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QUOTIENT STRUCTURES IN EQUALITY ALGEBRAS

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ABSTRACT. The notion of fuzzy ideal in bounded equality algebras is defined, and several properties are studied. Fuzzy ideal generated by a fuzzy set is established, and a fuzzy ideal is made by using the collection of ideals. Characterizations of fuzzy ideal are discussed. Conditions for a fuzzy ideal to attains its infimum on all ideals are provided. Homomorphic image and preimage of fuzzy ideal are considered. Finally, quotient structures of equality algebra induced by (fuzzy) ideal are studied.

1. INTRODUCTION

EQ-algebras were introduced by Novák et al [16]. Equality algebras were introduced by Jenei [7, 8] by removing the multiplication operation and as an extension of EQ-algebras. In [5, 9] the authors investigate the relation between equality algebra and BCK-meet-semilattice. Dvurečenskij et al. in [6] defined pseudo-equality algebra as an extension of equality algebra and study some properties of it. Borzooei et al. [4] introduced some types of filters of equality algebras and study the relation between them and moreover, they considered relations among equality algebras and some of the other logical algebras such as residuated lattice, MTL-algebra, BL-algebra, MV-algebra, and etc., in [22]. Since ideal theory is an important notion in logical algebras, Paad [17] introduced the notion of the ideal

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in bounded equality algebras and showed that there is a reciprocal correspondence between ideals and congruence relations.

Fuzzy sets were first introduced by Zadeh [21] and then studied by many mathematicians. Some mathematicians tried to overcome its shortcomings by presenting various extensions of fuzzy sets, and some other mathematicians studied fuzzy sets on various algebraic structures such as logical algebraic structures, groups, and loops. These results can be seen in the articles [1, 3, 2, 11, 12, 13, 18, 19].

The aim of this paper is to introduce the concept of fuzzy ideal in bounded equality algebras and study the fuzzy ideal generated by a fuzzy set. Using the collection of ideals, we make a fuzzy ideal. We discuss characterizations of fuzzy ideal and provide conditions for a fuzzy ideal to attains its infimum on all ideals. We consider homomorphic image and preimage of fuzzy ideal and study quotient structures of equality algebra induced by (fuzzy) ideal.

2. Preliminaries

In this section, we will refer to some of the previous works that we will use in this article.

Definition 2.1. [8] An algebraic structure $(\mathcal{E}, \curlywedge, \backsim, 1)$ is an *equality algebra*, if for any $\mathfrak{m}, \mathfrak{r}, \mathfrak{i} \in \mathcal{E}$ we have:

(E1) $(\mathcal{E}, \curlywedge, 1)$ is a commutative idempotent integral monoid,
(E2) the operation "$\backsim$" is commutative,
(E3) $\mathfrak{m} \backsim \mathfrak{m} = 1$,
(E4) $\mathfrak{m} \backsim 1 = \mathfrak{m}$,
(E5) if $\mathfrak{m} \precsim \mathfrak{r} \precsim \mathfrak{i}$, then $\mathfrak{m} \backsim \mathfrak{i} \precsim \mathfrak{r} \backsim \mathfrak{i}$ and $\mathfrak{m} \backsim \mathfrak{i} \precsim \mathfrak{m} \backsim \mathfrak{r}$,
(E6) $\mathfrak{m} \backsim \mathfrak{r} \precsim (\mathfrak{m} \curlywedge \mathfrak{i}) \backsim (\mathfrak{r} \curlywedge \mathfrak{i})$,
(E7) $\mathfrak{m} \backsim \mathfrak{r} \precsim (\mathfrak{m} \backsim \mathfrak{i}) \backsim (\mathfrak{r} \backsim \mathfrak{i})$,

where $\mathfrak{m} \precsim \mathfrak{r}$ iff $\mathfrak{m} \curlywedge \mathfrak{r} = \mathfrak{m}$.

In an equality algebra $(\mathcal{E}, \curlywedge, \backsim, 1)$, we introduce an operation $\dashrightarrow$ (*implication*) on $\mathcal{E}$ as follows:

$$\mathfrak{m} \dashrightarrow \mathfrak{r} := \mathfrak{m} \backsim (\mathfrak{m} \curlywedge \mathfrak{r}).$$

Note. From now on, $(\mathcal{E}, \curlywedge, \backsim, 1)$ or $\mathcal{E}$, for short, is an equality algebra.

Proposition 2.2. [8] *For any* $\mathfrak{m}, \mathfrak{r}, \mathfrak{i} \in \mathcal{E}$ *we have:*
(*i*) $\mathfrak{m} \dashrightarrow \mathfrak{r} = 1$ *iff* $\mathfrak{m} \precsim \mathfrak{r}$,
(*ii*) $\mathfrak{m} \dashrightarrow (\mathfrak{r} \dashrightarrow \mathfrak{i}) = \mathfrak{r} \dashrightarrow (\mathfrak{m} \dashrightarrow \mathfrak{i})$,
(*iii*) $1 \dashrightarrow \mathfrak{m} = \mathfrak{m}$, $\mathfrak{m} \dashrightarrow 1 = 1$ *and* $\mathfrak{m} \dashrightarrow \mathfrak{m} = 1$,
(*iv*) $\mathfrak{m} \precsim \mathfrak{r} \dashrightarrow \mathfrak{i}$ *iff* $\mathfrak{r} \precsim \mathfrak{m} \dashrightarrow \mathfrak{i}$,
(*v*) $\mathfrak{m} \precsim \mathfrak{r} \dashrightarrow \mathfrak{m}$,

(vi) $\mathfrak{m} \precsim (\mathfrak{m} \dashrightarrow \mathfrak{r}) \dashrightarrow \mathfrak{r}$,
(vii) $\mathfrak{m} \dashrightarrow \mathfrak{r} \precsim (\mathfrak{r} \dashrightarrow \mathfrak{i}) \dashrightarrow (\mathfrak{m} \dashrightarrow \mathfrak{i})$,
(viii) *if* $\mathfrak{r} \precsim \mathfrak{m}$, *then* $\mathfrak{m} \dashrightarrow \mathfrak{r} = \mathfrak{m} \backsim \mathfrak{r}$,
(ix) *if* $\mathfrak{m} \precsim \mathfrak{r}$, *then* $\mathfrak{r} \dashrightarrow \mathfrak{i} \precsim \mathfrak{m} \dashrightarrow \mathfrak{i}$ *and* $\mathfrak{i} \dashrightarrow \mathfrak{m} \precsim \mathfrak{i} \dashrightarrow \mathfrak{r}$.

$\mathcal{E}$ is *bounded* if $\mathcal{E}$ has a least element $0 \in \mathcal{E}$ such that $0 \precsim \mathfrak{m}$, for any $\mathfrak{m} \in \mathcal{E}$. Then we can introduce the negation "$\neg$" on $\mathcal{E}$ as

$$\neg\mathfrak{m} = \mathfrak{m} \dashrightarrow 0 = \mathfrak{m} \backsim 0,$$

for all $\mathfrak{m} \in \mathcal{E}$. If for all $\mathfrak{l} \in \mathcal{E}$, $\neg\neg\mathfrak{l} = \mathfrak{l}$, then $\mathcal{E}$ is called *involutive*.

Let $\mathcal{E}$ and M be sets and $h : \mathcal{E} \to M$ be a function. If ϖ is a fuzzy set in $\mathcal{E}$, then the *image* of ϖ under h is denoted by $h(\varpi)$ and is defined as follows.

$$h(\varpi) : M \to [0,1], \ \mathfrak{g} \mapsto \begin{cases} \sup\limits_{\mathfrak{l} \in h^{-1}(\mathfrak{g})} \varpi(\mathfrak{l}) & \text{if } h^{-1}(\mathfrak{g}) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

If θ is a fuzzy set in $h(\mathcal{E})$, then the *preimage* of θ under h is denoted by $h^{-1}(\theta)$ and is defined by

$$h^{-1}(\theta) : \mathcal{E} \to [0,1], \ \mathfrak{l} \mapsto \theta(h(\mathfrak{l})).$$

Definition 2.3. [17] Suppose $\mathcal{E}$ is bounded. Then $\emptyset \neq Q \subseteq \mathcal{E}$ is said to be an *ideal* of $\mathcal{E}$ if for any $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$,
(I_1) if $\mathfrak{l} \precsim \mathfrak{g}$ and $\mathfrak{g} \in Q$, then $\mathfrak{l} \in Q$,
(I_2) $\neg\mathfrak{l} \dashrightarrow \mathfrak{g} \in Q$, for all $\mathfrak{l}, \mathfrak{g} \in Q$.
All ideals of $\mathcal{E}$ is shown by $\mathcal{I}(\mathcal{E})$.

Lemma 2.4. [17] *Assume $\mathcal{E}$ is bounded and $\emptyset \neq Q \subseteq \mathcal{E}$. Then $Q \in \mathcal{I}(\mathcal{E})$ iff for any $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$ we have:*
(I_3) $0 \in Q$,
(I_4) *if* $\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l}) \in Q$ *and* $\mathfrak{g} \in Q$, *then* $\mathfrak{l} \in Q$.

Note. Now, we suppose ($\mathcal{E}$, $\curlywedge$, $\backsim$,0, 1) or $\mathcal{E}$ is a bounded equality algebra.

