.

Case (i): If $d_1, d_2, d_3, d_4 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U,S)\backslash U$ and $d_2, d_3, d_4 \in U$.

Then, by Theorem 1.1, d_1 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= y_m(a_{2m}d_2d_3d_4) \text{ (by zigzag equations)} \\ &= y_md_3a_{2m}d_2d_4 \text{ (by case (i))} \\ &= (y_md_3)a_{2m-1}t_md_2d_4 \text{ (by zigzag equations)} \\ &= t_my_md_3(a_{2m-1}d_2)d_4 \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5) \\ &= d_3t_my_ma_{2m-1}d_2d_4 \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5) \\ &= d_3t_m(y_{m-1}a_{2m-2})d_2d_4 \text{ (by zigzag equations)} \\ &= y_{m-1}a_{2m-2}d_3t_m(d_2d_4) \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5) \\ &= d_3y_{m-1}a_{2m-2}t_md_2d_4 \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5) \\ &= d_3y_{m-1}a_{2m-3}t_{m-1}d_2d_4 \text{ (by zigzag equations)} \end{aligned}$$

$$\begin{aligned}
&\vdots\\
&= d_3y_1a_1t_1d_2d_4\\
&= d_3a_ot_1d_2d_4 \text{ (by zigzag equations)}\\
&= d_3d_1d_2d_4 \text{ (by zigzag equations),}
\end{aligned}$$

as required.

Case (iii): $d_1, d_2 \in Dom(U,S)\backslash U$ and $d_3, d_4 \in U$.

Then, by Theorem 1.1, d_2 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned}
d_1d_2d_3d_4 &= d_1a_ot_1d_3d_4 \text{ (by zigzag equations)}\\
&= t_1d_1a_od_3d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5)\\
&= t_1d_1(y_1a_1)d_3d_4 \text{ (by zigzag equations)}\\
&= y_1a_1t_1d_1d_3d_4 \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5)\\
&= y_1a_2t_2d_1d_3d_4 \text{ (by zigzag equations)}\\
&= y_2a_3(t_2d_1)d_3d_4 \text{ (by zigzag equations)}\\
&= (t_2d_1)y_2a_3d_3d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5)\\
&= a_3t_2d_1y_2(d_3d_4) \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5)\\
&= d_1a_3t_2y_2d_3d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5)\\
&\vdots\\
&= d_1a_{2m-1}t_my_md_3d_4\\
&= d_1a_{2m}y_md_3d_4 \text{ (by zigzag equations)}\\
&= y_m(d_1a_{2m}d_3d_4) \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5)\\
&= y_md_3d_1a_{2m}d_4 \text{ (by case (ii))}\\
&= d_1y_md_3a_{2m}d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5)\\
&= d_3d_1y_ma_{2m}d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5)\\
&= d_3d_1d_2d_4 \text{ (by zigzag equations),}
\end{aligned}$$

as required.

Case (iv): $d_1, d_2, d_3 \in Dom(U,S)\backslash U$ and $d_4 \in U$.

Then, by Theorem 1.1, d_3 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned}
d_1d_2d_3d_4 &= d_1d_2a_ot_1d_4 \text{ (by zigzag equations)}\\
&= a_od_1d_2t_1d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5)\\
&= y_1a_1(d_1d_2)t_1d_4 \text{ (by zigzag equations)}\\
&= d_1d_2y_1a_1t_1d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5)
\end{aligned}$$

$$
\begin{aligned}
&= d_1d_2(y_1a_2)t_2d_4 \text{ (by zigzag equations)}\\
&= y_1a_2d_1d_2t_2d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5)\\
&= y_2a_3(d_1d_2)t_2d_4 \text{ (by zigzag equations)}\\
&= d_1d_2y_2a_3t_2d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5)\\
&\vdots\\
&= d_1d_2y_ma_{2m-1}t_md_4\\
&= d_1d_2y_ma_{2m}d_4 \text{ (by zigzag equations)}\\
&= y_m(d_1d_2a_{2m}d_4) \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5)\\
&= y_ma_{2m}d_1d_2d_4 \text{ (by case (iii))}\\
&= d_3d_1d_2d_4 \text{ (by zigzag equations),}
\end{aligned}
$$

as required.

