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ABSORBING PRIME MULTIPLICATION MODULES OVER A PULLBACK RING

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Abstract. The main purpose of this article is to present a new approach to the classification of all indecomposable absorbing prime multiplication modules with finite-dimensional top over pullback rings of two Dedekind domains. First, we give a complete description of the absorbing prime multiplication modules over a local Dedekind domain. In fact, we extend the definition and results given in [9] to a more general absorbing prime multiplication modules case. Next, we establish a connection between the absorbing prime multiplication modules and the pure-injective modules over such rings.

1. Introduction

One of the aims of the modern representation theory is to solve classification problems for subcategories of modules over a unitary ring R. Unfortunately, for the vast majority of rings, the classification of an arbitrary module is infeasible. For example, if R is a pullback of two local Dedekind domains over a common factor field, then the classification of all indecomposable pure-injective modules with infinite-dimensional top over $R/rad(R)$, for any module M over a ring R we define its top as $M/rad(R)M$, is somewhat difficult. Why do we consider pure-injective modules? Pure-injective modules are model-theoretically typical: for example classification of the complete theories

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of R-modules reduces to classifying the (complete theories of) pure-injective modules. Also, for some rings, “small” (finite-dimensional, finitely generated, ...) modules are classified and in many cases this classification can be extended to give a classification of (indecomposable) pure-injective modules. Indeed, there is sometimes a strong connection between infinitely generated pure-injective modules and some subclasses of finitely generated modules. Therefore, pure-injective modules are very important (see [12], [18] and [19]). One point of this paper is to introduce a subclass of pure-injective modules.

In the present article, we introduce a new class of R-modules, called absorbing prime multiplication modules (see Definition 2.5), and we study it in detail from the classification problem point of view. We are mainly interested in case either R is a Dedekind domain or R is a pullback of two local Dedekind domains. First, we give a complete description of the absorbing prime multiplication modules over a local Dedekind domain. Let R be a pullback of two local Dedekind domains over a common factor field. Next, the main purpose of this paper is to give a complete description of the indecomposable absorbing prime multiplication R-modules with finite-dimensional top over $R/Rad(R)$. In fact, we extend the definition and results given in [9] to a more general absorbing prime multiplication modules case.

First, we describe all indecomposable separated absorbing prime multiplication R-modules and then, using this list of separated absorbing prime multiplication modules, we show that non-separated indecomposable absorbing prime multiplication R-modules with finite-dimensional top are factor modules of finite direct sums of separated indecomposable absorbing prime multiplication R-modules. Then we use the classification of separated indecomposable absorbing prime multiplication modules from Section 3, together with results of Levy [13] on the possibilities for amalgamating finitely generated separated modules, to classify the non-separated indecomposable absorbing prime multiplication modules M with finite-dimensional top (see Theorem 4.8). We will see that the non-separated modules may be represented by certain amalgamation chains of separated indecomposable absorbing prime multiplication modules (where infinite length absorbing prime multiplication modules can occur only at the ends) and where adjacency corresponds to amalgamation in the socles of these separated absorbing prime multiplication modules.

For the sake of completeness, we state some definitions and notations used throughout. In this article all rings are commutative with identity and all modules unitary. Let $v_1 : R_1 \longrightarrow \bar{R}$ and $v_2 : R_2 \longrightarrow \bar{R}$ be homomorphisms of two local Dedekind domains R_i, $i = 1, 2$, onto a common field $\bar{R}$. Denote the pullback

$$R = \{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\}$$

by $(R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2)$. Then R is a ring under coordinate-wise multiplication. Denote the kernel of v_i, $i = 1, 2$, by P_i. Then

$$Ker(R \longrightarrow \bar{R}) = P = P_1 \times P_2,\ R/P \cong \bar{R} \cong R_2/P_2,$$

and $P_1P_2 = P_2P_1 = 0$ (so R is not a domain). Furthermore, for $i \neq j$,

$$0 \longrightarrow P_i \longrightarrow R \longrightarrow R_j \longrightarrow 0$$

is an exact sequence of R-modules (see [14]). An R-module S is defined to be separated if there exist R_i-modules S_i, $i = 1, 2$, such that S is a submodule of $S_1 \oplus S_2$ (the latter is made into an R-module by setting $(r_1, r_2)(s_1, s_2) = (r_1s_1, r_2s_2)$). Equivalently, S is separated if it is a pullback of an R_1-module and an R_2-module and then, using the same notation for pullbacks of modules as for rings,

$$S = (S/P_2S \longrightarrow S/PS \longleftarrow S/P_1S)$$

[14, Corollary 3.3] and $S \subseteq (S/P_2S) \oplus (S/P_1S)$. Also, S is separated if and only if $P_1S \cap P_2S = 0$ [14, Lemma 2.9].

If R is a pullback ring, then every R-module is an epimorphic image of a separated R-module, indeed every R-module has a "minimal" such representation: a separated representation of an R-module M is an epimorphism $\varphi : S \longrightarrow M$ of R-modules where S is separated and, if φ admits a factorization $\varphi : S \xrightarrow{f} S' \longrightarrow M$ with S' separated, then f is one-to-one. The module $K = Ker(\varphi)$ is then an $\bar{R}$-module, since $\bar{R} = R/P$ and $PK = 0$ [14, Proposition 2.3]. An exact sequence $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ of R-modules with S separated and K an $\bar{R}$-module is a separated representation of M if and only if $P_iS \cap K = 0$ for each i and $K \subseteq PS$ [14, Proposition 2.3]. Every module M has a separated representation, which is unique up to isomorphism [14, Theorem 2.8]. Moreover, R-homomorphisms lift to a separated representation, preserving epimorphisms and monomorphisms [14, Theorem 2.6].

Definition 1.1. (a) Let N be a submodule of an R-module M. Then the ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N : M)$. Moreover, $(0 : M)$ is the annihilator of M.

