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A KIND OF GRAPH STRUCTURE ASSOCIATED WITH ZERO-DIVISORS OF MONOID RINGS

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Abstract. Let R be an associative ring and M be a monoid. In this paper, we introduce new kind of graph structure asociated with zero-divisors of monoid ring $R[M]$, calling it the M-Armendariz graph of a ring R and denoted by $A(R, M)$. It is an undirected graph whose vertices are all non-zero zero-divisors of the monoid ring $R[M]$ and two distinct vertices $\alpha = a_1g_1 + \cdots + a_ng_n$ and $\beta = b_1h_1+\cdots+b_mh_m$ are adjacent if and only if $a_ib_j = 0$ or $b_ja_i = 0$ for all i, j. We investigate some graph properties of $A(R, M)$ such as diameter, girth, domination number and planarity. Also, we get some relations between diameters of the M-Armendariz graph $A(R, M)$ and that of zero divisor graph $\Gamma(R[M])$, where R is a reversible ring and M is a unique product monoid.

1. Introduction and preliminaries

The investigation of graphs related to various algebraic structures is one of useful methods to study the properties of algebraic structures. When one assigns a graph to an algebraic structure, numerous interesting algebraic problems arise from the translation of some graph-theoretic parameters such as diameter, girth and so on. There are a lot of papers which apply combinatorial methods to obtain algebraic results. In particular, zero-divisor graphs have

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attracted serious attention in the literature. The concept of zero-divisor graph was first introduced by Beck in [3]. In his work all elements of ring were vertices. Anderson and Livingston [2] redefined the notion of zero-divisor graph of commutative ring R and introduced the zero-divisor graph $\Gamma(R)$ whose vertices are non-zero zero-divisors of R and two distinct vertices a, b joined by an edge if and only if $ab = 0$. Redmond [16] studied the zero-divisor graph of a non-commutative ring. Let R be an associated ring with identity. The set of zero-divisors of R denoted by $Z(R)$. Then undirected graph $\Gamma(R)$ is a graph with vertices in $Z(R)^* = Z(R) \setminus \{0\}$, where $a - b$ is an edge between two distinct vertices a and b if and only if $ab = 0$ or $ba = 0$.

For two distinct vertices a, b in the undirected graph Γ, the *distance* between a, b denoted by $d(a, b)$, is the length of shortest path connecting a and b, if such a path exists, otherwise we put $d(a, b) = \infty$. Recall that the *diameter* of a graph Γ is defined as follows:

$$\mathrm{diam}(\Gamma) = \mathrm{Sup}\{d(a, b) \mid a \text{ and } b \text{ are distinct vertices of } \Gamma\}.$$

The *girth* of Γ, denoted by $gr(\Gamma)$, is the length of shortest cycle in Γ, if Γ contains a cycle. If there is no cycle in Γ, then $gr(\Gamma) = \infty$. A graph is said to be *connected* if there is a path between any two distinct vertices. A graph in which each pair of distinct vertices is joined by an edge is called a *complete* graph. We use K_n for the complete graph with n vertices. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$.

Let us recall some ring theory terminologies. In what follows R will always denote an associative ring and, unless otherwise specified, with identity. Also, M will denote a monoid written multiplicatively, and $R[M]$ will denote the monoid ring with coefficients in R and exponents in M. As usual, the rings of integers and integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively. Moreover, $\mathbb{N}$ will denote the natural numbers and we denote by C_n the cyclic group of order n. All other terminology is standard, and definitions can be found in any ring theory monograph. According to Cohn [6], a ring R is called *reversible* if $ab = 0$ implies that $ba = 0$ for each $a, b \in R$. Clearly reduced ring (i.e., rings with no non-zero nilpotent elements) and commutative rings are reversible. In [12], Liu introduced the concept of Armendariz ring relative to monoid. A ring R is called *M-Armendariz* ring, if whenever elements $\alpha = a_1g_1 + \cdots + a_ng_n,\ \beta = b_1h_1 + \cdots + b_mh_m \in R[M]$ satisfy $\alpha\beta = 0$, then $a_ib_j = 0$ for each i, j.