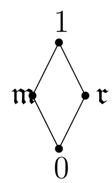
3. Fuzzy ideals in bounded equality algebras

In this part, we define the concept of fuzzy ideal in bounded equality algebras and study some attributes of it. We establish fuzzy ideal generated by a fuzzy set and discuss characterizations of fuzzy ideal. Conditions for a fuzzy ideal to attains its infimum on all ideals are provided.

Definition 3.1. A fuzzy set ϖ in $\mathcal{E}$ is said to be a *fuzzy ideal* of $\mathcal{E}$ if for any $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$:
(FI_1) $\varpi(0) \gtrsim \varpi(\mathfrak{l})$,

(FI_2) $\varpi(\mathfrak{l}) \gtrsim \min\{\varpi(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})), \varpi(\mathfrak{g})\}$.
All fuzzy ideals of $\mathcal{E}$ is denoted by $\mathcal{FI}(\mathcal{E})$.

Example 3.2. Consider $\mathcal{E} = \{0, \mathfrak{m}, \mathfrak{r}, 1\}$ is a set with the next Hasse diagram.



Then $(\mathcal{E}, \curlywedge, 1)$ is a commutative monoid. Define an operation $\backsim$ on $\mathcal{E}$ by Table 1.

TABLE 1. Cayley table for the implication "$\backsim$"

$\backsim$	0	$\mathfrak{m}$	$\mathfrak{r}$	1
0	1	$\mathfrak{r}$	$\mathfrak{m}$	0
$\mathfrak{m}$	$\mathfrak{r}$	1	0	$\mathfrak{m}$
$\mathfrak{r}$	$\mathfrak{m}$	0	1	$\mathfrak{r}$
1	0	$\mathfrak{m}$	$\mathfrak{r}$	1

Then $(\mathcal{E}, \curlywedge, \backsim, 1)$ is an equality algebra, and the implication "$\dashrightarrow$" is certained by Table 2.

TABLE 2. Cayley table for the implication "$\dashrightarrow$"

$\dashrightarrow$	0	$\mathfrak{m}$	$\mathfrak{r}$	1
0	1	1	1	1
$\mathfrak{m}$	$\mathfrak{r}$	1	$\mathfrak{r}$	1
$\mathfrak{r}$	$\mathfrak{m}$	$\mathfrak{m}$	1	1
1	0	$\mathfrak{m}$	$\mathfrak{r}$	1

Define a fuzzy set ϖ in $\mathcal{E}$ by:

$$\varpi : \mathcal{E} \to [0,1], \ \mathfrak{l} \mapsto \begin{cases} 0.7 & \text{if } \mathfrak{l} = 0, \\ 0.5 & \text{if } \mathfrak{l} = \mathfrak{m}, \\ 0.3 & \text{if } \mathfrak{l} \in \{\mathfrak{r}, 1\}. \end{cases}$$

Clearly, $\varpi \in \mathcal{FI}(\mathcal{E})$.

Theorem 3.3. *Assume $S \subseteq \mathcal{E}$ and consider a fuzzy set ϖ_S in $\mathcal{E}$ as follows:*

$$\varpi_S : \mathcal{E} \to [0,1],\ \mathfrak{l} \mapsto \begin{cases} \mathfrak{s} & \text{if } \mathfrak{l} \in S, \\ \mathfrak{t} & \text{otherwise,} \end{cases} \tag{3.1}$$

for $\mathfrak{s}, \mathfrak{t} \in [0,1]$ with $\mathfrak{s} > \mathfrak{t}$. Then $\varpi_S \in \mathcal{F}I(\mathcal{E})$ iff $S \in \mathcal{I}(\mathcal{E})$. Moreover $S = \{\mathfrak{l} \in \mathcal{E} \mid \varpi_S(\mathfrak{l}) = \varpi_S(0)\}$.

Proof. ($\Rightarrow$) Consider $\varpi_S \in \mathcal{F}I(\mathcal{E})$. Clearly, by (FI_1), $\varpi_S(0) = \mathfrak{s}$. Thus $0 \in S$. Let $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$ such that $\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l}) \in S$ and $\mathfrak{g} \in S$. Then $\varpi_S(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})) = \mathfrak{s} = \varpi_S(\mathfrak{g})$. From (FI_2) we obtain

$$\varpi_S(\mathfrak{l}) \gtrsim \min\{\varpi_S(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})), \varpi_S(\mathfrak{g})\} = \mathfrak{s}.$$

Hence $\mathfrak{l} \in S$, and therefore $S \in \mathcal{I}(\mathcal{E})$.

($\Leftarrow$) From $S \in \mathcal{I}(\mathcal{E})$, we get $0 \in S$, and so $\varpi_S(0) = \mathfrak{s} \gtrsim \varpi_S(\mathfrak{l})$ for all $\mathfrak{l} \in \mathcal{E}$. For every $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$, if $\min\{\varpi_S(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})), \varpi_S(\mathfrak{g})\} = \mathfrak{s}$, then it means that $\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l}) \in S$ and $\mathfrak{g} \in S$. Since $S \in \mathcal{I}(\mathcal{E})$, we get $\mathfrak{l} \in S$. Hence

$$\varpi_S(\mathfrak{l}) = \mathfrak{s} \gtrsim \min\{\varpi_S(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})), \varpi_S(\mathfrak{g})\}.$$

If $\min\{\varpi_S(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})), \varpi_S(\mathfrak{g})\} = \mathfrak{t}$, then it means that

$$\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l}) \notin S \text{ or } \mathfrak{g} \notin S,$$

and so

$$\mathfrak{t} = \min\{\varpi_S(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})), \varpi_S(\mathfrak{g})\} \lesssim \varpi_S(\mathfrak{l}).$$

Hence, $\varpi_S \in \mathcal{F}I(\mathcal{E})$. Therefore, $S = \{\mathfrak{l} \in \mathcal{E} \mid \varpi_S(\mathfrak{l}) = \varpi_S(0)\}$. ☐

Lemma 3.4 ([17])**.** *If F is a filter of $\mathcal{E}$, then*

$$N(F) := \{\mathfrak{l} \in \mathcal{E} \mid \neg\mathfrak{l} \in F\} \in \mathcal{I}(\mathcal{E}). \tag{3.2}$$

Corollary 3.5. *Consider $S \subseteq \mathcal{E}$ and a fuzzy set ϖ in $\mathcal{E}$ such that:*

$$\varpi : \mathcal{E} \to [0,1],\ \mathfrak{l} \mapsto \begin{cases} \mathfrak{s} & \text{if } \mathfrak{l} \in \{\mathfrak{l} \in \mathcal{E} \mid \neg\mathfrak{l} \in S\}, \\ \mathfrak{t} & \text{otherwise,} \end{cases} \tag{3.3}$$

for $\mathfrak{s}, \mathfrak{t} \in [0,1]$ with $\mathfrak{s} > \mathfrak{t}$. If S is a filter of $\mathcal{E}$, then $\varpi \in \mathcal{F}I(\mathcal{E})$.

Proposition 3.6. *Every fuzzy ideal of $\mathcal{E}$ is order reversing.*

Proof. Suppose $\varpi \in \mathcal{F}I(\mathcal{E})$ and $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$ such that $\mathfrak{l} \lesssim \mathfrak{g}$. Then by Proposition 2.2(vii), $1 = \mathfrak{l} \dashrightarrow \mathfrak{g} \lesssim \neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l}$, and so $\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l} = 1$.

Hence

$$\begin{aligned}\varpi(\mathfrak{l}) &\gtrsim \min\{\varpi(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})), \varpi(\mathfrak{g})\}\\ &= \min\{\varpi(\neg 1), \varpi(\mathfrak{g})\}\\ &= \min\{\varpi(0), \varpi(\mathfrak{g})\}\\ &= \varpi(\mathfrak{g}).\end{aligned}$$

Hence, ϖ is order reversing. □

Proposition 3.7. *If $\mathcal{E}$ is involutive, then for every $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$ and every $\varpi \in \mathcal{FI}(\mathcal{E})$, the following statements hold:*
(i) $\varpi(\neg\mathfrak{l} \dashrightarrow \mathfrak{g}) \gtrsim \min\{\varpi(\mathfrak{l}), \varpi(\mathfrak{g})\}$,
(ii) $\varpi(\mathfrak{l}) \gtrsim \min\{\varpi(\neg(\mathfrak{l} \dashrightarrow \mathfrak{g})), \varpi(\mathfrak{g})\}$.
(iii) $\varpi(1) \precsim \varpi(\mathfrak{l})$.