(b) A proper submodule N of an R-module M is said to be primary (resp., prime) if whenever $rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $r^n \in (N : M)$ for some positive integer n (resp., $m \in N$ or $r \in (N : M)$), so $Rad(N : M) = P$ (resp., $(N : M) = P'$) is a prime ideal of R, and N is said to be a

P-primary (resp., P'-prime) submodule. The set of all primary submodules (resp., prime submodules) in an R-module M is denoted $pSpec(M)$ (resp., $Spec(M)$).

(c) A proper submodule N of an R-module M is called 2-absorbing, if for $a, b \in R$ and $m \in M$, $abm \in N$ implies that $ab \in (N : M)$ or $am \in N$ or $bm \in N$. So $(N : M)$ is a 2-absorbing ideal of R. The set of all 2-absorbing submodules in an R-module M is denoted by $2 - abSpec(M)$ (see [17]).

(d) A proper ideal I of a commutative ring R is said to be 1-absorbing prime, if for all non-unit elements $a, b, c \in R$, $abc \in I$, then $ab \in I$ or $c \in I$ [21].

(e) A proper submodule N of an R-module M is said to be 1-absorbing prime, if for all non-unit elements $a, b \in R$ and $m \in M$, $abm \in N$, then $m \in N$ or $ab \in (N : M)$. The set of all 1-absorbing prime submodules in an R-module M is denoted by $abpSpec(M)$.

An R-module M is called 1-absorbing prime, if its zero submodule is a 1-absorbing prime submodule of M.

(f) An R-module M is defined to be a multiplication module if for each submodule N of M, $N = IM$, for some ideal I of R. In this case, we can take $I = (N : M)$.

(g) An R-module M is defined to be a weak multiplication module if $Spec(M) = \emptyset$ or for every prime submodule N of M, $N = IM$, for some ideal I of R (see [9]).

(h) An R-module M is defined to be a primary multiplication module if $pSpec(M) = \emptyset$ or for every primary submodule N of M, $N = IM$, for some ideal I of R [6].

(i) An R-module M is defined to be a 2-absorbing multiplication module if $2 - abSpec(M) = \emptyset$ or for every 2-absorbing submodule N of M, $N = IM$, for some ideal I of R (see [10]).

(j) An R-module M is defined to be a semiprime multiplication module if for every semiprime submodule N of M, $N = IM$, for some ideal I of R (see [7]).

(k) A submodule N of an R-module M is called pure submodule, if any finite system of equations over N which is solvable in M is also solvable in N. A submodule N of an R-module M is called relatively divisible (or an RD-submodule) in M if $rN = N \cap rM$ for all $r \in R$ (see [18, 19]).

(l) An exact sequence of R-modules $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is pure exact, if the image of α is pure in B.

(m) An R-module M is said to be injective relative to an exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ if for any homomorphism $f : A \longrightarrow M$ there exists a homomorphism $g : B \longrightarrow M$ such that $g \circ \alpha = f$.

(n) An R-module M is pure-injective if it has the injective property relative to all pure exact sequences (see [18, 19]).

(o) An R-module M is called algebraically compact, if every finitely solvable family of linear equations over R in M has a simultaneous.

Remark 1.2. (i) An R-module is pure-injective if and only if it is algebraically compact (see [20]).

(ii) Let R be a Dedekind domain, M an R-module and N a submodule of M. Then N is pure in M if and only if $IN = N \cap IM$ for each ideal I of R. Moreover, N is pure in M if and only if N is an RD-submodule of M [19].

2. Absorbing prime multiplication modules over a local Dedekind domain

The aim of this section is to classify absorbing prime multiplication modules over a local Dedekind domain. First, we collect basic properties of absorbing prime multiplication modules.

Note that every prime submodule is a 1-absorbing prime submodule and every 1-absorbing prime submodule is a 2-absorbing prime submodule. But the converse of each of them does not necessarily hold in general. See the following examples.

Example 2.1. (**1-absorbing prime submodule that is not prime**) Consider $\mathbb{Z}_4$-module $\mathbb{Z}_4[X]$ and the submodule $N = \langle X \rangle$. Thus N is a 1-absorbing prime submodule, but N is not a prime submodule of $\mathbb{Z}_4[X]$.

Example 2.2. (**2-absorbing submodule that is not 1-absorbing prime**) Consider $\mathbb{Z}$-module $\mathbb{Z}_{30}$ and the cyclic submodule $N = \langle 6 \rangle$. It is clear that $N = \langle 6 \rangle$ is a 2-absorbing submodule of $\mathbb{Z}_{30}$, but it is not a 1-absorbing prime submodule of $\mathbb{Z}_{30}$. Indeed, $2 \times 2 \times 3 \in \langle 6 \rangle$ but $4 \notin (N : \mathbb{Z}_{30})$ and $3 \notin \langle 6 \rangle$.

Proposition 2.3. *Let M be an R-module. Then*

(i) *If N is a 1-absorbing prime submodule of M, then $(N : M)$ is a 1-absorbing prime ideal of R.*

(ii) *If N is a 1-absorbing prime submodule of M and I, J are ideals of R and K is a submodule of M such that $IJK \subseteq N$, then $K \subseteq N$ or $IJ \subseteq (N : M)$.*

(iii) *Let $K \subset N$ be submodules of M. Then N is a 1-absorbing prime submodule of M if and only if N/K is a 1-absorbing prime submodule of M/K.*

(iv) *If N is a 1-absorbing prime submodule of M, then M/N is a 1-absorbing prime R-module.*

Proof. The proof is straightforward. □

Lemma 2.4. *Let M be an R-module, N a 1-absorbing prime submodule of M and I an ideal of R with $I \subset (0 : M)$. Then N is a 1-absorbing prime submodule of M as an R/I-module.*

Proof. Let $(a+I)(b+I)m \in N$ for some $m \in M$ and $a+I, b+I \in R/I$. Then $abm \in N$, hence $m \in N$ or $ab \in (N : M)$ since N a 1-absorbing prime submodule of M. Thus $m \in N$ or $ab + I \in (N :_{R/I} M)$. □

Definition 2.5. Let R be a commutative ring. An R-module M is said to be an absorbing prime multiplication module, if $abpSpec(M) = \emptyset$ or for every 1-absorbing prime submodule N of M, $N = IM$, for some ideal I of R.