Case (v): $d_1, d_2, d_3, d_4 \in Dom(U, S)\backslash U$.

Then, by Theorem 1.1, d_4 has zigzag equations of type (1.1) in S over U of length m. Now

$$
\begin{aligned}
d_1d_2d_3d_4 &= d_1d_2d_3a_ot_1 \text{ (by zigzag equations)}\\
&= d_3d_1d_2a_ot_1 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_4x_5)\\
&= d_3d_1d_2d_4 \text{ (by zigzag equations),}
\end{aligned}
$$

as required. Thus the proof of the theorem is completed. □

Theorem 2.8. *Let U be a right externally commutative sub-semigroup of a dual right semi-commutative semigroup S. Then $Dom(U, S)$ is right externally commutative semigroup.*

Proof. Let U be a right externally commutative sub-semigroup of a dual right semi-commutative semigroup S. Then we have to show that $Dom(U, S)$ is also right externally commutative semigroup.

Case (i): If $d_1, d_2, d_3, d_4 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U, S)\backslash U$ and $d_2, d_3, d_4 \in U$.

Then, by Theorem 1.1, d_1 has zigzag equations of type (1.1) in S over U of length m. Now

$$
\begin{aligned}
d_1d_2d_3d_4 &= y_m(a_{2m}d_2d_3d_4) \text{ (by zigzag equations)}\\
&= y_ma_{2m}d_4d_3d_2 \text{ (by case (i))}\\
&= d_1d_4d_3d_2 \text{ (by zigzag equations),}
\end{aligned}
$$

as required.

Case (iii): $d_1, d_2 \in Dom(U, S)\backslash U$ and $d_3, d_4 \in U$.

Then, by Theorem 1.1, d_2 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= d_1y_ma_{2m}d_3d_4 \text{ (by zigzag equations)} \\ &= d_1d_4d_3y_ma_{2m} \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_5x_4x_2x_3) \\ &= d_1d_4d_3d_2 \text{ (by zigzag equations)}, \end{aligned}$$

as required.

Case (iv): $d_1, d_2, d_3 \in Dom(U,S)\backslash U$ and $d_4 \in U$.

Then, by Theorem 1.1, d_3 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= d_1d_2y_ma_{2m}d_4 \text{ (by zigzag equations)} \\ &= d_1d_4a_{2m}d_2y_m \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_5x_4x_2x_3) \\ &= d_1(d_4a_{2m-1}t_md_2y_m) \text{ (by zigzag equations)} \\ &= (d_1d_4y_md_2a_{2m-1})t_m \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_5x_4x_2x_3) \\ &= d_1a_{2m-1}d_2d_4(y_mt_m) \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_5x_4x_2x_3) \\ &= d_1y_mt_m(d_4a_{2m-1})d_2 \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_5x_4x_2x_3) \\ &= (d_1d_2d_4a_{2m-1}y_m)t_m \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_5x_4x_2x_3) \\ &= d_1y_ma_{2m-1}d_2d_4t_m \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_5x_4x_2x_3) \\ &= (d_1y_{m-1}a_{2m-2}d_2d_4)t_m \text{ (by zigzag equations)} \\ &= d_1d_4d_2y_{m-1}a_{2m-2}t_m \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_5x_4x_2x_3) \\ &= d_1d_4d_2y_{m-1}(a_{2m-3}t_{m-1}) \text{ (by zigzag equations)} \\ &= (d_1a_{2m-3}t_{m-1}y_{m-1}d_4)d_2 \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_5x_4x_2x_3) \\ &= d_1d_4y_{m-1}a_{2m-3}t_{m-1}d_2 \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_5x_4x_2x_3) \\ &\ \ \vdots \\ &= d_1d_4y_1a_1t_1d_2 \\ &= d_1d_4a_0t_1d_2 \text{ (by zigzag equations)} \end{aligned}$$