There is considerable interest in studying if and how certain graph-theoretic properties of rings are preserved under various ring-theoretic extensions. The first such extensions that come to mind are those of polynomial and power series extensions. In [8], Hashemi and Alhevaz studied the zero-divisor graph of a monoid ring $R[M]$ over noncommutative ring R. Armendariz graph is another graph structure associated with commutative rings which is introduced in [1]. The *Armendariz graph* of a commutative ring R, denoted by $A(R)$, is an undirected graph having all non-zero zero-divisors of $R[x]$ as its vertex set, and two distinct vertices $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ joined by an edge if and only if $a_i b_j = 0$, for all i, j. As it is shown in [1], $A(R)$ is connected, $\operatorname{diam}(A(R)) \in \{1, 2, 3\}$ and $gr(A(R)) \leq 4$. Since we are especially interested in non-commutative aspects of the theory, we do not wish to restrict ourselves to commutative cases. Let R be an associative ring and M be a monoid. We introduce a new kind of graph structure asociated with zero-divisors of monoid ring $R[M]$, calling it the M-Armendariz graph of a ring R and denoted by $A(R, M)$. It is an undirected graph whose vertices are all non-zero zero-divisors of the monoid ring $R[M]$ and two distinct vertices $\alpha = a_1 g_1 + \cdots + a_n g_n$ and $\beta = b_1 h_1 + \cdots + b_m h_m$ are adjacent if and only if $a_i b_j = 0$ for all i, j. It is clear that, in the special case of $M = (\mathbb{N} \cup \{0\}, +)$, the monoid ring $R[M]$ is isomorphic to polynomial ring $R[x]$ and hence M-Armendariz graph $A(R, M)$ coincides with the usual Armendariz graph $A(R)$.

In this paper we study some basic properties of the graph $A(R, M)$, where R is a reversible ring and M is a unique product monoid. We will show that under what conditions M-Armendariz graph $A(R, M)$ is connected. Then we investigate some graph properties of $A(R, M)$ such as diameter, girth, domination number and planarity. Moreover, we get some relations between diameters of $A(R, M)$ and $\Gamma(R[M])$, where R is a reversible ring and M is a unique product monoid.

2. Main results

Let P and Q be nonempty subsets of a monoid M. An element s is called a *u.p.-element* (*unique product element*) of

$$PQ = \{pq : p \in P\ ,\ q \in Q\}$$

if it is uniquely presented in the form $s = pq$ where $p \in P$ and $q \in Q$. Recall that a monoid M is called a *u.p.-monoid* (*unique product monoid*) if for any two non-empty finite subsets $P, Q \subseteq M$ there exist a u.p.-element in PQ. Unique product monoids and groups play an important role in ring theory, for example providing a positive case in the zero-divisor problem for group rings (see also [4]), and their structural properties have been extensively studied (see [4, 14, 15]). The class of u.p.-monoids includes the right and the left totally ordered monoids, sub- monoids of a free group, and torsion-free nilpotent groups. Every u.p.-monoid M is cancellative and has no non-unity element of finite order. Thus if M is a u.p.-monoid and $1 \neq g \in M$, then the set $\{1, g, g^2, \dots\}$ is infinite.

A monoid M (written multiplicatively) equipped with an order $\leq$ is called an *ordered monoid* if for any $s_1, s_2, t \in M$, $s_1 \leq s_2$ implies $s_1 t \leq s_2 t$ and $ts_1 \leq ts_2$. Moreover, if $s_1 < s_2$ implies $s_1 t < s_2 t$ and $ts_1 < ts_2$, then $(M, \leq)$ is said to be *strictly ordered*.