We have the class of 2-absorbing multiplication modules contains the class of absorbing prime multiplication modules, and the class of absorbing prime multiplication modules contains the class of weak multiplication modules.

Lemma 2.6. *Let M be an absorbing prime multiplication module over a commutative ring R. Then the following hold:*

(i) *If I is an ideal of R and N is a non-zero R-submodule of M with $I \subseteq (N : M)$, then M/N is an absorbing prime multiplication R/I-module.*

(ii) *If N is a submodule of M, then M/N is an absorbing prime multiplication R-module.*

(iii) *Every direct summand of M is an absorbing prime multiplication module.*

Proof. (i) Let K/N be a 1-absorbing prime submodule of M/N. Then by Proposition 2.3, K is a 1-absorbing prime submodule of M, then $K = (K : M)M$. An inspection will show that

$$K/N = (K/N :_{R/I} M/N)M/N.$$

(ii) Take $I = 0$ in (i).

(iii) It follows from (ii). □

Lemma 2.7. *Let R and R' be commutative rings, $f : R \to R'$ an epimorphism and M an R'-module. Then the following hold:*

(i) *If M is a 1-absorbing prime as an R-module, then M is 1-absorbing prime as an R'-module.*
(ii) *If N is a 1-absorbing prime R-submodule of M, then N is a 1-absorbing prime R'-submodule of M.*
(iii) *If M is an absorbing prime multiplication R'-module, then M is an absorbing prime multiplication R-module.*

Proof. (i) It is obvious.

(ii) Clearly, M/N is a 1-absorbing prime R-module, so M/N is a 1-absorbing prime R'-module by (i), hence N is a 1-absorbing prime R'-submodule of M.

(iii) Let N be a 1-absorbing prime R-submodule of M. Then N is a 1-absorbing prime R'-submodule of M by (ii), so $N = I'M$ for some ideal I' of R'. Set $I = f^{-1}(I')$. Then I is an ideal of R and $f(I) = f(f^{-1}(I')) = I' \cap f(R) = I'$, hence $IM = f(I)M = N$. □

Proposition 2.8. *Let R be a local Dedekind domain with unique maximal ideal $P = \langle p \rangle$. Then*

(i) *$E = E(R/P)$, the injective hull of R/P, is an absorbing prime multiplication R-module.*
(ii) *$Q(R)$, the field of fractions of R, is an absorbing prime multiplication R-module.*

Proof. (i) By [3, Lemma 2.6], every non-zero proper submodule L of E is of the form $L = A_n = (0 :_E P^n)$ $(n \geq 1)$, $L = A_n = Ra_n$ and $PA_{n+1} = A_n$. However, no A_n is a 1-absorbing prime submodule of E, for if n is a positive integer, then $P^2 A_{n+2} = A_n$, but $A_{n+2} \nsubseteq A_n$ and $P^2 \nsubseteq (A_n : E) = 0$. Now we conclude that $abpSpec(E) = \emptyset$. Thus E is an absorbing prime multiplication module.

(ii) Clearly, 0 is a 1-absorbing prime (prime) submodule of $Q(R)$. To show that 0 is the only 1-absorbing prime submodule of $Q(R)$, we assume the contrary and let N be a non-zero 1-absorbing prime submodule of $Q(R)$. Since N is a non-zero submodule, there exists $0/1 \neq a/b$, where $a, b \in R$, so that $a/b \in N$. Clearly, $1/ab \notin N$ (otherwise, $ab/ab = 1/1 \in N$, which is a contradiction). Now we have $a^2(1/ab) \in N$, but $1/ab \notin N$ and $a^2 \notin (N :_R Q(R)) = 0$. Thus $abpSpec(Q(R)) = \{0\}$, and hence $Q(R)$ is an absorbing prime multiplication module. □

Proposition 2.9. *Let M be an absorbing prime multiplication module over an integral domain R (which is not a field). Then M is either torsion or torsion-free.*

Proof. Assume that $T(M)$ is the torsion submodule of M, and let $T(M) \neq M$. Then $T(M)$ is a prime submodule (so a 1-absorbing prime submodule) of M with $(T(M) : M) = 0$ by [16, Lemma 3.8]. It follows that $T(M) = (T(M) : M)M = 0$. Thus M is a torsion-free module and this completes the proof. $\square$

Example 2.10. Let R be a domain and let $Q(R)$ be the field of fractions of R. By Proposition 2.8, the only 1-absorbing prime submodule of $Q(R)$ is 0, hence $Q(R)$ is an absorbing prime multiplication module which is not a multiplication module, but we have the following result:

Theorem 2.11. *Let R be a local Dedekind domain with a unique maximal ideal $P = \langle p \rangle$. Then the following is a complete list, up to isomorphism, of the indecomposable absorbing prime multiplication modules:*

(i) *R;*
(ii) *R/P^n $(n \geq 1)$ the indecomposable torsion modules;*
(iii) *$E(R/P)$, the injective hull of R/P;*
(iv) *$Q(R)$, the field of fractions of R.*

Proof. First, we note that each of the preceding modules is indecomposable (by [2, Proposition 1.3]) and absorbing prime multiplication module. Clearly, R and R/P^n $(n \geq 1)$ are multiplication modules, so they are absorbing prime multiplication modules. Moreover, $Q(R)$ and $E(R/P)$ are absorbing prime multiplication modules by Proposition 2.8. Now let M be an indecomposable absorbing prime multiplication module, and choose any non-zero element $a \in M$. Let $h(a) = sup\{n \mid a \in P^n M\}$ (so $h(a)$ is a non-negative integer or ∞). Also, $(0 : a) = \{r \in R \mid ra = 0\}$, thus $(0 : a)$ is an ideal of the form P^n or 0. Because $(0 : a) = P^{m+1}$ implies that $P^m a \neq 0$ and $P(P^m a) = 0$, we can choose a such that $(0 : a) = P$ or 0. Now we consider the various possibilities for $h(a)$ and $(0 : a)$.