Proof. (*i*) Let $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$. Since $\mathcal{E}$ is involutive, by Proposition 2.2(vi), (vii) and (ix), we have

$$\neg(\neg\mathfrak{g} \dashrightarrow \neg(\neg\mathfrak{l} \dashrightarrow \mathfrak{g})) \precsim \neg((\neg\mathfrak{l} \dashrightarrow \mathfrak{g}) \dashrightarrow \mathfrak{g}) \precsim \neg\neg\mathfrak{l},$$

thus by Proposition 3.6, $\varpi(\neg(\neg\mathfrak{g} \dashrightarrow \neg(\neg\mathfrak{l} \dashrightarrow \mathfrak{g}))) \gtrsim \varpi(\neg\neg\mathfrak{l})$. Moreover, from $\mathcal{E}$ is involutive, by (FI_2) we obtain

$$\begin{aligned}\varpi(\neg\mathfrak{l} \dashrightarrow \mathfrak{g}) &\gtrsim \min\{\varpi(\neg(\neg\mathfrak{g} \dashrightarrow \neg(\neg\mathfrak{l} \dashrightarrow \mathfrak{g}))), \varpi(\mathfrak{g})\}\\ &\gtrsim \min\{\varpi(\neg\neg\mathfrak{l}), \varpi(\mathfrak{g})\}\\ &\gtrsim \min\{\varpi(\mathfrak{l}), \varpi(\mathfrak{g})\}.\end{aligned}$$

(*ii*) By Proposition 2.2(ix), $\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l}) \precsim \neg(\mathfrak{l} \dashrightarrow \mathfrak{g})$, then by Proposition 3.6 we have

$$\varpi(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})) \gtrsim \varpi(\neg(\mathfrak{l} \dashrightarrow \mathfrak{g})).$$

From (FI_2) we get

$$\begin{aligned}\varpi(\mathfrak{l}) &\gtrsim \min\{\varpi(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})), \varpi(\mathfrak{g})\}\\ &\gtrsim \min\{\varpi(\neg(\mathfrak{l} \dashrightarrow \mathfrak{g})), \varpi(\mathfrak{g})\}.\end{aligned}$$

(*iii*) If we put $\mathfrak{g} = 1$ in (ii), then by Proposition 2.2(iii) and (FI_1), we get

$$\begin{aligned}\varpi(\mathfrak{l}) &\gtrsim \min\{\varpi(\neg(\mathfrak{l} \dashrightarrow 1), \varpi(1)\}\\ &= \min\{\varpi(\neg 1), \varpi(1)\}\\ &= \min\{\varpi(0), \varpi(1)\}\\ &= \varpi(1).\end{aligned}$$

□

We make a fuzzy ideal using the family of ideals.

Theorem 3.8. *Assume* $\{Q_i \mid i \in \Lambda \subseteq [0,1]\}$ *is a family of ideals of* $\mathcal{E}$ *such that* $\mathcal{E} = \bigcup_{i \in \Lambda} Q_i$ *and for any* $i, j \in \Lambda$,

$$i \gtrsim j \text{ iff } Q_i \subseteq Q_j.$$

A fuzzy set ϖ *on* $\mathcal{E}$ *defined by*

$$\varpi : \mathcal{E} \to [0,1],\ \mathfrak{l} \mapsto \sup\{i \in \Lambda \mid \mathfrak{l} \in Q_i\} \in \mathcal{F}I(\mathcal{E}). \tag{3.4}$$

Proof. Obviously, $\varpi(0) \gtrsim \varpi(\mathfrak{l})$, for all $\mathfrak{l} \in \mathcal{E}$. Let $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$ such that

$$\varpi(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})) = \sup\{i \in \Lambda \mid \neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l}) \in Q_i\} = \mathfrak{t}$$

and $\varpi(\mathfrak{g}) = \sup\{i \in \Lambda \mid \mathfrak{g} \in Q_i\} = \mathfrak{k}$. Suppose $\mathfrak{t} \gtrsim \mathfrak{k}$. Then by assumption $Q_{\mathfrak{t}} \subseteq Q_{\mathfrak{k}}$. Since $\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l}) \in Q_{\mathfrak{t}} \subseteq Q_{\mathfrak{k}}$ and $\mathfrak{g} \in Q_{\mathfrak{k}}$, we get $\mathfrak{l} \in Q_{\mathfrak{k}}$. Hence

$$\begin{aligned}\varpi(\mathfrak{l}) &= \sup\{i \in \Lambda \mid \mathfrak{l} \in Q_i\} \\ &\gtrsim \mathfrak{k} \\ &= \min\{\varpi(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})), \varpi(\mathfrak{g})\}.\end{aligned}$$

Thus $\varpi \in \mathcal{F}I(\mathcal{E})$. □

Theorem 3.9. *Given a fuzzy set* ϖ *in* $\mathcal{E}$, *the next conditions are equivalent.*

(i) $\varpi \in \mathcal{F}I(\mathcal{E})$.

(ii) *The nonempty set*

$$U(\varpi, \mathfrak{t}) := \{\mathfrak{l} \in \mathcal{E} \mid \varpi(\mathfrak{l}) \gtrsim \mathfrak{t}\} \in \mathcal{I}(\mathcal{E}),\ \mathfrak{t} \in [0,1],$$

which is called a $\mathfrak{t}$*-level ideal of* ϖ.

Proof. (⇒) Consider $\varpi \in \mathcal{F}I(\mathcal{E})$ and $\mathfrak{t} \in [0,1]$ such that $U(\varpi, \mathfrak{t}) \neq \emptyset$. Then there exists $\mathfrak{l} \in U(\varpi, \mathfrak{t})$ and so $\varpi(\mathfrak{l}) \gtrsim \mathfrak{t}$. Since $\varpi \in \mathcal{F}I(\mathcal{E})$, we get $\varpi(0) \gtrsim \varpi(\mathfrak{l}) \gtrsim \mathfrak{t}$. Hence $0 \in U(\varpi, \mathfrak{t})$. Let $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$ such that $\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l}) \in U(\varpi, \mathfrak{t})$ and $\mathfrak{g} \in U(\varpi, \mathfrak{t})$. Then $\varpi(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})) \gtrsim \mathfrak{t}$ and $\varpi(\mathfrak{g}) \gtrsim \mathfrak{t}$. From $\varpi \in \mathcal{F}I(\mathcal{E})$, we obtain

$$\varpi(\mathfrak{l}) \gtrsim \min\{\varpi(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})), \varpi(\mathfrak{g})\} \gtrsim \mathfrak{t}.$$

Hence $\mathfrak{l} \in U(\varpi, \mathfrak{t})$. Therefore, $U(\varpi, \mathfrak{t}) \in \mathcal{I}(\mathcal{E})$.

(⇐) For $\mathfrak{l} \in \mathcal{E}$, let $\varpi(\mathfrak{l}) = \mathfrak{t}$. Then $\mathfrak{l} \in U(\varpi, \mathfrak{t})$. Since $U(\varpi, \mathfrak{t}) \in \mathcal{I}(\mathcal{E})$, we get $0 \in U(\varpi, \mathfrak{t})$, and so $\varpi(0) \gtrsim \mathfrak{t} = \varpi(\mathfrak{l})$, for all $\mathfrak{l} \in \mathcal{E}$. For any $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$, let

$$\mathfrak{t} := \min\{\varpi(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})), \varpi(\mathfrak{g})\}.$$

Then $\varpi(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})) \gtrsim \mathfrak{t}$ and $\varpi(\mathfrak{g}) \gtrsim \mathfrak{t}$, that is,

$$\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l}) \in U(\varpi, \mathfrak{t})$$

and $\mathfrak{g} \in U(\varpi, \mathfrak{t})$. Since $U(\varpi, \mathfrak{t}) \in \mathcal{I}(\mathcal{E})$, we have $\mathfrak{l} \in U(\varpi, \mathfrak{t})$, and so

$$\varpi(\mathfrak{l}) \gtrsim \min\{\varpi(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})), \varpi(\mathfrak{g})\} = \mathfrak{t}.$$

Therefore, $\varpi \in \mathcal{F}I(\mathcal{E})$. □

Lemma 3.10. *Suppose $\varpi \in \mathcal{F}I(\mathcal{E})$ and $\mathfrak{t}, \mathfrak{s} \in [0,1]$ with $\mathfrak{t} > \mathfrak{s}$. Then*

(i) *$U(\varpi, \mathfrak{t}) \subseteq U(\varpi, \mathfrak{s})$.*
(ii) *If $\mathfrak{t}, \mathfrak{s} \in Im(\varpi)$, then $U(\varpi, \mathfrak{t})$ is a proper subset of $U(\varpi, \mathfrak{s})$.*
(iii) *$U(\varpi, \mathfrak{t}) = U(\varpi, \mathfrak{s})$ iff there isn't $\mathfrak{l} \in \mathcal{E}$ such that $\mathfrak{t} \precsim \varpi(\mathfrak{l}) < \mathfrak{s}$.*

Theorem 3.11. *Assume $\varpi \in \mathcal{F}I(\mathcal{E})$ in which $Im(\varpi) = \{\mathfrak{t}_i \mid i \in \lambda\}$ and $\mathcal{I} = \{U(\varpi, \mathfrak{t}_i) \mid i \in \Lambda\}$ for any index set Λ. Then*

(i) *for any $i \in \Lambda$, there exists $j \in \Lambda$ such that $\mathfrak{t}_j \gtrsim \mathfrak{t}_i$.*
(ii) *$U(\varpi, \varpi(0)) = \bigcap_{i\in\Lambda} U(\varpi, \mathfrak{t}_i) = U(\varpi, \mathfrak{t}_j)$.*
(iii) *$\mathcal{E} = \bigcup_{i\in\Lambda} U(\varpi, \mathfrak{t}_i)$.*
(iv) *$(\mathcal{I}, \subseteq)$ is a chain.*
(v) *Every $\mathfrak{t}$-level ideal of ϖ is contained in $\mathcal{I}$ iff ϖ attains its infimum on all ideals of $\mathcal{E}$.*

Proof. (i) Since $\varpi(0) \in Im(\varpi)$, there exists a unique $j \in \Lambda$ such that $\varpi(0) = \mathfrak{t}_j$. By (FI_1), $\varpi(\mathfrak{l}) \precsim \varpi(0) = \mathfrak{t}_j$, for all $\mathfrak{l} \in \mathcal{E}$. Hence $\mathfrak{t}_j \gtrsim \mathfrak{t}_i$, for all $i \in \Lambda$.