Marks et. al. [13], introduced and studied new classes of u.p.-monoids as follows: An ordered set $(S, \leq)$ is called *artinian* if every strictly decreasing sequence of elements of S is finite, and $(S, \leq)$ is called *narrow* if every subset of pairwise order-incomparable elements of S is finite. It is easy to see that $(S, \leq)$ is artinian an narrow if and only if every nonempty subset of S contains at least one, but only a finite number, of minimal elements. For a partially ordered set X we write $\min X$ to denote the set of minimal elements of X.

An ordered monoid $(M, \leq)$ is called an *artinian narrow unique product monoid* (or an *a.n.u.p.-monoid*, or simply *a.n.u.p.*) if for every two artinian and narrow subsets A and B of M there exists a u.p. element in the product AB. Also an ordered monoid $(M, \leq)$ is called a *minimal artinian narrow unique product monoid* (or a *m.a.n.u.p.-monoid*, or simply *m.a.n.u.p.*) if for every two artinian and narrow subsets A and B of M there exist $a \in \min A$ and $b \in \min B$ such that ab is a u.p. element of AB.

For an ordered monoid $(M, \leq)$, Marks et. al. [13] proved the following implications:

$$\begin{gathered} M \text{ is a commutative, torsion-free, cancellative monoid} \\ \Downarrow \\ (M, \leq) \text{ is quasitotally ordered} \\ \Downarrow \\ (M, \leq) \text{ is a m.a.n.u.p.-monoid} \\ \Downarrow \\ (M, \leq) \text{ is an a.n.u.p.-monoid} \\ \Downarrow \\ M \text{ is a u.p.-monoid} \end{gathered}$$

Also they showed that all of the implications in diagram above are irreversible. So, the class of unique product monoids is large and very important.

Example 2.1. Let R be a ring and M a monoid. If R is a M-Armendariz ring, then the M-Armendariz graph of R and the zero divisor graph of $R[M]$, denoted by $\Gamma(R[M])$, are coincide. Specially, for a reduced ring R and u.p. monoid M, R is M-Armendariz by [12, Proposition 1.1], hence $A(R, M)$ and $\Gamma(R[M])$ are coincide.

Let R be a ring. Define a ring $T(R)$ as follows:

$$T(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}.$$

Example 2.2. Let R be a ring and let

$$T_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a & a_{23} & \dots & a_{2n} \\ 0 & 0 & a & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

where $n \geq 4$ is a positive integer. Let M be a monoid with $|M| \geq 2$. We denote the identity element in M by e. Take $e \neq g \in M$. Let $\alpha = e_{12}e + (e_{12} - e_{13})g$ and $\beta = e_{34}e + (e_{24} + e_{34})g$ be in $T_n(R)[M]$, where e_{ij}'s are the matrix units in $T_n(R)$. According to [12, Remark 1.8], $\alpha\beta = 0$, but $(e_{12} - e_{13})e_{34} \neq 0$. Thus α and β are adjacent in zero-divisor graph $\Gamma(R[M])$ but are not adjacent in M-Armendariz graph $A(R, M)$. Therefore these two graph are different.

Example 2.3. Let $M = (\mathbb{N} \cup \{0\}, +)$ and $R = \mathbb{Z}_8 \times \mathbb{Z}_8$ be the trivial extension of the ring $\mathbb{Z}_8$ by the module $\mathbb{Z}_8$. The product is defined by $(a, m)(b, m) = (ab, an + bm)$. Let $f(x) = (4, 2) + (4, 1)x$, $g(x) = (4, 0) + (4, 1)x \in R[x]$. As it is shown in [1, Example 2], $f(x)$ and $g(x)$ are adjacent in $\Gamma(R[x])$ while they are not adjacent in the Armendariz graph $A(R)$. Therefore these two graph are different. Therefore $A(R, M)$ and $\Gamma(R[M])$ are different.