Case 1. $abpSpec(M) = \emptyset$. Since $Spec(M) \subseteq abpSpec(M)$, it follows from [15, Lemma 1.3, Proposition 1.4] that M is a torsion divisible R-module with $PM = M$ and M is not finitely generated. We may assume that $(0 : a) = P$. By an argument like that in [3, Proposition 2.7, Case 2(a)], $M \cong E(R/P)$. So we may assume that $abpSpac(M) \neq \emptyset$.

Case 2. $h(a) = n$, $(0 : a) = 0$. Say $a = p^n b$. Then $rb = 0$ implies $ra = 0$ and so $r = 0$. Thus $Rb \cong R$. We also have that Rb is pure in M (see [1, Theorem 2.12, Case 1]). As M is a torsion-free R-module by Proposition 2.9, we must have Rb is a prime submodule of M (see Remark 1.2) (so 1-absorbing prime submodule), hence by the hypothesis $R \cong Rb = P^t M$ for some t. Then there is an element

$m \in M$ such that $b = p^t m$, whence $a = p^{n+t}m$. Therefore, $t = 0$, and $R \cong Rb = P^0 M = RM = M$.

Case 3. $h(a) = n$, $(0 : a) = P$. Say $a = p^n b$. Then we have $Rb \cong R/P^{n+1}$. Furthermore, Rb is pure in M. Now since Rb is a pure submodule of bounded order of M, we obtain Rb is a direct summand of M by [11, Theorem 5], hence $M = Rb \cong R/P^{n+1}$.

Case 4. $h(a) = \infty$, $(0 : a) = P$. By an argument like that in [3, Proposition 2.7, Case 2(a)], we get $M \cong E(R/P)$, hence $abpSpac(M) = \emptyset$ by Proposition 2.8, which is a contradiction.

Case 5. $h(a) = \infty$, $(0 : a) = 0$. By an argument like that in [3, Proposition 2.7, Case 2(b)], we obtain $M \cong Q(R)$. □

Corollary 2.12. *Let $R \neq M$ be an absorbing prime multiplication module over a local Dedekind domain with maximal ideal P. Then M is of the form $M = N \oplus K$, where N is a direct sum of copies of R/P^n ($n \geq 1$) and K is a direct sum of copies of $E(R/P)$ and $Q(R)$. In particular, every absorbing prime multiplication R-module not isomorphic with R is pure-injective.*

Proof. Let M_i denote an indecomposable summand of M. Then by Lemma 2.6(*iii*), M_i is an indecomposable absorbing prime multiplication module. Now the assertion follows from Theorem 2.11 and [2, Proposition 1.3]. □

3. The separated absorbing prime multiplication modules

Throughout this section, we shall assume unless otherwise stated, that

$$R = (R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2) \qquad (1)$$

is the pullback of two local Dedekind domains R_1, R_2 with maximal ideals P_1, P_2 generated, respectively, by p_1, p_2, P denotes $P_1 \oplus P_2$ and

$$R_1/P_1 \cong R_2/P_2 \cong R/P \cong \bar{R}$$

is a field. In particular, R is a commutative Noetherian local ring with unique maximal ideal P. The other prime ideals of R are easily seen to be P_1 (that is $P_1 \oplus 0$) and P_2 (that is $0 \oplus P_2$).

Remark 3.1. ([8, Remark 3.1]) Let R be the pullback ring as in (1) and let T be an R-submodule of a separated module $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$, with projection maps $\pi_i : S \to S_i$, $i = 1, 2$. Set

$$T_1 = \{t_1 \in S_1 : (t_1, t_2) \in T \text{ for some } t_2 \in S_2\}$$

and

$$T_2 = \{t_2 \in S_2 : (t_1, t_2) \in T \text{ for some } t_1 \in S_1\}.$$

Then for each i, $i = 1, 2$, T_i is an R_i-submodule of S_i and $T \leq T_1 \oplus T_2$. Moreover, we can define a mapping $\acute{\pi_1} = \pi_1|T : T \to T_1$ by sending (t_1, t_2) to t_1, hence

$$\begin{aligned} T_1 &\cong T/((0 \oplus Ker(f_2)) \cap T) \\ &\cong T/(T \cap P_2S) \\ &\cong (T + P_2S)/P_2S \\ &\subseteq S/P_2S. \end{aligned}$$

Thus we may assume that T_1 is a submodule of S_1. Similarly, we may assume that T_2 is a submodule of S_2 (note that $Ker(f_1) = P_1S_1$ and $Ker(f_2) = P_2S_2$).

We need the following proposition proved in [10, Proposition 3.2].