(ii) By (i) and (FI_1), we have

$$\begin{aligned} U(\varpi, \mathfrak{t}_j) &= \{\mathfrak{l} \in \mathcal{E} \mid \varpi(\mathfrak{l}) \gtrsim \mathfrak{t}_j\} \\ &= \{\mathfrak{l} \in \mathcal{E} \mid \varpi(\mathfrak{l}) = \mathfrak{t}_j\} \\ &= \{\mathfrak{l} \in \mathcal{E} \mid \varpi(\mathfrak{l}) = \varpi(0)\} \\ &= U(\varpi, \varpi(0)). \end{aligned}$$

Since $\mathfrak{t}_j \gtrsim \mathfrak{t}_i$ for all $i \in \Lambda$, by using Lemma 3.10(i) we get

$$U(\varpi, \mathfrak{t}_j) \subseteq U(\varpi, \mathfrak{t}_i)$$

for all $i \in \Lambda$. Hence $U(\varpi, \varpi(0)) \subseteq \bigcap_{i\in\Lambda} U(\varpi, \mathfrak{t}_i)$, and so

$$U(\varpi, \varpi(0)) = \bigcap_{i\in\Lambda} U(\varpi, \mathfrak{t}_i).$$

(iii) Obviously $\bigcup_{i\in\Lambda} U(\varpi, \mathfrak{t}_i) \subseteq \mathcal{E}$. If $\mathfrak{l} \in \mathcal{E}$, then $\varpi(\mathfrak{l}) \in Im(\varpi)$ and so there exists $i_{\mathfrak{l}} \in \Lambda$ such that $\varpi(\mathfrak{l}) = \mathfrak{t}_{i_{\mathfrak{l}}}$. This induces

$$\mathfrak{l} \in U(\varpi, \mathfrak{t}_{i_{\mathfrak{l}}}) \subseteq \bigcup_{i\in\Lambda} U(\varpi, \mathfrak{t}_i).$$

(iv) Since either $i \gtrsim j$ or $i \precsim j$ for all $i, j \in \Lambda$, by Lemma 3.10(i), the proof is clear.

(v) $(\Rightarrow)$ Assume that every $\mathfrak{t}$-level ideal of ϖ is contained in $\mathcal{I}$ and let $S \in \mathcal{I}(\mathcal{E})$. If ϖ is constant on S, then nothing left to prove. Now, suppose ϖ is not constant. If $S = \mathcal{E}$, then

$$\mathfrak{s} := \inf\{\mathfrak{t}_i \mid i \in \Lambda\} \precsim \mathfrak{t}_i$$

for all $i \in \Lambda$ which implies that $U(\varpi, \mathfrak{t}_i) \subseteq U(\varpi, \mathfrak{s})$ for all $i \in \Lambda$. Note that $U(\varpi, 0) = \mathcal{E} \in \mathcal{I}$. Hence there exists $i \in \Lambda$ such that $\mathfrak{t}_i \in Im(\varpi)$ and $U(\varpi, \mathfrak{t}_i) = \mathcal{E}$. Thus $\mathcal{E} = U(\varpi, \mathfrak{t}_i) \subseteq U(\varpi, \mathfrak{s})$, and so $U(\varpi, \mathfrak{s}) = U(\varpi, \mathfrak{t}_i) = \mathcal{E}$. Now, we show $\mathfrak{s} = \mathfrak{t}_i$. If $\mathfrak{s} < \mathfrak{t}_i$, then $\mathfrak{t}_j \in Im(\varpi)$ and $\mathfrak{s} \precsim \mathfrak{t}_j < \mathfrak{t}_i$ for some $j \in \Lambda$. This induces

$$\mathcal{E} = U(\varpi, \mathfrak{t}_i) \subsetneq U(\varpi, \mathfrak{t}_j),$$

which is a contradiction. So $\mathfrak{s} = \mathfrak{t}_i$. Assume that S is a proper subset of $\mathcal{E}$. Then by Theorem 3.3, the restriction ϖ_S of ϖ to $S \in \mathcal{FI}(\mathcal{E})$. Let

$$\Lambda_S := \{i \in \Lambda \mid \varpi(\mathfrak{g}) = \mathfrak{t}_i \text{ for some } \mathfrak{g} \in \mathcal{E}\}$$

and $\mathcal{I}_S = \{U(\varpi_S, \mathfrak{t}_i) \mid i \in \Lambda_S\}$. Then all $\mathfrak{t}_i$-level ideals of ϖ_S are contained in $\mathcal{I}_S$. Hence $\varpi_S(\mathfrak{j}) = \inf\{\varpi_S(\mathfrak{l}) \mid \mathfrak{l} \in S\}$ for some $\mathfrak{j} \in S$, and so $\varpi(\mathfrak{j}) = \inf\{\varpi(\mathfrak{l}) \mid \mathfrak{l} \in S\}$.

($\Leftarrow$) By assumption, if $\mathfrak{t} = \mathfrak{t}_i$ for some $i \in \Lambda$, then $U(\varpi, \mathfrak{t}) \in \mathcal{I}$. If $\mathfrak{t} \neq \mathfrak{t}_i$ for all $i \in \Lambda$, then there is not $\mathfrak{l} \in \mathcal{E}$ such that $\varpi(\mathfrak{l}) = \mathfrak{t}$. Let $S = \{\mathfrak{l} \in \mathcal{E} \mid \varpi(\mathfrak{l}) > \mathfrak{t}\}$. Clearly $0 \in S$. Let $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$ such that $\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l}) \in S$ and $\mathfrak{g} \in S$. Then $\varpi(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})) > \mathfrak{t}$ and $\varpi(\mathfrak{g}) > \mathfrak{t}$. Using (FI_2), we have

$$\varpi(\mathfrak{l}) \gtrsim \min\{\varpi(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})), \varpi(\mathfrak{g})\} > \mathfrak{t}.$$

and so $\mathfrak{l} \in S$. Therefore $\varpi \in \mathcal{FI}(\mathcal{E})$. The hypothesis induces

$$\varpi(\mathfrak{g}) = \inf\{\varpi(\mathfrak{l}) \mid \mathfrak{l} \in S\}$$

for some $\mathfrak{g} \in S$. Since $\varpi(\mathfrak{g}) \in Im(\varpi)$, we have

$$\inf\{\varpi(\mathfrak{l}) \mid \mathfrak{l} \in S\} = \varpi(\mathfrak{g}) = \mathfrak{t}_i$$

for some $i \in \Lambda$. Obviously $\mathfrak{t} \precsim \mathfrak{t}_i$ and so $\mathfrak{t} < \mathfrak{t}_i$ by hypothesis. So there is not any $\mathfrak{j} \in \mathcal{E}$ such that $\mathfrak{t} \precsim \varpi(\mathfrak{j}) < \mathfrak{t}_i$. Then by Lemma 3.10(iii), $U(\varpi, \mathfrak{t}) = U(\varpi, \mathfrak{t}_i) \in \mathcal{I}$. □

Lemma 3.12. *Assume $f : \mathcal{E} \to M$ is an onto $\dashrightarrow$-homomorphism of bounded equality algebras. If $S \in \mathcal{I}(\mathcal{E})$, then $f(S) \in \mathcal{I}(M)$.*

Theorem 3.13. *Suppose $f : \mathcal{E} \to M$ is an onto $\dashrightarrow$-homomorphism of bounded equality algebras. Then*

(i) *If $\varpi \in \mathcal{FI}(\mathcal{E})$, then $f(\varpi)$ of $\varpi \in \mathcal{FI}(M)$ under f.*
(ii) *If $\theta \in \mathcal{FI}(M)$, then $f^{-1}(\theta)$ of $\theta \in \mathcal{FI}(\mathcal{E})$ under f.*

Proof. (i) Let $\varpi \in \mathcal{FI}(\mathcal{E})$. We first show that for any $\mathfrak{t} \in (0, 1]$,

$$U(f(\varpi), \mathfrak{t}) = \bigcap_{0<\mathfrak{s}<\mathfrak{t}} f(U(\varpi, \mathfrak{t} - \mathfrak{s})). \tag{3.5}$$

For $\mathfrak{g} = f(\mathfrak{l}) \in M$, if $\mathfrak{g} \in U(f(\varpi), \mathfrak{t})$, then

$$\mathfrak{t} \precsim f(\varpi)(\mathfrak{g}) = f(\varpi)(f(\mathfrak{l})) = \sup_{\mathfrak{j} \in f^{-1}(f(\mathfrak{l}))} \varpi(\mathfrak{j}).$$