Let M be a monoid and R a ring. According to [7], R is called *right M-McCoy* if whenever

$$0 \neq \alpha = a_1g_1 + \cdots + a_ng_n,\ 0 \neq \beta = b_1h_1 + \cdots + b_mh_m \in R[M],$$

with $g_i, h_j \in M$ and $a_i, b_j \in R$ satisfy $\alpha\beta = 0$, then $\alpha r = 0$ for some non-zero element $r \in R$. Left M-McCoy rings are defined similarly. If R is both left and right M-McCoy then R is called *M-McCoy*.

In [7, Proposition 1.2], among other results it was shown that if M is a u.p.-monoid, then a reversible ring R is M-McCoy. Now, we present it here giving an alternative straightforward proof.

Theorem 2.4. *Let M be a u.p.-monoid and R be a reversible ring. Then R is M-McCoy.*

Proof. Let β have the smallest length such that $\alpha\beta = 0$, where $\alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m \in R[M]$. Suppose that $a_i \neq 0 \neq b_j$ for each i, j. Since M is a *u.p.*-monoid, without loss of generality, we can assume that g_1h_1 is uniquely presented by considering two subsets $A_1 = \{g_1, \ldots, g_n\}$ and $B = \{h_1, \ldots, h_m\}$ of M. Thus $a_1b_1 = 0$, and so $b_1a_1 = 0$, since R is reversible. It follows that $\text{length}(\beta a_1) \leq \text{length}(\beta)$. Since $\alpha\beta a_1 = 0$, we have $\beta a_1 = 0$, and thus $a_1\beta = 0$. Therefore $0 = \alpha\beta = (a_2g_2 + \cdots a_ng_n)\beta$. Hence there exist i, j with $2 \leq i \leq n$ and $1 \leq j \leq m$ such that g_ih_j is uniquely presented by considering two subsets $A_2 = \{g_2, \ldots, g_n\}$ and B of M. Without loss of generality, put $i = 2$ and $j = 1$. Thus $a_2b_1 = b_1a_2 = 0$. Now, $\alpha\beta a_2 = 0$ and $\text{length}(\beta a_2) \leq \text{length}(\beta)$ imply that $\beta a_2 = 0$, and so $a_2\beta = 0$. By continuing this process, we get $a_i\beta = 0$ for each $i = 1, \ldots, n$. Hence R is right M-McCoy, since $\alpha b_1 = 0$. By a similar argument one can prove that R is left M-McCoy. □

The following example shows that the condition "M is a u.p.-monoid" in Theorem 2.4 is not superfluous.

Example 2.5. Let $R = \mathbb{Z}$ and M be a monoid with two elements, namely $G = \{e, g\}$ with $g^2 = e$. It is clear that R is reversible and M is not a u.p.-monoid. Let $\alpha = e + g$ and $\beta = e - g$ be elements of $R[M]$. Then $\alpha\beta = 0$ in $R[M]$ but $\alpha r \neq 0$ for each $0 \neq r \in \mathbb{Z}$.

Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then by [9, Remark 2.2], the graph $\Gamma(R)$ is complete, whereas the graph $A(R, M)$ is not complete. Let $\alpha = (1, 0)g_1$, $\beta = (1, 0)g_2$ for $g_1, g_2 \in M$. In this case, $\alpha - (0, 1) - \beta$ is a path in $A(R, M)$ but α, β are not adjacent. However, by the following theorem, we will show that there is a relationship between the zero-divisor graph and M-Armendariz graph of a reversible ring R.

Theorem 2.6. *Let $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a reversible ring and M a u.p. monoid. Then the following are equivalent:*

1) $\Gamma(R[M])$ *is complete.*
2) $A(R, M)$ *is complete.*
3) $\Gamma(R)$ *is complete.*

Proof. Observe that $\Gamma(R)$ is a subgraph of $A(R, M)$, which is a subgraph of $\Gamma(R[M])$. Thus $1 \Rightarrow 2 \Rightarrow 3$ is clear.