Proposition 3.2. *Let $T = (T_1 \longrightarrow \bar{T} \longleftarrow T_2)$ be a proper submodule of a separated module $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ over the pullback ring as in (1). Then the following hold:*

- (i) *$Rad(T : S) = I \oplus J$ if and only if $Rad(T_1 : S_1) = I$ and $Rad(T_2 : S_2) = J$, where $I \neq 0$ and $J \neq 0$.*
- (ii) *$Rad(T : S) = P_1 \oplus 0$ if and only if $Rad(T_1 : S_1) = P_1$ and $Rad(T_2 : S_2) = 0$.*
- (iii) *$Rad(T : S) = 0 \oplus P_2$ if and only if $Rad(T_1 : S_1) = 0$ and $Rad(T_2 : S_2) = P_2$.*

Lemma 3.3. *Let I be a 1-absorbing prime ideal of R. Then $Rad(I) = P$ is a prime ideal of R such that $P^2 \subseteq I$.*

Proof. It follows from [21, Theorem 2.3 and Lemma 2.8]. □

Proposition 3.4. *Let R be the pullback ring as in (1) and let*

$$S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$$

be a separated R-module. Then the following hold:

- (i) *If S has a 1-absorbing prime submodule $T = (T_1 \longrightarrow \bar{T} \longleftarrow T_2)$ with $Rad(T : S) = P = P_1 \oplus P_2$ and $P^2 \subseteq (T : S)$, then T_1 is a 1-absorbing prime submodule of S_1 with $Rad(T_1 : S_1) = P_1$ and T_2 is a 1-absorbing prime submodule of S_2 with $Rad(T_2 : S_2) = P_2$.*
- (ii) *If S has a 1-absorbing prime submodule $T = (T_1 \longrightarrow \bar{T} \longleftarrow T_2)$ with $Rad(T : S) = P_1 \oplus 0$ and $(P_1 \oplus 0)^2 \subseteq (T : S)$, then T_1 is a 1-absorbing prime submodule of S_1 with $Rad(T_1 : S_1) = P_1$ and T_2 is a 1-absorbing prime submodule of S_2 with $Rad(T_2 : S_2) = 0$.*

(iii) *If S has a 1-absorbing prime submodule $T = (T_1 \longrightarrow \bar{T} \longleftarrow T_2)$ with $Rad(T : S) = 0 \oplus P_2$ and $(0 \oplus P_2)^2 \subseteq (T : S)$, then T_1 is a 1-absorbing prime submodule of S_1 with $Rad(T_1 : S_1) = 0$ and T_2 is a 1-absorbing prime submodule of S_2 with $Rad(T_2 : S_2) = P_2$.*

Proof. (i) Take non-unit elements $a_1, b_1 \in R_1$ and $s_1 \in S_1$ such that $a_1 b_1 s_1 \in T_1$. Then $v_1(a_1) = v_2(a_2)$, $v_1(b_1) = v_2(b_2)$ and $f_1(s_1) = f_2(s_2)$ for some $a_2, b_2 \in R_2$ and $s_2 \in S_2$. Hence $(a_1, a_2), (b_1, b_2) \in R$ are non-unit elements. We have $(a_1, a_2)(b_1, b_2)(s_1, s_2) \in P^2 S \subseteq T$. Since T is a 1-absorbing prime submodule of S, so $(s_1, s_2) \in T$ or

$$(a_1, a_2)(b_1, b_2) \in (T :_R S).$$

Thus $s_1 \in T_1$ or $a_1 b_1 \in (T_1 : S_1)$. Similarly, T_2 is a 1-absorbing prime submodule of S_2.

(ii) Take non-unit elements $a_1, b_1 \in R_1$ and $s_1 \in S_1$ such that $a_1 b_1 s_1 \in T_1$. Since $a_1, b_1 \in P_1$, so we have $v_1(a_1) = 0 = v_2(0)$, $v_1(b_1) = 0 = v_2(0)$. Then $(a_1, 0), (b_1, 0) \in R$ and there exist $s_2 \in S_2$ such that $f_1(s_1) = f_2(s_2)$. Therefore

$$(a_1, 0)(b_1, 0)(s_1, s_2) \in (P_1 \oplus 0)^2 S \subseteq T$$

by hypothesise. So $(a_1, 0)(b_1, 0(s_1, s_2) \in T$ and thus $(s_1, s_2) \in T$ or $(a_1, 0)(b_1, 0) \in (T :_R S)$. Hence $s_1 \in T_1$ or $a_1 b_1 \in (T_1 :_{R_1} S_1)$. Therefore, T_1 is a 1-absorbing prime submodule of S_1. Now we show that T_2 is a 1-absorbing prime submodule of S_2. Take non-unit elements $a_2, b_2 \in R_2$ and $s_2 \in S_2$ such that $a_2 b_2 s_2 \in T_2$ and $a_2 b_2 \notin (T_2 : S_2) = 0$. Thus there exists $s_1 \in S_1$ such that $(s_1, s_2) \in S$. Since $p_1^2 s_1 \in T_1 \cap P_1 S_1$ ($p_1 \in P_1 = (T_1 : S_1)$), $a_2 b_2 s_2 \in T_2 \cap P_2 S_2$ ($a_2 b_2 \in P_2$) and

$$f_1(p_1^2 s_1) = 0 = f_2(a_2 b_2 s_2),$$

we give $(p_1, a_2)(p_1, b_2)(s_1, s_2) \in T$. Hence $(s_1, s_2) \in T$, because T is a 1-absorbing prime submodule of S. Hence $s_2 \in T_2$, so T_2 is a 1-absorbing prime submodule of S_2.

(iii) It is similar to that (ii). □

Proposition 3.5. *Let*

$$S = (S/P_2 S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1 S)$$

be a separated module over the pullback ring as (1). Then $abpSpec(S) = \emptyset$ if and only if $abpSpec(S_i) = \emptyset$ for $i = 1, 2$.