So, for every $\mathfrak{s} \in \mathbb{R}$ with $0 < \mathfrak{s} < \mathfrak{t}$, there exists $\mathfrak{l}_0 \in f^{-1}(\mathfrak{g})$ such that $\varpi(\mathfrak{l}_0) > \mathfrak{t} - \mathfrak{s}$. Hence $\mathfrak{g} = f(\mathfrak{l}_0) \in f(U(\varpi, \mathfrak{t} - \mathfrak{s}))$. Hence $\mathfrak{g} \in \bigcap_{0<\mathfrak{s}<\mathfrak{t}} f(U(\varpi, \mathfrak{t} - \mathfrak{s}))$, and so

$$U(f(\varpi), \mathfrak{t}) \subseteq \bigcap_{0<\mathfrak{s}<\mathfrak{t}} f(U(\varpi, \mathfrak{t} - \mathfrak{s})).$$

Conversely, if $\mathfrak{g} \in \bigcap_{0<\mathfrak{s}<\mathfrak{t}} f(U(\varpi, \mathfrak{t} - \mathfrak{s}))$, then $\mathfrak{g} \in f(U(\varpi, \mathfrak{t} - \mathfrak{s}))$ for every $0 < \mathfrak{s} < \mathfrak{t}$. Hence $\mathfrak{g} = f(\mathfrak{l}_0)$ for some $\mathfrak{l}_0 \in U(\varpi, \mathfrak{t} - \mathfrak{s})$. Thus $\varpi(\mathfrak{l}_0) \gtrsim \mathfrak{t} - \mathfrak{s}$ and $\mathfrak{l}_0 \in f^{-1}(\mathfrak{g})$. Hence

$$f(\varpi)(\mathfrak{g}) = \sup_{\mathfrak{j} \in f^{-1}(\mathfrak{g})} \varpi(\mathfrak{j}) \gtrsim \mathfrak{t} - \mathfrak{s}.$$

Since $\mathfrak{s}$ is arbitrary, $\mathfrak{g} \in U(f(\varpi), \mathfrak{t})$, and so (3.5) is proved. Note that $U(f(\varpi), \mathfrak{t}) = M$ for $\mathfrak{t} = 0$. If $\mathfrak{t} \in (0, 1]$, then

$$U(f(\varpi), \mathfrak{t}) = \bigcap_{0<\mathfrak{s}<\mathfrak{t}} f(U(\varpi, \mathfrak{t} - \mathfrak{s}))$$

by (3.5). Since $U(\varpi, \mathfrak{t} - \mathfrak{s}) \in \mathcal{I}(\mathcal{E})$, we know that $f(U(\varpi, \mathfrak{t} - \mathfrak{s})) \in \mathcal{I}(M)$ by Lemma 3.12. Therefore

$$U(f(\varpi), \mathfrak{t}) = \bigcap_{0<\mathfrak{s}<\mathfrak{t}} f(U(\varpi, \mathfrak{t} - \mathfrak{s})) \in \mathcal{I}(M).$$

By Theorem 3.9, the image $f(\varpi)$ of $\varpi \in \mathcal{FI}(M)$ under f.

(ii) Let $\theta \in \mathcal{FI}(\mathcal{E})$. Then

$$f^{-1}(\theta)(0_{\mathcal{E}}) = \theta(f(0_{\mathcal{E}})) = \theta(0_M) \gtrsim \theta(f(\mathfrak{l})) = f^{-1}(\theta)(\mathfrak{l}),$$

for all $\mathfrak{l} \in \mathcal{E}$ where $0_{\mathcal{E}}$ and 0_M are units of $\mathcal{E}$ and M, respectively. Since $\theta \in \mathcal{FI}(M)$, by (FI_2) we have

$$f^{-1}(\theta)(\mathfrak{l}) = \theta(f(\mathfrak{l})) \gtrsim \min\{\theta(\neg(\neg\mathfrak{i} \dashrightarrow \neg f(\mathfrak{l}))), \theta(\mathfrak{i})\},$$

for all $\mathfrak{l} \in \mathcal{E}$ and $\mathfrak{i} \in M$. Since f is onto, there exists $\mathfrak{g} \in \mathcal{E}$ such that $f(\mathfrak{g}) = \mathfrak{i}$. Since $f(0_{\mathcal{E}}) = 0_M$, we get $f(\neg\mathfrak{l}) = \neg f(\mathfrak{l})$ for all $\mathfrak{l} \in \mathcal{E}$. Hence

$$\begin{aligned}
f^{-1}(\theta)(\mathfrak{l}) &\gtrsim \min\{\theta(\neg(\neg\mathfrak{i} \dashrightarrow \neg f(\mathfrak{l}))), \theta(\mathfrak{i})\} \\
&= \min\{\theta(\neg(\neg f(\mathfrak{g}) \dashrightarrow \neg f(\mathfrak{l}))), \theta(f(\mathfrak{g}))\} \\
&= \min\{\theta(\neg(f(\neg\mathfrak{g}) \dashrightarrow f(\neg\mathfrak{l}))), \theta(f(\mathfrak{g}))\} \\
&= \min\{\theta(\neg f(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l})), \theta(f(\mathfrak{g}))\} \\
&= \min\{\theta(f(\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l}))), \theta(f(\mathfrak{g}))\} \\
&= \min\{f^{-1}(\theta)((\neg(\neg\mathfrak{g} \dashrightarrow \neg\mathfrak{l}))), f^{-1}(\theta)(\mathfrak{g})\},
\end{aligned}$$

and therefore the preimage $f^{-1}(\theta)$ of $\theta \in \mathcal{FI}(\mathcal{E})$ under f. □

4. Quotient equality algebras via fuzzy ideals

In continuing, by using the concept of ideals we define a congruence relation on bounded equality algebra. Then the quotient structures of equality algebra induced by (fuzzy) ideal are studied.

Definition 4.1. Suppose $Q \in \mathcal{I}(\mathcal{E})$. For any $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$, define the relation "$\equiv_Q$" on $\mathcal{E}$ as

$$\mathfrak{l} \equiv_Q \mathfrak{g} \text{ iff } \neg(\mathfrak{l} \dashrightarrow \mathfrak{g}) \in Q \text{ and } \neg(\mathfrak{g} \dashrightarrow \mathfrak{l}) \in Q. \tag{4.1}$$

Lemma 4.2. *The binary relation $\equiv_Q$ in (4.1) is a congruence relation on $\mathcal{E}$.*

Proof. Obviously, $\equiv_Q$ is both reflexive and symmetric. Let $\mathfrak{l}, \mathfrak{g}, \mathfrak{j} \in \mathcal{E}$ such that $\mathfrak{l} \equiv_Q \mathfrak{g}$ and $\mathfrak{g} \equiv_Q \mathfrak{j}$. Then $\neg(\mathfrak{l} \dashrightarrow \mathfrak{g}) \in Q$, $\neg(\mathfrak{g} \dashrightarrow \mathfrak{l}) \in Q$, $\neg(\mathfrak{g} \dashrightarrow \mathfrak{j}) \in Q$ and $\neg(\mathfrak{j} \dashrightarrow \mathfrak{g}) \in Q$. Since

$$\begin{aligned}\neg(\neg(\neg(\mathfrak{g} \dashrightarrow \mathfrak{l})) \dashrightarrow \neg(\neg(\mathfrak{j} \dashrightarrow \mathfrak{l}))) &\precsim \neg(\neg(\mathfrak{j} \dashrightarrow \mathfrak{l}) \dashrightarrow \neg(\mathfrak{g} \dashrightarrow \mathfrak{l}))\\ &\precsim \neg((\mathfrak{g} \dashrightarrow \mathfrak{l}) \dashrightarrow (\mathfrak{j} \dashrightarrow \mathfrak{l}))\\ &\precsim \neg(\mathfrak{j} \dashrightarrow \mathfrak{g}),\end{aligned}$$

from (I_1), we have $\neg(\neg(\neg(\mathfrak{g} \dashrightarrow \mathfrak{l})) \dashrightarrow \neg(\neg(\mathfrak{j} \dashrightarrow \mathfrak{l}))) \in Q$. Hence $\neg(\mathfrak{j} \dashrightarrow \mathfrak{l}) \in Q$ by (I_4). The similar way induces $\neg(\mathfrak{l} \dashrightarrow \mathfrak{j}) \in Q$, and so $\mathfrak{l} \equiv_Q \mathfrak{j}$. Therefore $\equiv_Q$ is an equivalence relation on $\mathcal{E}$.