(3) $\Rightarrow$ (1): By [9, Remark 2.2], $ab = 0$ for all $a, b \in Z(R)$. On the other hand $Z(R[M]) \subseteq Z(R)[M]$, since R is M-McCoy by Theorem 2.4. Thus $\Gamma(R[M])$ is complete. □

Theorem 2.7. *Let R be a reversible ring and M a u.p. monoid. Then there is a vertex of $A(R,M)$ which is adjacent to every other vertex if and only if $Z(R)$ is an annihilator ideal of R.*

Proof. Let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n \in Z(R[M])$ which is adjacent to every other vertex in $A(R,M)$. Then, for all $r \in Z(R) \subseteq Z(R[M])$, $ra_i = 0$ for all $1 \leq i \leq n$. Therefore $R \cong \mathbb{Z}_2 \times D$ where D is a domain or $Z(R)$ is an annihilator ideal by [9, Remark 2.1]. If R is isomorphic to $\mathbb{Z}_2 \times D$, then the coefficients of α are either $(1,0)$ or $(0,a)$ in $A(R,M)$ where $a \in D$, Since $Z(R[M]) \subseteq Z(R)[M]$, by Theorem 2.4. In any case α is not adjacent to either $(1,0)$ or $(0,a)$ in $A(R,M)$, which is a contradiction. Thus R is not isomorphic to $\mathbb{Z}_2 \times D$ and hence $Z(R)$ is an annihilator ideal.

Conversely, let $Z(R)$ be an annihilator ideal in R and let $Z(R) = ann(a)$ for some non-zero element $a \in R$. Since R is M-McCoy by Theorem 2.4, all coefficients of non-zero zero-divisors of $R[M]$ contained in $Z(R)$. Therefore a is adjacent to every other vertices in $A(R,M)$. □

Proposition 2.8. *Let R be a reversible ring and M a u.p. monoid. If $\alpha = a_1g_1 + \cdots + a_ng_n$ is a vertex of $A(R,M)$, then there exists a cycle of length 3 or 4 in $A(R,M)$ including α as one vertex.*

Proof. There exist $0 \neq \beta = b_1h_1 + \cdots + b_mh_m \in Z(R[M])$ such that $a_ib_j = 0$ for $1 \leq i \leq n, 1 \leq j \leq m$. R is M-McCoy by Theorem 2.4, there exist $0 \neq r, s \in R$ such that $ra_i = sb_j = 0$ for $1 \leq i \leq n$, $1 \leq j \leq m$. If $r = s$, then $r - \alpha - \beta - r$ is a cycle. If $r \neq s$ and $rs = 0$, then $r - \alpha - \beta - s - r$ is a cycle. If $rs \neq 0$, but $sa_i = 0$ for all i, then $s - \alpha - \beta - s$ is a cycle. If $sa_i = r \neq 0$ for all i, $r - \alpha - \beta - r$ is a cycle. If both $rs \neq 0$ and $sa_i \neq 0$, $sa_i \neq r$ for some i, then $r - \alpha - \beta - sa_i - r$ is a cycle in $A(R,M)$. □

Theorem 2.9. *Let R be a reversible ring and M a u.p. monoid. Then $A(R,M)$ is connected and* $\operatorname{diam}(A(R,M)) \leq 3$.

Proof. Let α, β be two distinct elements in $Z(R[M]) \setminus \{0\}$. If they are adjacent, we are done. Otherwise, there exist $0 \neq r, s \in R$ such that $r\alpha = 0 = s\beta$, since R is M-McCoy by Theorem 2.4. If $rs \neq 0$, then $\alpha - rs - \beta$ is a path in $A(R,M)$, since R is reversible. If $rs = 0$, then $\alpha - r - s - \beta$ is a path in $A(R,M)$. If $r = s$, then $\alpha - r - \beta$ is a path in $A(R,M)$. Hence $A(R,M)$ is connected and $\operatorname{diam}(A(R,M)) \leq 3$. □

According to [11], a ring R has *right (resp., left) Property (A)* if every finitely generated two sided ideal of R consisting entirely of left (resp., right) zero-divisors has right (resp., left) non-zero annihilator.