Proof. For the necessarily, assume that $abpSpec(S) = \emptyset$ and let π be the projection map of R onto R_i. Suppose that $abpSpec(S_1) \neq \emptyset$ and let T_1 be a 1-absorbing prime submodule of S_1, so T_1 is a 1-absorbing prime R-submodule of $S/(0 \oplus P_2)S \cong S_1$, hence $abpSpec(S) \neq \emptyset$ by

Proposition 2.3, which is a contradiction. Similarly, $abpSpec(S_2) = \emptyset$. The sufficiency by Proposition 3.4. □

Theorem 3.6. *Let*

$$S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1S)$$

be a separated module over the pullback ring as (1)*. Then S is an absorbing prime multiplication R-module if and only if S_i is an absorbing prime multiplication R_i-module, $i = 1, 2$.*

Proof. By Proposition 3.5, $abpSpec(S) = \emptyset$ if and only if $abpSpec(S_i) = \emptyset$ for $i = 1, 2$. So we may assume that $abpSpec(S) \neq \emptyset$. Let S be a separated absorbing prime multiplication R-module. If $\bar{S} = 0$, then by [2, Lemma 2.7], $S = S_1 \oplus S_2$, hence for each i, S_i is an absorbing prime multiplication module by Lemma 2.6. So we may assume that $\bar{S} \neq 0$. Since $(0 \oplus P_2) \subseteq ((0 \oplus P_2)S : S)$, Lemma 2.6, show

$$S_1 \cong S/(0 \oplus P_2)S$$

is an absorbing prime multiplication $R/(0 \oplus P_2) \cong R_1$-module. Similarly, S_2 is an absorbing prime multiplication R_2-module. Conversely, assume that each S_i is an absorbing prime multiplication R_i-module and let $T = (T_1 \to \bar{T} \leftarrow T_2)$ be a 1-absorbing submodule of S. We may assume that $(T : S) \neq 0$. Now we split the proof into two cases for $Rad(T : S)$. *Case 1.* $Rad(T : S) = P$. Then $S_i \neq 0$ for $i = 1, 2$. By Proposition 3.4, we have T_1 is a 1-absorbing prime submodule of S_1 and T_2 is a 1-absorbing prime submodule of S_2. Hence $T_1 = P_1^m S_1 \subseteq P_1S_1$ and $T_2 = P_2^n S_2 \subseteq P_1S_1$ since S_1, S_2 are absorbing prime multiplication modules. Let $k = min\{m, n\}$. Therefore,

$$T \subseteq T_1 \oplus T_2 \subseteq P^{k-1}(P_1S_1 \oplus P_2S_2) \subseteq P^kS.$$

For the containment, assume that

$$s = (p_1^k, p_2^k)(s_1, s_2) = (p_1^k s_1, p_2^k s_2) \in P^kS.$$

Then $s \in T$ since $p_1^k s_1 \in T_1$, $p_2^k s_2 \in T_2$ and $f_1(p_1^k s_1) = 0 = f_2(p_2^k s_2)$ (note that $Ker(f_1) = P_1S_1$ and $Ker(f_2) = P_2S_2$).

Case 2. $Rad(T : S) = P_1 \oplus 0$. So by Proposition 3.4, T_1 is a 1-absorbing prime submodule of S_1 with $Rad(T_1 : S_1) = P_1$ and T_2 is a 1-absorbing prime submodule of S_2 with $(T_2 : S_2) = 0$ ($Rad(T_2 : S_2) = 0$). So $T_2 = 0$ since S_2 is an absorbing prime multiplication R_2-module and also, $T_1 = P_1^m S_1$. Therefore,

$$T \subseteq T_1 \oplus T_2 \subseteq (P_1 \oplus 0)^{m-1}(P_1S_1 + P_2S_2) = (P_1 \oplus 0)^m S.$$

Now let $s = (p_1^m, 0)(s_1, s_2) \in (P_1 \oplus 0)^m S$, then $s \in T$ since $p_1^m s_1 \in T_1$ and $f_1(p_1^m s_1) = 0 = f_2(0)$. Hence $T = (P_1 \oplus 0)^m S$, so S is an absorbing prime multiplication module. Similarly, if $Rad(T : S) = 0 \oplus P_2$, then we get S is an absorbing prime multiplication module. □

We need the following lemma proved in [6, Lemma 4.3].

Lemma 3.7. *Let $R \neq S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ be a separated module over the pullback ring as* (1)*. If S_1 or S_2 is torsion-free, then $\bar{S} = 0$.*

Lemma 3.8. *Let R be the pullback ring as in* (1)*. The following separated R-modules are indecomposable and absorbing prime multiplication modules:*

(1) $R = (R_1 \longrightarrow \bar{R} \longleftarrow R_2)$*;*
(2) $S = (E(R_1/P_1) \longrightarrow 0 \longleftarrow 0)$*,* $(0 \longrightarrow 0 \longleftarrow E(R_2/P_2))$*, where $E(R_i/P_i)$ is the R_i-injective hull of R_i/P_i for $i = 1, 2$;*
(3) $S = (Q(R_1) \longrightarrow 0 \longleftarrow 0)$*,* $(0 \longrightarrow 0 \longleftarrow Q(R_2))$*, where $Q(R_i)$ is the field of fractions of R_i for $i = 1, 2$;*
(4) $R = (R_1/P_1^n \longrightarrow \bar{R} \longleftarrow R_2/P_2^m)$ *for all positive integers m, n.*

Proof. By [2, Lemma 2.8], these modules are indecomposable, absorbing prime multiplicativity follows from Theorem 2.11 and Theorem 3.6. □

We refer to modules of type (2) in Lemma 3.8 as P_1-Prüfer and P_2-Prüfer, respectively.