Now, let $\mathfrak{l}, \mathfrak{g}, m, n \in \mathcal{E}$ such that $\mathfrak{l} \equiv_Q m$ and $\mathfrak{g} \equiv_Q n$. Then

$$\neg(\mathfrak{l} \dashrightarrow m) \in Q,$$

$\neg(m \dashrightarrow \mathfrak{l}) \in Q$, $\neg(\mathfrak{g} \dashrightarrow n) \in Q$ and $\neg(n \dashrightarrow \mathfrak{g}) \in Q$. Note that

$$\neg((n \dashrightarrow \mathfrak{l}) \dashrightarrow (\mathfrak{g} \dashrightarrow \mathfrak{l})) \precsim \neg(\mathfrak{g} \dashrightarrow n)$$

and $\neg((\mathfrak{g} \dashrightarrow \mathfrak{l}) \dashrightarrow (n \dashrightarrow \mathfrak{l})) \precsim \neg(n \dashrightarrow \mathfrak{g})$. By (I_1) we obtain $\neg((n \dashrightarrow \mathfrak{l}) \dashrightarrow (\mathfrak{g} \dashrightarrow \mathfrak{l})) \in Q$ and $\neg((\mathfrak{g} \dashrightarrow \mathfrak{l}) \dashrightarrow (n \dashrightarrow \mathfrak{l})) \in Q$. Hence $(\mathfrak{g} \dashrightarrow \mathfrak{l}) \equiv_Q (n \dashrightarrow \mathfrak{l})$. Also, note that

$$\neg((n \dashrightarrow m) \dashrightarrow (n \dashrightarrow \mathfrak{l})) \precsim \neg(m \dashrightarrow \mathfrak{l})$$

and $\neg((n \dashrightarrow \mathfrak{l}) \dashrightarrow (n \dashrightarrow m)) \precsim \neg(\mathfrak{l} \dashrightarrow m)$, which imply from (I_1) that $\neg((n \dashrightarrow m) \dashrightarrow (n \dashrightarrow \mathfrak{l})) \in Q$ and

$$\neg((n \dashrightarrow \mathfrak{l}) \dashrightarrow (n \dashrightarrow m)) \in Q.$$

Thus $(n \dashrightarrow \mathfrak{l}) \equiv_Q (n \dashrightarrow m)$. By the transitivity of $\equiv_Q$, we conclude that $(\mathfrak{g} \dashrightarrow \mathfrak{l}) \equiv_Q (n \dashrightarrow m)$. □

Denote by $Q_{\mathfrak{l}}$ the set $\{\mathfrak{g} \in \mathcal{E} \mid \mathfrak{g} \equiv_Q \mathfrak{l}\}$ and $\mathcal{E}/Q$ the set $\{Q_{\mathfrak{l}} \mid \mathfrak{l} \in \mathcal{E}\}$. Define a binary operation $\multimap$ on $\mathcal{E}/Q$ as follows:

$$(\forall Q_{\mathfrak{l}}, Q_{\mathfrak{g}} \in \mathcal{E}/Q)(Q_{\mathfrak{l}} \multimap Q_{\mathfrak{g}} = Q_{\mathfrak{g} \dashrightarrow \mathfrak{l}}). \tag{4.2}$$

Theorem 4.3. *If $\mathcal{E}$ is involutive and $Q \in \mathcal{I}(\mathcal{E})$, then $Q_0 \in \mathcal{I}(\mathcal{E})$ and $Q_0 = Q$.*

Proof. We have

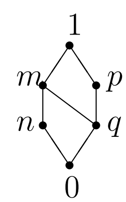
$$\begin{aligned} Q_0 &= \{\mathfrak{l} \in \mathcal{E} \mid \mathfrak{l} \equiv_Q 0\} \\ &= \{\mathfrak{l} \in \mathcal{E} \mid \neg(\mathfrak{l} \dashrightarrow 0) \in Q\} \\ &= \{\mathfrak{l} \in \mathcal{E} \mid \neg\neg\mathfrak{l} \in Q\} \\ &= \{\mathfrak{l} \in \mathcal{E} \mid \mathfrak{l} \in Q\} \\ &= Q. \end{aligned}$$

□

Theorem 4.4. *If $Q \in \mathcal{I}(\mathcal{E})$, then $(\mathcal{E}/Q, \multimap, Q_1)$ is a BCK-algebra.*

Proof. It is straightforward. □

Example 4.5. Consider $\mathcal{E} = \{0, m, n, p, q, 1\}$ with the next Hasse diagram.



Define an operation $\backsim$ on $\mathcal{E}$ by Table 3.

TABLE 3. Cayley table for the implication "$\backsim$"

$\backsim$	0	m	n	p	q	1
0	1	q	p	n	m	0
m	q	1	m	q	p	m
n	p	m	1	0	q	n
p	n	q	0	1	m	p
q	m	p	q	m	1	q
1	0	m	n	p	q	1

Then $(\mathcal{E}, \curlywedge, \backsim, 1)$ is a bounded equality algebra, and the implication "$\dashrightarrow$" is given by Table 4.

TABLE 4. Cayley table for the implication "$\dashrightarrow$"

$\dashrightarrow$	0	m	n	p	q	1
0	1	1	1	1	1	1
m	q	1	m	p	p	1
n	p	1	1	p	p	1
p	n	m	n	1	m	1
q	m	1	m	1	1	1
1	0	m	n	p	q	1

Given an ideal $Q = \{0, n\}$ of $\mathcal{E}$, we have $Q_p = Q_1 = \{1, p\}$, $Q_m = Q_q = \{0, m, q\}$ and $Q_0 = Q_n = \{0, n\}$. Thus $\mathcal{E}/Q = \{Q_0, Q_m, Q_1\}$ which is a BCK-algebra.

Theorem 4.6. *If $Q \in \mathcal{I}(\mathcal{E})$, then $\mathcal{E}/Q = \{Q_1\}$ iff $Q = \mathcal{E}$.*

Proof. Obviously, if $Q = \mathcal{E}$, then $\mathcal{E}/Q = \{Q_1\}$. Assume that $\mathcal{E}/Q = \{Q_1\}$. If $\mathfrak{l} \in \mathcal{E}$, then $Q_{\mathfrak{l}} \in \mathcal{E}/Q$ and so $Q_{\mathfrak{l}} = Q_1$. Thus $\mathfrak{l} \equiv_Q 1$, and so $0 \in Q$ and $\neg\mathfrak{l} \in Q$. Since $\neg\mathfrak{l} \in Q \subseteq \mathcal{E}$, we have $Q_{\neg\mathfrak{l}} \in \mathcal{E}/Q = \{Q_1\}$ and thus $Q_{\neg\mathfrak{l}} = Q_1$. Hence $\neg\mathfrak{l} \equiv_Q 1$, that is, $\neg(\neg\mathfrak{l} \dashrightarrow 1) \in Q$ and $\neg(1 \dashrightarrow \neg\mathfrak{l}) \in Q$. Thus $\neg\neg\mathfrak{l} \in Q$. Since $Q \in \mathcal{I}(\mathcal{E})$ and $\mathfrak{l} \leq \neg\neg\mathfrak{l}$, we get $\mathfrak{l} \in Q$. □

Let Q be an ideal of an involutive equality algebra $\mathcal{E}$. Define the operations "$\bar{\curlywedge}$" and "$\approx$" on $\mathcal{E}/Q$ by

$$Q_{\mathfrak{l}} \,\bar{\curlywedge}\, Q_{\mathfrak{g}} = Q_{\mathfrak{l} \curlywedge \mathfrak{g}} \text{ and } Q_{\mathfrak{l}} \approx Q_{\mathfrak{g}} = Q_{\mathfrak{l} \sim \mathfrak{g}}$$

for all $Q_{\mathfrak{l}}, Q_{\mathfrak{g}} \in \mathcal{E}/Q$.

Theorem 4.7. *If $\mathcal{E}$ is involutive and $Q \in \mathcal{I}(\mathcal{E})$, then $(\mathcal{E}/Q, \bar{\curlywedge}, \approx, Q_1)$ is an equality algebra.*

Proof. The proof is routine, so we omit the proof. □

Now, we define a binary relation on a bounded equality algebra $\mathcal{E}$ by using a fuzzy ideal. Let ϖ be a non-constant fuzzy ideal of a bounded equality algebra $\mathcal{E}$. For any $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$, define a binary relation "$\approx_\varpi$" on $\mathcal{E}$ by

$$\mathfrak{l} \approx_\varpi \mathfrak{g} \text{ iff } \varpi(\neg(\mathfrak{l} \dashrightarrow \mathfrak{g})) \neq 0 \neq \varpi(\neg(\mathfrak{g} \dashrightarrow \mathfrak{l})). \tag{4.3}$$

Proposition 4.8. *The binary relation $\approx_\varpi$ in (4.3) is a congruence relation on $\mathcal{E}$.*

Proof. Since ϖ is non-constant and $\varpi(0) \gtrsim \varpi(\mathfrak{l})$ for all $\mathfrak{l} \in \mathcal{E}$, we have $\varpi(0) \neq 0$. Let $\mathfrak{l}, \mathfrak{g}, \mathfrak{j} \in \mathcal{E}$. Then

$$\varpi(\neg(\mathfrak{l} \dashrightarrow \mathfrak{l})) = \varpi(\neg 1) = \varpi(0) \neq 0,$$

and so $\mathfrak{l} \approx_{\varpi} \mathfrak{l}$. Obviously, if $\mathfrak{l} \approx_{\varpi} \mathfrak{g}$, then $\mathfrak{g} \approx_{\varpi} \mathfrak{l}$. Suppose that $\mathfrak{l} \approx_{\varpi} \mathfrak{g}$ and $\mathfrak{g} \approx_{\varpi} \mathfrak{j}$. Then $\varpi(\neg(\mathfrak{l} \dashrightarrow \mathfrak{g})) \neq 0 \neq \varpi(\neg(\mathfrak{g} \dashrightarrow \mathfrak{l}))$ and $\varpi(\neg(\mathfrak{g} \dashrightarrow \mathfrak{j})) \neq 0 \neq \varpi(\neg(\mathfrak{j} \dashrightarrow \mathfrak{g}))$.