A ring R is said to has Property (A) if R has right and left Property (A).

Theorem 2.10. *Let R be a reversible ring and M a u.p. monoid. Then*

1) $\operatorname{diam}(A(R,M)) = 2$ *if and only if* $\operatorname{diam}(\Gamma(R[M])) = 2$.
2) $\operatorname{diam}(A(R,M)) = 3$ *if and only if* $\operatorname{diam}(\Gamma(R[M])) = 3$.

Proof. 1) Suppose that $\operatorname{diam}(A(R,M)) = 2$. Since $A(R,M)$ is an induced subgraph of $\Gamma(R[M])$ with the same vertices, hence we have $\operatorname{diam}(\Gamma(R[M])) \leq \operatorname{diam}(A(R,M)) = 2$. If $\operatorname{diam}(\Gamma(R[M])) = 1$, then $\operatorname{diam}(A(R,M))) = 1$, by Theorem 2.6, which is a contradiction. Therefore we have $\operatorname{diam}(\Gamma(R,M)) = 2$.

Conversely, suppose that $\operatorname{diam}(\Gamma(R[M])) = 2$. Then R is a reduced ring with exactly two minimal primes, or R has right Property (A) and $Z(R)$ is an ideal of R with $Z(R)^2 \neq 0$, by [8, Theorem 3.12]. If R is reduced, then R is M-Armendariz by [12, Proposition 1.1]. Hence $\Gamma(R[M])$ and $A(R,M)$ are coincide by Example 2.1. Now, assume that R has right Property (A) and $Z(R)$ is an ideal of R with $Z(R)^2 \neq 0$. There exist

$$\alpha, \beta \in Z(R[M]) \setminus \{0\}$$

such that $\alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m$ and $\alpha\beta \neq 0$, otherwise $\operatorname{diam}(\Gamma(R[M])) = 1$, which is a contradiction. By Theorem 2.4, $Z(R[M]) \subseteq Z(R)[M]$. Hence the finite generated ideal $\langle a_1, \ldots, a_n, b_1, \ldots, b_m\rangle$ contained in $Z(R)$. Therefore there exists nonzero element $r \in R$ such that $\langle a_1, \ldots, a_n, b_1, \ldots, b_m\rangle r = 0$, since R has right Property (A). Thus $\alpha - r - \beta$ is a path in $A(R,M)$.

2) Assume that $\operatorname{diam}(A(R,M)) = 3$. Since $A(R,M)$ is a subgraph of $\Gamma(R[M])$, hence

$$\operatorname{diam}(\Gamma(R[M])) \subseteq \operatorname{diam}(A(R,M)).$$

If $\operatorname{diam}(\Gamma(R[M])) = 1$, then $\operatorname{diam}(A(R,M)) = 1$, by Theorem 2.6, which is a contradiction. Also, if $\operatorname{diam}(\Gamma(R[M])) = 2$, then we have $\operatorname{diam}(A(R,M)) = 2$ by statement (1), which is a contradiction. Therefore $\operatorname{diam}(\Gamma(R[M])) = 3$.

Conversely, suppose that $\operatorname{diam}(\Gamma(R[M])) = 3$. Since

$$\operatorname{diam}(\Gamma(R[M])) \leq \operatorname{diam}(A(R,M)) \leq 3,$$

then $\operatorname{diam}(A(R,M)) = 3$. ∎

Corollary 2.11. *Let R be a reversible ring and M a u.p. monoid. Then*

$$\operatorname{diam}(A(R,M)) = \operatorname{diam}(\Gamma(R[M])).$$

Proof. The result follows from Theorems 2.6, 2.10. □

Example 2.12. Let $R = \mathbb{Z}_9$ and $M = C_4$. Then R is a non-reduced reversible ring and $Z(R) = \{0,3\}$ is an ideal of R with $(Z(R))^2 = 0$ and that $R[M] \cong R \oplus R \oplus R[x]/(x^2+1)$. Clearly,

$$Z(R[M]) = \{(a,b,c) \in R[M] \mid a = 0 \text{ or } b = 0 \text{ or } c = 0\}.$$