Theorem 3.9. *Let $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ be an indecomposable separated absorbing prime multiplication module over the pullback ring as* (1)*. Then S is isomorphic to one of the modules listed in Lemma 3.8. In particular, every indecomposable separated absorbing prime multiplication R-module which is not isomorphism with R is pure-injective.*

Proof. We may assume that $S \neq R$. If $abpSpec(S) = \emptyset$, then by Proposition 3.5 $abpSpec(S_i) = \emptyset$ for each $i = 1, 2$. So $S_i = P_i S_i$ for each $i = 1, 2$ by Theorem 2.11, hence

$$S = PS = P_1 S_1 \oplus P_2 S_2 = S_1 \oplus S_2.$$

Therefore, $S = S_1$ or S_2 and so S is of type (2) in the list of Lemma 3.8 by Theorem 2.11. Now we may assume that $abpSpec(S) \neq \emptyset$. Suppose that $PS = S$. Then by [2, Lemma 2.7], $S = S_1$ or S_2 and so S is an indecomposable absorbing prime multiplication R_i-modules for some i and since $PS = S$, then S is of type (3) in the list of Lemma 3.8, by Theorem 2.11. So we may assume that $S/PS \neq 0$. By

Theorem 3.6, S_i is an absorbing prime multiplication R_i-module, for each $i = 1, 2$. Now by Theorem 2.11, for each i, S_i is torsion and it is not divisible R_i-module. Hence by the structure of absorbing prime multiplication modules over a local Dedekind domain (see Corollary 2.12), $S_i = M_i \oplus N_i$ where N_i is a direct sum of copies of R_i/P_i^n $(n \geq 1)$ and M_i is a direct sum of copies $E(R_i/P_i)$ and $Q(R_i)$. Then we have

$$S = (N_1 \longrightarrow \bar{S} \longleftarrow N_2) \oplus (M_1 \longrightarrow 0 \longleftarrow 0) \oplus (0 \longrightarrow 0 \longleftarrow M_2).$$

Since S is indecomposable and $S/PS \neq 0$ it follows that

$$S = (N_1 \longrightarrow \bar{S} \longleftarrow N_2).$$

We will see that each $S_i (= N_i)$ is indecomposable. There exist positive integers m, n and k such that $P_1^m S_1 = 0$, $t = (t_1, t_2)$ such that $o(t) = n$, $o(t_1) = m$ and $o(t_2) = k$. Then $R_i t_i$ is pure in S_i for $i = 1, 2$ (see [2, Theorem 2.9]). Therefore, $R_1 t_1 \cong R_1/P_1^m$ (resp. $R_2 t_2 \cong R_2/P_2^k$) is a direct summand of S_1 (resp. S_2) since for each i, $R_i t_i$ is pure-injective [2, Lemma 2.8]. Let $\bar{M}$ be the $\bar{R}$-subspace of $\bar{S}$ generated by $\bar{t}$. Then $\bar{M} \cong \bar{R}$. Let $M = (R_1 t_1 = M_1 \longrightarrow \bar{M} \longleftarrow M_2 = R_2 t_2)$. Then M is an R-submodule of S which is an absorbing prime multiplication module by Lemma 3.8, and it is a direct summand of S, this implies that $S = M$, and S is as in (4) (see [2, Lemma 2.8]). □

Corollary 3.10. *Let R be the pullback ring as in* (1)*, and let $S \neq R$ be a separated absorbing prime multiplication R-module. Then S is of the form $S = M \oplus N$, where M is a direct sum of copies of the modules as in* (3)*, N is a direct sum of copies of the modules as in* $(1) - (2)$ *of the Lemma 3.8. In particular, every separated absorbing prime multiplication R-module not isomorphic with R is pure-injective.*

Proof. Apply Theorem 3.9 and [2, Lemma 2.8]. □

4. The non-separated absorbing prime multiplication modules

We continue to use the notions already established, so R is the pullback ring as in (1). In this section, we find the indecomposable non-separated absorbing prime multiplication modules with finite-dimensional top. It turns out that each can be obtained by amalgamating finitely many separated indecomposable absorbing prime multiplication modules.

Proposition 4.1. *Let R be the pullback ring as in* (1)*. Then $E(R/P)$ is a non-separated absorbing prime multiplication R-module.*

Proof. It suffices to show that $abpSpac(E(R/P)) = \emptyset$. Assume that L is any submodule of $E(R/P)$ described in [9, Proposition 3.1]. However, no L, say $E_1 + A_n$ is a 1-absorbing prime submodule of $E(R/P)$, for if n is any positive integer, then $P^2(E_1 + A_{n+2}) = E_1 + A_n$, but

$$(E_1 + A_{n+2}) \nsubseteq E_1 + A_n,$$

and $P^2 \nsubseteq (E_1 + A_n : E(R/P)) = 0$. Therefore, $E(R/P)$ is a non-separated absorbing prime multiplication R-module (see [2, p. 4053]). □

Proposition 4.2. *Let R be the pullback ring as in* (1)*, and let M be an R-module. Let*

$$0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$$

be a separated representation of M. Then $abpSpec(S) = \emptyset$ if and only if $abpSpec(M) = \emptyset$.

Proof. First suppose that $abpSpec(S) = \emptyset$ and let $abpSpec(M) \neq \emptyset$. So $M \cong S/K$ has a 1-absorbing prime submodule, say T/K, where T is a 1-absorbing prime submodule of S by Proposition 2.3, which is a contradiction. Next suppose that $abpSpec(M) = \emptyset$ and let $abpSpec(S) \neq \emptyset$. Let T be a 1-absorbing prime submodule of S. Then by [5, Proposition 4.3], $K \subseteq T$; hence T/K is a 1-absorbing prime submodule of M, which is a contradiction. □

Theorem 4.3. *Let R be the pullback ring as in* (1)*, and let M be a non-separated R-module. Let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of M. Then S is an absorbing prime multiplication module if and only if M is an absorbing prime multiplication module.*

Proof. By Proposition 4.2, we may assume that $abpSpec(S) \neq \emptyset$. Suppose that M is an absorbing prime multiplication R-module and let T be a non-zero 1-absorbing prime submodule of S. Then by [5, Proposition 4.3], $K \subseteq T$ and so T/K is a 1-absorbing prime submodule of S/K. Since $M \cong S/K$ is an absorbing prime multiplication module, we must have

$$T/K = P^n(S/K) = (P^nS + K)/K = (P^nS)/K$$

(note that $K \subseteq P^nS$ by [5, Proposition 4.3]), hence $T = P^nS$. Thus S is an absorbing prime multiplication module. Conversely, if S is an absorbing prime multiplication module, then $M \cong S/K$ is an absorbing prime multiplication module by Lemma 2.6, and this completes the proof. □