$$= d_1d_4d_3d_2 \text{ (by zigzag equations)},$$

as required.
Case (v): $d_1, d_2, d_3, d_4 \in Dom(U,S)\backslash U$.
Then, by Theorem 1.1, d_4 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= d_1d_2d_3a_0t_1 \text{ (by zigzag equations)} \\ &= d_1t_1a_od_2d_3 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_5x_4x_2x_3) \\ &= d_1t_1(y_1a_1)d_2d_3 \text{ (by zigzag equations)} \\ &= d_1d_3d_2t_1y_1a_1 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_5x_4x_2x_3) \\ &= d_1d_3d_2t_1a_0 \text{ (by zigzag equations)} \\ &= d_1a_0t_1d_3d_2 \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_1x_5x_4x_2x_3) \\ &= d_1d_4d_3d_2 \text{ (by zigzag equations)}, \end{aligned}$$

as required. Thus the proof of the theorem is completed. □

Theorem 2.9. *Let U be a left externally commutative sub-semigroup of a middle left externally commutative semigroup S. Then $Dom(U,S)$ is left externally commutative semigroup.*

Proof. Let U be a left externally commutative sub-semigroup of a middle left externally commutative semigroup S. Then we have to show that $Dom(U,S)$ is also left externally commutative semigroup.

Case (i): If $d_1, d_2, d_3, d_4 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U,S)\backslash U$ and $d_2, d_3, d_4 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= y_m(a_{2m}d_2d_3d_4) \text{ (by zigzag equations)} \\ &= y_md_3d_2a_{2m}d_4 \text{ (by case (i))} \\ &= y_m(d_3d_2a_{2m-1}t_md_4) \text{ (by zigzag equations)} \\ &= y_ma_{2m-1}d_2d_3t_md_4 \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5) \\ &= (y_{m-1}a_{2m-2})d_2d_3t_md_4 \text{ (by zigzag equations)} \\ &= d_3d_2y_{m-1}a_{2m-2}t_md_4 \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5) \\ &= d_3d_2y_{m-1}a_{2m-3}t_{m-1}d_4 \text{ (by zigzag equations)} \\ &\quad \vdots \end{aligned}$$

$$= d_3d_2y_1a_1t_1d_4$$
$$= d_3d_2a_ot_1d_4 \text{ (by zigzag equations)}$$
$$= d_3d_2d_1d_4 \text{ (by zigzag equations)},$$

as required.

Case (iii): $d_1, d_2 \in Dom(U,S)\backslash U$ and $d_3, d_4 \in U$.

Then, by Theorem 1.1, d_2 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned}
d_1d_2d_3d_4 &= d_1y_ma_{2m}d_3d_4 \text{ (by zigzag equations)}\\
&= a_{2m}y_md_1d_3d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= a_{2m-1}(t_my_m)d_1d_3d_4 \text{ (by zigzag equations)}\\
&= d_1t_my_m(a_{2m-1}d_3)d_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= y_mt_m(d_1a_{2m-1}d_3d_4)\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= y_mt_md_3(a_{2m-1}d_1)d_4 \text{ (by case (ii))}\\
&= d_3t_my_ma_{2m-1}d_1d_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= d_3t_my_{m-1}(a_{2m-2}d_1)d_4 \text{ (by zigzag equations)}\\
&= y_{m-1}t_m(d_3a_{2m-2})d_1d_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= d_3a_{2m-2}t_my_{m-1}d_1d_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= (d_3a_{2m-3}t_{m-1}y_{m-1}d_1)d_4 \text{ (by zigzag equations)}\\
&= t_{m-1}a_{2m-3}(d_3y_{m-1})d_1d_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= d_3y_{m-1}a_{2m-3}t_{m-1}d_1d_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&\ \vdots\\
&= d_3y_1a_1t_1d_1d_4\\
&= d_3a_ot_1d_1d_4 \text{ (by zigzag equations)}\\
&= d_3d_2d_1d_4 \text{ (by zigzag equations)},
\end{aligned}$$

as required.

Case (iv): $d_1, d_2, d_3 \in Dom(U,S)\backslash U$ and $d_4 \in U$.