Let $\alpha = (1,1,0)$ and $\beta = (0,1,1) \in R[M]$. Then $d(\alpha,\beta) = 3$ and so $\operatorname{diam}(\Gamma(R[M])) = 3$. Since $A(R,M)$ is an induced subgraph of $\Gamma(R[M])$ with the same vertices, $\operatorname{diam}(\Gamma(R[M])) \leq \operatorname{diam}(A(R,M))$. Hence by Theorem 2.9, we get $\operatorname{diam}(A(R,M)) = 3$.

Theorem 2.13. *Let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a reversible ring and M a u.p. monoid. Then*

$$\operatorname{diam}(A(R,M)) = \operatorname{diam}(\Gamma(R[M])) = \operatorname{diam}(\Gamma(R)).$$

Proof. Since $A(R,M)$ and $\Gamma(R[M])$ have the same vertex set, hence $\operatorname{diam}(\Gamma(R[M])) \leq \operatorname{diam}(A(R,M))$. Following [9], $\operatorname{diam}(\Gamma(R)) \leq 3$. If $\operatorname{diam}(\Gamma(R)) = 1$, then by Theorem 2.6 we get

$$\operatorname{diam}(A(R,M)) = \operatorname{diam}(\Gamma(R[M])) = \operatorname{diam}(\Gamma(R)) = 1.$$

If $\operatorname{diam}(\Gamma(R)) = 2$, then $\operatorname{diam}(\Gamma(R[M])) = 2$ by [8, Theorems 3.9, 3.12]. Therefore $\operatorname{diam}(A(R,M)) = 2$, by Theorem 2.10. Now suppose that $\operatorname{diam}(\Gamma(R)) = 3$. Since

$$\operatorname{diam}(\Gamma(R)) \leq \operatorname{diam}(\Gamma(R[M])) \leq \operatorname{diam}(A(R,M)) \leq 3,$$

then

$$\operatorname{diam}(A(R,M)) = \operatorname{diam}(\Gamma(R[M])) = 3.$$

□

The following theorem concerns the girth of the graph $A(R,M)$.

Theorem 2.14. *Let R be a reversible ring and M a monoid. Then $gr(A(R,M)) \leq 4$. If R is not reduced, then $gr(A(R,M)) = 3$.*

Proof. If there exists two distinct non-zero elements $a, b \in R$ such that $ab = 0$, then $a - b - ag_1 - bg_2 - a$ is a cycle in $A(R,M)$ for $g_1, g_2 \in M$. If a is a nilpotent element in R such that $a^n = 0$ but $a^{n-1} \neq 0$, then $a - a^{n-1}g_1 - a^{n-2}g_2 - a$ is a cycle in $A(R,M)$. □

Let G be a graph with vertex set V. A nonempty subset S of V is called a *dominating set* of G if every vertex in $V \setminus S$ is adjacent to at least a vertex in S. In other words, S dominates the vertices outside S. A γ*-set* of G is a minimum dominating set of G, that is, a dominating set of G whose cardinality is minimum. The *domination number* of

G, denoted by $\gamma(G)$, is the cardinality of a γ-set of G. The following theorem concerns the domination number of the graph $A(R,M)$.

Theorem 2.15. *Let R be a reversible and M a u.p. monoid. Then* $\gamma(\Gamma(R)) \leq \gamma(A(R,M)) \leq \mathbf{card}(Z(R)^*)$.