$$\begin{aligned}\neg(\neg(\neg(\mathfrak{g} \dashrightarrow \mathfrak{l})) \dashrightarrow \neg(\neg(\mathfrak{j} \dashrightarrow \mathfrak{l}))) &\precsim \neg(\neg(\mathfrak{j} \dashrightarrow \mathfrak{l}) \dashrightarrow \neg(\mathfrak{g} \dashrightarrow \mathfrak{l}))\\ &\precsim \neg((\mathfrak{g} \dashrightarrow \mathfrak{l}) \dashrightarrow (\mathfrak{j} \dashrightarrow \mathfrak{l}))\\ &\precsim \neg(\mathfrak{j} \dashrightarrow \mathfrak{g}),\end{aligned}$$

from (FI_2) and Proposition 3.6 that

$$\begin{aligned}\varpi(\neg(\mathfrak{j} \dashrightarrow \mathfrak{l})) &\gtrsim \min\{\varpi(\neg(\neg(\neg(\mathfrak{g} \dashrightarrow \mathfrak{l})) \dashrightarrow \neg(\neg(\mathfrak{j} \dashrightarrow \mathfrak{l})))), \varpi(\neg(\mathfrak{g} \dashrightarrow \mathfrak{l}))\}\\ &\gtrsim \min\{\varpi(\neg(\mathfrak{j} \dashrightarrow \mathfrak{g})), \varpi(\neg(\mathfrak{g} \dashrightarrow \mathfrak{l}))\} \neq 0.\end{aligned}$$

Similarly, we have $\varpi(\neg(\mathfrak{l} \dashrightarrow \mathfrak{j})) \neq 0$. Hence $\mathfrak{l} \approx_{\varpi} \mathfrak{j}$, and therefore $\mathfrak{l} \approx_{\varpi} \mathfrak{j}$ is an equivalence relation on $\mathcal{E}$. Assume that $\mathfrak{l} \approx_{\varpi} \mathfrak{g}$ for all $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$ and let $\mathfrak{j} \in \mathcal{E}$. Since

$$\neg(\neg(\mathfrak{g} \dashrightarrow \mathfrak{l})) \precsim \neg(\neg((\mathfrak{j} \dashrightarrow \mathfrak{g}) \dashrightarrow (\mathfrak{j} \dashrightarrow \mathfrak{l}))),$$

we have

$$\begin{aligned}\varpi(\neg((\neg(\neg(\mathfrak{g} \dashrightarrow \mathfrak{l}))) \dashrightarrow (\neg(\neg((\mathfrak{j} \dashrightarrow \mathfrak{g}) \dashrightarrow (\mathfrak{j} \dashrightarrow \mathfrak{l})))))) &= \varpi(\neg 1)\\ &= \varpi(0)\\ &\neq 0.\end{aligned}$$

Since $\varpi \in \mathcal{FI}(\mathcal{E})$, by (FI_2) we obtain

$$\begin{aligned}&\varpi(\neg((\mathfrak{j} \dashrightarrow \mathfrak{g}) \dashrightarrow (\mathfrak{j} \dashrightarrow \mathfrak{l})))\\ &\gtrsim \min\{\varpi(\neg((\neg(\neg(\mathfrak{g} \dashrightarrow \mathfrak{l}))) \dashrightarrow (\neg(\neg((\mathfrak{j} \dashrightarrow \mathfrak{g}) \dashrightarrow (\mathfrak{j} \dashrightarrow \mathfrak{l})))))), \varpi(\neg(\mathfrak{g} \dashrightarrow \mathfrak{l}))\}\\ &\neq 0.\end{aligned}$$

Similarly, we get $\varpi(\neg((\mathfrak{j} \dashrightarrow \mathfrak{l}) \dashrightarrow (\mathfrak{j} \dashrightarrow \mathfrak{g}))) \neq 0$. Hence

$$(\mathfrak{j} \dashrightarrow \mathfrak{l}) \approx_{\varpi} (\mathfrak{j} \dashrightarrow \mathfrak{g}).$$

Since $\neg(\neg(\mathfrak{l} \dashrightarrow \mathfrak{g})) \dashrightarrow \neg(\neg((\mathfrak{g} \dashrightarrow \mathfrak{j}) \dashrightarrow (\mathfrak{l} \dashrightarrow \mathfrak{j}))) = 1$, we have

$$\begin{aligned}\varpi(\neg(\neg(\neg(\mathfrak{l} \dashrightarrow \mathfrak{g})) \dashrightarrow \neg(\neg((\mathfrak{g} \dashrightarrow \mathfrak{j}) \dashrightarrow (\mathfrak{l} \dashrightarrow \mathfrak{j}))))) &= \varpi(\neg 1)\\ &= \varpi(0)\\ &\neq 0.\end{aligned}$$

Hence

$$\begin{aligned}&\varpi(\neg((\mathfrak{g} \dashrightarrow \mathfrak{j}) \dashrightarrow (\mathfrak{l} \dashrightarrow \mathfrak{j})))\\ &\gtrsim \min\{\varpi(\neg((\neg(\neg(\mathfrak{l} \dashrightarrow \mathfrak{g}))) \dashrightarrow (\neg(\neg((\mathfrak{g} \dashrightarrow \mathfrak{j}) \dashrightarrow (\mathfrak{l} \dashrightarrow \mathfrak{j})))))),\\ &\qquad \varpi(\neg(\mathfrak{l} \dashrightarrow \mathfrak{g}))\}\\ &\neq 0\end{aligned}$$

by (FI_2). Similarly, we have $\varpi(\neg((\mathfrak{l} \dashrightarrow \mathfrak{j}) \dashrightarrow (\mathfrak{g} \dashrightarrow \mathfrak{j}))) \neq 0$. Thus $(\mathfrak{l} \dashrightarrow \mathfrak{j}) \approx_\varpi (\mathfrak{g} \dashrightarrow \mathfrak{j})$. Therefore, if $\mathfrak{l} \approx_\varpi m$ and $\mathfrak{g} \approx_\varpi n$ for all $\mathfrak{l}, \mathfrak{g}, m, n \in \mathcal{E}$, then $(\mathfrak{l} \dashrightarrow \mathfrak{g}) \approx_\varpi (m \dashrightarrow n)$. □

Denote by $[\mathfrak{l}]_\varpi$ the set $\{\mathfrak{g} \in \mathcal{E} \mid \mathfrak{l} \approx_\varpi \mathfrak{g}\}$ and $\mathcal{E}/\varpi$ the set $\{[\mathfrak{l}]_\varpi \mid \mathfrak{l} \in \mathcal{E}\}$.

Theorem 4.9. *If $\varpi \in \mathcal{FI}(\mathcal{E})$, then $(\mathcal{E}/\varpi, \dashrightarrow_\varpi, [1]_\varpi)$ is a BCK-algebra where $\dashrightarrow_\varpi$ is a binary operation on $\mathcal{E}/\varpi$ defined by*

$$(\forall [\mathfrak{l}]_\varpi, [\mathfrak{g}]_\varpi \in \mathcal{E}/\varpi)([\mathfrak{l}]_\varpi \dashrightarrow_\varpi [\mathfrak{g}]_\varpi = [\mathfrak{g} \dashrightarrow \mathfrak{l}]_\varpi). \tag{4.4}$$

Proof. We show $\dashrightarrow_\varpi$ on $\mathcal{E}/\varpi$ is well-defined. Let $\mathfrak{l}, \mathfrak{g}, m, n \in \mathcal{E}$ such that $[\mathfrak{l}]_\varpi = [m]_\varpi$ and $[\mathfrak{g}]_\varpi = [n]_\varpi$. Let $w \in [\mathfrak{l}]_\varpi \dashrightarrow_\varpi [\mathfrak{g}]_\varpi$. Then $w \dashrightarrow_\varpi (\mathfrak{g} \dashrightarrow \mathfrak{l})$. Hence $(\mathfrak{g} \dashrightarrow \mathfrak{l}) \dashrightarrow_\varpi (n \dashrightarrow m)$ implies $w \dashrightarrow_\varpi (n \dashrightarrow m)$, and so $w \in [m]_\varpi \dashrightarrow_\varpi [n]_\varpi$. Hence

$$[\mathfrak{l}]_\varpi \dashrightarrow_\varpi [\mathfrak{g}]_\varpi \subseteq [m]_\varpi \dashrightarrow_\varpi [n]_\varpi.$$

Similarly, we have the reverse inclusion and hence

$$[\mathfrak{l}]_\varpi \dashrightarrow_\varpi [\mathfrak{g}]_\varpi = [m]_\varpi \dashrightarrow_\varpi [n]_\varpi.$$