Then, by Theorem 1.1, d_3 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned}
d_1d_2d_3d_4 &= d_1d_2y_ma_{2m}d_4 \text{ (by zigzag equations)}\\
&= y_md_2d_1a_{2m}d_4 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= (y_md_2)d_1a_{2m-1}t_md_4 \text{ (by zigzag equations)}\\
&= a_{2m-1}(d_1y_md_2t_md_4)\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= a_{2m-1}d_2(y_md_1)t_md_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= y_md_1d_2a_{2m-1}t_md_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= y_m(d_1d_2a_{2m}d_4) \text{ (by zigzag equations)}\\
&= y_ma_{2m}d_2d_1d_4 \text{ (by case (iii))}\\
&= (y_ma_{2m-1})t_md_2d_1d_4 \text{ (by zigzag equations)}\\
&= a_{2m-1}(y_md_2t_md_1d_4)\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= (d_2t_m)y_ma_{2m-1}d_1d_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= a_{2m-1}(t_md_2)y_md_1d_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= y_mt_md_2(a_{2m-1}d_1)d_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= d_2t_my_ma2m-1d_1d_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= d_2t_m(ym-1a_{2m-2})d_1d_4 \text{ (by zigzag equations)}\\
&= y_{m-1}a_{2m-2}t_md_2d_1d_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_2x_1x_4x_5)\\
&= y_{m-1}a_{2m-3}t_{m-1}d_2d_1d_4 \text{ (by zigzag equations)}\\
&\;\;\vdots\\
&= y_1a_1t_1d_2d_1d_4\\
&= a_ot_1d_2d_1d_4 \text{ (by zigzag equations)}\\
&= d_3d_2d_1d_4 \text{ (by zigzag equations),}
\end{aligned}$$

as required.
Case (v): $d_1, d_2, d_3, d_4 \in Dom(U, S)\backslash U$.
Then, by Theorem 1.1, d_4 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= (d_1d_2d_3a_0)t_1 \text{ (by zigzag equations)} \\ &= d_3d_2d_1a_ot_1 \text{ (by Case (iv))} \\ &= d_3d_2d_1d_4 \text{ (by zigzag equations)}, \end{aligned}$$

as required. Thus the proof of the theorem is completed. □

Theorem 2.10. *Let U be a bi-commutative sub-semigroup of a left dual-cyclic right semi-commutative semigroup S. Then $Dom(U, S)$ is bi-commutative semigroup.*

Proof. Let U be a bi-commutative sub-semigroup of a left dual-cyclic right semi-commutative semigroup S. Then we have to show that $Dom(U, S)$ is also bi-commutative semigroup.

Case (i): If $d_1, d_2, d_3, d_4 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U, S)\backslash U$ and $d_2, d_3, d_4 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= y_ma_{2m}d_2d_3d_4 \text{ (by zigzag equations)} \\ &= d_2y_ma_{2m}d_4d_3 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4) \\ &= d_2d_1d_4d_3 \text{ (by zigzag equations)}, \end{aligned}$$

as required.
Case (iii): $d_1, d_2 \in Dom(U, S)\backslash U$ and $d_3, d_4 \in U$.
Then, by Theorem 1.1, d_2 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3d_4 &= d_1y_ma_{2m}d_3d_4 \text{ (by zigzag equations)} \\ &= a_{2m}d_1y_md_4d_3 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4) \\ &= (a_{2m-1}t_m)d_1y_md_4d_3 \text{ (by zigzag equations)} \\ &= y_ma_{2m-1}t_md_1d_3d_4 \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4) \\ &= y_{m-1}(a_{2m-2}t_m)d_1d_3d_4 \text{ (by zigzag equations)} \\ &= d_1y_{m-1}a_{2m-2}t_md_4d_3 \\ &\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4) \end{aligned}$$