Proof. Since R is M-McCoy by Theorem 2.4, then for any zero-divisor $\alpha \in R[M]$ there exists a non-zero element $r \in R$ such that $\alpha r = 0$. Thus $Z(R)^*$ is a dominating set in $A(R,M)$ and hence

$$\gamma(A(R,M)) \leq \mathbf{card}(Z(R)^*).$$

Let S be a dominating set in $A(R,M)$. Then for all

$$r_i \in Z(R) \subseteq Z(R[M])$$

there exists $\alpha_i = a_{1_i}g_{1_i} + \cdots + a_{n_i}g_{n_i} \in S$ such that $\alpha_i r_i = 0$. Especially, $a_{1_i}r_i = 0$. Set

$$T = \{a_{1_i} : \alpha_i = a_{1_i}g_{1_i} + \cdots + a_{n_i}g_{n_i} \in Z(R[M])\}.$$

Then T is a dominating set for $\Gamma(R)$ and $\mathbf{card}(T) \leq \mathbf{card}(S)$. Thus

$$\gamma(\Gamma(R)) \leq \mathbf{card}(T) \leq \mathbf{card}(S) \leq \gamma(A(R,M)) \leq \mathbf{card}(Z(R)^*).$$

□

We conclude by investigation of the planarity of the new defined graph $A(R,M)$. Recall that a graph is called *planar* if it can be drawn on the plane without edge crossing except at endpoints. Kuratowski's Theorem states that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ (see [5, p. 24]).

Theorem 2.16. *For any ring R with at least one non-zero zero-divisor, the graph $A(R,M)$ is not planar.*

Proof. Let $a \in Z(R)^*$. There exists $0 \neq b \in R$ such that $ab = 0$. Consider the subgraph $\langle S \rangle$ induced by $S = S_1 \cup S_2$ where $S_1 = \{a, ag_1, ag_2\}$ and $S_2 = \{b, bg_3, bg_4\}$ such that $g_i \neq g_j$ for all i, j. Clearly $\langle S \rangle$ contains a complete bipartite graph $K_{3,3}$ with S_1 and S_2 being the two partite sets. Therefore as $A(R,M)$ contains $K_{3,3}$ as a subgraph, hence $A(R,M)$ is not planar. □

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A KIND OF GRAPH STRUCTURE ASSOCIATED WITH ZERO–DIVISORS OF MONOID RINGS

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نوعی از ساختار گرافی متناظر با مقسوم‌علیه‌های صفر حلقه‌های مونوییدی

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فرض کنید R یک حلقه شرکت‌پذیر و M یک مونویید باشد. در این مقاله نوع جدیدی از ساختار گرافی را معرفی خواهیم کرد که متناظر با مقسوم‌علیه‌های صفر حلقه‌ی مونوییدی $R[M]$ می‌باشد، که ما آن را گراف M–آرمنداریز حلقه R نامیده و به صورت $A(R, M)$ نمایش می‌دهیم. این گراف یک گراف بدون‌جهت بوده که مجموعه راسی آن تمام مقسوم‌علیه‌های صفر $R[M]$ که ناصفر می‌باشند را در بر دارد و دو راس متمایز $\alpha = a_1g_1 + \cdots + a_ng_n$ و $\beta = b_1h_1 + \cdots + b_mh_m$ مجاورند اگر و تنها اگر $a_ib_j = 0$ یا $b_ja_i = 0$ برای هر i, j. برخی از خواص گرافی این گراف جدید به مانند قطر، کمر، عدد احاطه گری و همچنین مسطح‌پذیری را مورد مطالعه قرار می‌دهیم. همچنین، برخی ارتباط‌هایی بین قطر گراف M–آرمنداریز حلقه‌ی R یعنی $A(R, M)$ و قطر گراف مقسوم‌علیه صفر حلقه‌ی $R[M]$ یعنی $\Gamma(R[M])$، جایی‌که R حلقه‌ای برگشت‌پذیر و M یک مونویید حاصلضرب یکتا می‌باشد، به‌دست می‌آوریم.

کلمات کلیدی: گراف M–آرمنداریز، قطر، مونویید حاصلضرب یکتا، حلقه مونوییدی، عدد احاطه‌گری.