Therefore $\dashrightarrow_\varpi$ is well-defined. Let $[\mathfrak{l}]_\varpi, [\mathfrak{g}]_\varpi, [\mathfrak{j}]_\varpi \in \mathcal{E}/\varpi$. Then

$$\begin{aligned}&((([\mathfrak{l}]_\varpi \dashrightarrow_\varpi [\mathfrak{g}]_\varpi) \dashrightarrow_\varpi ([\mathfrak{l}]_\varpi \dashrightarrow_\varpi [\mathfrak{j}]_\varpi)) \dashrightarrow_\varpi ([\mathfrak{j}]_\varpi \dashrightarrow_\varpi [\mathfrak{g}]_\varpi)\\ &= [(\mathfrak{g} \dashrightarrow \mathfrak{j}) \dashrightarrow ((\mathfrak{j} \dashrightarrow \mathfrak{l}) \dashrightarrow (\mathfrak{g} \dashrightarrow \mathfrak{l}))]_\varpi\\ &= [1]_\varpi,\end{aligned}$$

$$\begin{aligned}[\mathfrak{g}]_\varpi \dashrightarrow_\varpi ((([\mathfrak{g}]_\varpi \dashrightarrow_\varpi [\mathfrak{l}]_\varpi) \dashrightarrow_\varpi [\mathfrak{l}]_\varpi) &= [(\mathfrak{l} \dashrightarrow (\mathfrak{l} \dashrightarrow \mathfrak{g})) \dashrightarrow \mathfrak{g}]_\varpi\\ &= [1]_\varpi,\end{aligned}$$

$$[\mathfrak{l}]_\varpi \dashrightarrow_\varpi [\mathfrak{l}]_\varpi = [\mathfrak{l} \dashrightarrow \mathfrak{l}]_\varpi = [1]_\varpi.$$

If $[\mathfrak{l}]_\varpi \dashrightarrow_\varpi [\mathfrak{g}]_\varpi = [1]_\varpi$ and $[\mathfrak{g}]_\varpi \dashrightarrow_\varpi [\mathfrak{l}]_\varpi = [1]_\varpi$, then

$$[\mathfrak{g} \dashrightarrow \mathfrak{l}]_\varpi = [1]_\varpi = [\mathfrak{l} \dashrightarrow \mathfrak{g}]_\varpi$$

and so $\varpi(\neg(\mathfrak{g} \dashrightarrow \mathfrak{l})) = \varpi(\neg(1 \dashrightarrow (\mathfrak{g} \dashrightarrow \mathfrak{l}))) \neq 0$ and

$$\varpi(\neg(\mathfrak{l} \dashrightarrow \mathfrak{g})) = \varpi(\neg(1 \dashrightarrow (\mathfrak{l} \dashrightarrow \mathfrak{g}))) \neq 0.$$

Hence $\mathfrak{l} \dashrightarrow_\varpi \mathfrak{g}$, and so $[\mathfrak{l}]_\varpi = [\mathfrak{g}]_\varpi$. Finally

$$[1]_\varpi \dashrightarrow_\varpi [\mathfrak{l}]_\varpi = [\mathfrak{l} \dashrightarrow 1]_\varpi = [1]_\varpi.$$

Therefore $(\mathcal{E}/\varpi, \dashrightarrow_\varpi, [1]_\varpi)$ is a BCK-algebra. □

Theorem 4.10. *If $\varpi \in \mathcal{FI}(\mathcal{E})$, then we have the following epimorphism of BCI-algebras.*

$$f : \mathcal{E}/U(\varpi, t) \to \mathcal{E}/\varpi,\ U(\varpi, t)_{\mathfrak{l}} \mapsto [\mathfrak{l}]_{\varpi} \tag{4.5}$$

where $\mathfrak{t} \in (0, 1]$ *with* $U(\varpi, \mathfrak{t}) \neq \emptyset$.

Proof. Assume that $U(\varpi, \mathfrak{t})_{\mathfrak{l}} = U(\varpi, \mathfrak{t})_{\mathfrak{g}}$ for $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$. Then $\mathfrak{l} \dashrightarrow_{U(\varpi,\mathfrak{t})} \mathfrak{g}$ and so $\neg(\mathfrak{l} \dashrightarrow \mathfrak{g}) \in U(\varpi, \mathfrak{t})$ and $\neg(\mathfrak{g} \dashrightarrow \mathfrak{l}) \in U(\varpi, \mathfrak{t})$. It follows that $\varpi(\neg(\mathfrak{l} \dashrightarrow \mathfrak{g})) \gtrsim \mathfrak{t} > 0$ and $\varpi(\neg(\mathfrak{g} \dashrightarrow \mathfrak{l})) \gtrsim \mathfrak{t} > 0$, that is, $\mathfrak{l} \approx_{\varpi} \mathfrak{g}$. Hence $[\mathfrak{l}]_{\varpi} = [\mathfrak{g}]_{\varpi}$, and therefore the mapping is well-defined. Clearly, f is onto. For any $U(\varpi, \mathfrak{t})_{\mathfrak{l}}, U(\varpi, \mathfrak{t})_{\mathfrak{g}} \in \mathcal{E}/U(\varpi, \mathfrak{t})$, we have

$$\begin{aligned} f\left(U(\varpi, \mathfrak{t})_{\mathfrak{l}} \multimap U(\varpi, \mathfrak{t})_{\mathfrak{g}}\right) &= U(\varpi, \mathfrak{t})_{\mathfrak{g} \dashrightarrow \mathfrak{l}} \\ &= [\mathfrak{g} \dashrightarrow \mathfrak{l}]_{\varpi} = [\mathfrak{l}]_{\varpi} \dashrightarrow_{\varpi} [\mathfrak{g}]_{\varpi} \\ &= f\left(U(\varpi, \mathfrak{t})_{\mathfrak{l}}\right) \dashrightarrow_{\varpi} f\left(U(\varpi, \mathfrak{t})_{\mathfrak{g}}\right), \end{aligned}$$

which shows that f is a homomorphism. This completes the proof. □

Theorem 4.11. *For any $\varpi \in \mathcal{FI}(\mathcal{E})$ and $\mathfrak{t} \in (0, 1]$, if $\varpi(\mathfrak{j}) = 0$ whenever $\mathfrak{j} \in \mathcal{E} \setminus U(\varpi, \mathfrak{t})$, then $\mathcal{E}/U(\varpi, \mathfrak{t})$ is isomorphic to $\mathcal{E}/\varpi$.*

Proof. By Theorem 4.10, the mapping

$$f : \mathcal{E}/U(\varpi, \mathfrak{t}) \to \mathcal{E}/\varpi,\ U(\varpi, \mathfrak{t})_{\mathfrak{l}} \mapsto [\mathfrak{l}]_{\varpi}$$

is an epimorphism of BCK-algebras. It is sufficient to show that f is one-one. Assume that $U(\varpi, \mathfrak{t})_{\mathfrak{l}} = U(\varpi, \mathfrak{t})_{\mathfrak{g}}$ for $\mathfrak{l}, \mathfrak{g} \in \mathcal{E}$. Then $\mathfrak{l} \dashrightarrow_{U(\varpi,\mathfrak{t})} \mathfrak{g}$, and so $\neg(\mathfrak{l} \dashrightarrow \mathfrak{g}) \in U(\varpi, \mathfrak{t})$ and $\neg(\mathfrak{g} \dashrightarrow \mathfrak{l}) \in U(\varpi, \mathfrak{t})$. It follows that $\varpi(\neg(\mathfrak{l} \dashrightarrow \mathfrak{g})) \gtrsim \mathfrak{t} > 0$ and $\varpi(\neg(\mathfrak{g} \dashrightarrow \mathfrak{l})) \gtrsim \mathfrak{t} > 0$. Hence $\mathfrak{l} \approx_{\varpi} \mathfrak{g}$ which implies that

$$f\left(U(\varpi, \mathfrak{t})_{\mathfrak{l}}\right) = [\mathfrak{l}]_{\varpi} = [\mathfrak{g}]_{\varpi} = f\left(U(\varpi, \mathfrak{t})_{\mathfrak{g}}\right).$$

Thus f is injective, and therefore f is an isomorphism. □

5. Conclusion

In this paper, the notion of fuzzy ideal on bounded equality algebras is defined and conditions for a fuzzy ideal to attains its infimum on all ideals are provided. Also, homomorphic image and preimage of fuzzy ideal are investigated. Then by using the notion of fuzzy ideal, a congruence relation on bounded equality algebra is introduced and by introducing an operation on the set of all congruence classes of it, showed that the quotient structure is made a BCK-algebra.

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QUOTIENT STRUCTURES IN EQUALITY ALGEBRAS

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ساختارهای خارج‌قسمت در جبرهای تساوی

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مفهوم ایده‌آل فازی در جبرهای تساوی کراندار تعریف شده و چندین ویژگی آن مورد مطالعه قرار گرفته است. ایده‌آل فازی تولید شده توسط یک مجموعه فازی به دست آمده و با استفاده از مجموعه ایده‌آل‌ها، یک ایده‌آل فازی ساخته شده است. خصوصیات ایده‌آل فازی مورد بحث قرار گرفته و شرایط لازم برای به دست آوردن یک ایده‌آل فازی به کمک بزرگترین کران پایین تمام ایده‌آل‌ها فراهم شده است. تصویر همریختی و تصویر وارون یک ایده‌آل فازی مورد مطالعه قرار گرفته است. در نهایت، ساختارهای خارج‌قسمت جبر تساوی، القا شده توسط ایده‌آل (فازی)، مورد مطالعه قرار می‌گیرند.

کلمات کلیدی: جبر تساوی کراندار، مجموعه‌های فازی، ایده‌آل، ایده‌آل فازی و ساختارهای خارج‌قسمت.