$$
\begin{aligned}
&= d_1 y_{m-1}(a_{2m-3}t_{m-1})d_4 d_3 \text{ (by zigzag equations)}\\
&= (a_{2m-3}t_{m-1})d_1 y_{m-1} d_3 d_4\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4)\\
&= y_{m-1}a_{2m-3}t_{m-1}d_1 d_4 d_3\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4)\\
&\vdots\\
&= y_1 a_1 t_1 d_1 d_4 d_3\\
&= a_0 t_1 d_1 d_4 d_3 \text{ (by zigzag equations)}\\
&= d_2 d_1 d_4 d_3 \text{ (by zigzag equations)},
\end{aligned}
$$

as required.
Case (iv): $d_1, d_2, d_3 \in Dom(U,S)\setminus U$ and $d_4 \in U$.
Then, by Theorem 1.1, d_3 has zigzag equations of type (1.1) in S over U of length m. Now

$$
\begin{aligned}
d_1d_2d_3d_4 &= d_1 d_2 y_m a_{2m} d_4 \text{ (by zigzag equations)}\\
&= y_m d_1 d_2 d_4 a_{2m} \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4)\\
&= (y_m d_1 d_2 d_4 a_{2m-1})t_m \text{ (by zigzag equations)}\\
&= (d_2 y_m d_1 a_{2m-1} d_4)t_m\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4)\\
&= d_1(d_2 y_m d_4 a_{2m-1} t_m)\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4)\\
&= (d_1 d_4 d_2 y_m t_m)a_{2m-1}\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4)\\
&= d_2 d_1 d_4 t_m y_m a_{2m-1}\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4)\\
&= d_2 d_1 d_4 t_m (y_{m-1} a_{2m-2}) \text{ (by zigzag equations)}\\
&= d_4 d_2 d_1 y_{m-1} a_{2m-2} t_m\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4)\\
&= d_4 d_2 d_1 (y_{m-1} a_{2m-3}) t_{m-1} \text{ (by zigzag equations)}\\
&= d_1 d_4 d_2 t_{m-1} y_{m-1} a_{2m-3}\\
&\quad \text{(as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4)\\
&\vdots\\
&= d_1 d_4 d_2 t_1 y_1 a_1
\end{aligned}
$$

$$= d_1 d_4 d_2 t_1 a_0 \text{ (by zigzag equations)}$$
$$= d_2 d_1 d_4 a_0 t_1 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4)$$
$$= d_2 d_1 d_4 d_3 \text{ (by zigzag equations)},$$

as required.

Case (v): $d_1, d_2, d_3, d_4 \in Dom(U, S) \backslash U$.

Then, by Theorem 1.1, d_4 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned}
d_1d_2d_3d_4 &= (d_1d_2d_3a_o)t_1 \text{ (by zigzag equations)}\\
&= d_2d_1a_od_3t_1 \text{ (by Case (iv))}\\
&= a_0d_2d_1t_1d_3 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4)\\
&= (y_1a_1)d_2d_1t_1d_3 \text{ (by zigzag equations)}\\
&= d_1(y_1a_1)d_2d_3t_1 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4)\\
&= d_2d_1y_1a_1t_1d_3 \text{ (as } S \text{ satisfies } x_1x_2x_3x_4x_5 = x_3x_1x_2x_5x_4)\\
&= d_2d_1a_0t_1d_3 \text{ (by zigzag equations)}\\
&= d_2d_1d_4d_3 \text{ (by zigzag equations)},
\end{aligned}$$

as required. Thus the proof of the theorem is completed. □

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VARIETIES OF PERMUTATIVE SEMIGROUPS
CLOSED UNDER DOMINIONS

H. MAQBOOL AND M. Y. BHAT

انواع نیم‌گروه‌های جایگشتی تحت قلمروها بسته

حمیرا مقبول[۱] و محمد یونس بهات[۲]

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در این مقاله با استفاده از قضیه زیگزاگی ایزابل، یکی از نتایج ایزابل در کلاس نیم‌گروه‌های جابجایی را برای کلاسی تعمیم‌یافته از نیم‌گروه‌های جابجایی با استفاده از مجموعه احاطه‌گر ثابت می کنیم.

کلمات کلیدی: معادلات زیگ‌زاگی، قلمرو، انواع، همانی.