Proposition 4.4. *Let R be the pullback ring as in* (1)*, and let M be an indecomposable absorbing prime multiplication non-separated R-module with finite dimensional top over $\bar{R}$. Let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of M. Then S is pure-injective.*

Proof. By [2, Proposition 2.6 (i)], $S/PS \cong M/PM$, so S has finite-dimensional top. Now the assertion follows from Theorem 4.3 and Corollary 3.10. □

Lemma 4.5. *Let R be the pullback ring as in* (1)*, and let M be an indecomposable absorbing prime multiplication non-separated R-module with finite dimensional top over $\bar{R}$, and let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of M. Then R does not occur among the direct summand of S.*

Proof. Let $S = R \oplus T$ for some submodule T of S. Then since $Soc(R) = 0$, we must have $K \subseteq T$. Therefore, $M \cong T/K \oplus R$, a contradiction since M is indecomposable and non-separated. □

Let R be the pullback ring as in (1), and let M be an indecomposable absorbing prime multiplication non-separated R-module with finite-dimensional top over $\bar{R}$. Consider the separated representation $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$. By Proposition 4.4, S is a pure-injective module. So by the proofs of Lemma 3.1, Proposition 3.2 and proposition 3.4 of [2] (here the pure-injectivity of M implies the pure-injectivity of S by [2, Proposition 2.6 (ii)]), we can replace the statement "M is an indecomposable pure-injective non-separated R-module" by "M is an indecomposable absorbing prime multiplication non-separated R-module", because the main key in those results are the pure-injectivity of S, the indecomposability and the non-separability of M. So we have the following result.

Corollary 4.6. *Let R be the pullback ring as in* (1)*, and let M be an indecomposable absorbing prime multiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let*

$$0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$$

be a separated representation of M. Then the following hold:

(i) *The quotient fields $Q(R_1)$ and $Q(R_2)$ of R_1 and R_2 do not occur among the direct summand of S;*
(ii) *S is a direct sum of finitely many indecomposable absorbing prime multiplication modules;*
(iii) *At most two copies of modules of finite length can occur among the indecomposable summands of S.*

Recall that every indecomposable R-module of finite length is an absorbing prime multiplication module since it is a quotient of a multiplication R-module. So by Corollary 4.6(iii), the infinite length non-separated indecomposable absorbing prime multiplication modules are obtained in just the same way as the deleted cycle type indecomposable ones are, except that at least one of the two "*end*" modules must be a separated indecomposable absorbing prime multiplication of infinite length (that is, P_1-Prüfer and P_2-Prüfer). Note that one can not have, for instance, a P_1-Prüfer module at each end (consider the alternation of primes P_1, P_2 along the amalgamation chain). So, apart from any finite length modules: we have amalgamations involving two Prüfer modules as well as modules of finite length (the injective hull $E(R/P)$ is the simplest module of this type), a P_1-Prüfer module and a P_2-Prüfer module. If the P_1-Prüfer module and the P_2-Prüfer module are direct summands of S then we will describe these modules as **doubly infinite**. Those where S has just one infinite length summand we will call **singly infinite** (the reader is referred to [2], [4] and [9] for more details). It remains to show that the modules obtained by these amalgamations are, indeed, indecomposable absorbing prime multiplication modules.

Theorem 4.7. *Let $R = (R_1 \longrightarrow \bar{R} \longleftarrow R_2)$ be the pullback ring of two discrete valuation domains R_1, R_2 with common factor field $\bar{R}$. Then the class of indecomposable non-separated absorbing prime multiplication modules with finite-dimensional top consists of the following:*

(i) *The indecomposable modules of finite length (apart from R/P which is separated);*
(ii) *The doubly infinite absorbing prime multiplication modules as described above;*
(iii) *The singly infinite absorbing prime multiplication modules as described above, except the two Prüfer modules (2) in Lemma 3.8.*

Proof. Let M be an indecomposable non-separated absorbing prime multiplication R-module with finite-dimensional top, and let

$$0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$$

be a separated representation of M.

(i) Clearly, M is an absorbing prime multiplication R-module. The indecomposability follows from [14].

(ii) and (iii) (involving one or two Prüfer modules) M is an absorbing prime multiplication module since they are a quotient of an absorbing prime multiplication R-module. Finally, the indecomposability follows from [2, Theorem 3.5]. □

Corollary 4.8. *Let R be the pullback ring as described in Theorem 4.7. Then every indecomposable absorbing prime multiplication R-module with finite-dimensional top is pure-injective.*

Proof. Apply [2, Theorem 3.5] and Theorem 4.7. □

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ABSORBING PRIME MULTIPLICATION MODUIES OVER A PULLBACK RING

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مدول‌های ضربی اول جاذب روی یک حلقه پولبک

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هدف اصلی این مقاله ارائه یک رویکرد جدید به طبقه‌بندی همه مدول‌های ضربی اول جاذب تجزیه‌ناپذیر با تاپ متناهی‌البعد روی حلقه‌های پولبک از دو دامنه‌ی ددکیند می‌باشد. ابتدا، شرح کاملی از مدول‌های ضربی اول جاذب را روی یک دامنه ددکیند موضعی بیان می‌کنیم. در حقیقت، تعریف و نتایج ارائه شده در مرجع ۵ را به یک مدول ضربی اول جاذب تعمیم می‌دهیم. در مرحله بعد، ارتباط بین مدول‌های اول جاذب و مدول‌های انژکتیو محض روی این نوع از حلقه‌ها را برقرار می‌کنیم.

کلمات کلیدی: مدول‌های ضربی اول جاذب، دامنه‌های ددکیند، مدول‌های جداپذیر، مدول‌های انژکتیو محض، حلقه‌های پولبک.