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SOME PROPERTIES OF SUPER-GRAPH OF $(\mathscr{C}(R))^c$ AND ITS LINE GRAPH

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Abstract. Let R be a commutative ring with identity $1 \neq 0$. The comaximal ideal graph of R is the simple, undirected graph whose vertex set is the set of all proper ideals of the ring R not contained in the Jacobson radical of R and two vertices I and J are adjacent in this graph if and only if $I + J = R$. In this article, we have discussed the graph $G(R)$ whose vertex set is the set of all proper ideals of ring R and two vertices I and J are adjacent in this graph if and only if $I + J \neq R$. In this article, we have discussed some interesting results about $G(R)$ and its line graph.

1. Introduction

The rings considered in this article are commutative with identity $1 \neq 0$ which are not fields. The idea of associating a graph with certain subsets of a commutative ring and exploring the interplay between the ring-theoretic properties of a ring and the graph-theoretic properties of the graph associated with it began with the work of I. Beck in [7].

For a commutative ring R, we denote the set of all maximal ideals of R by $Max(R)$. $I(R)$ denotes the set of all proper ideals of a ring R. We denote the cardinality of a set A using the notation $|A|$. Let R be a ring. In [26], M. Ye and T. Wu introduced and investigated a graph called the *comaximal ideal graph* of R, denoted by $\mathscr{C}(R)$. It is an undirected graph whose vertex set is the set of all proper ideals I of R

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such that $I \nsubseteq J(R)$ and distinct vertices I_1, I_2 are joined by an edge in this graph if and only if $I_1 + I_2 = R$. In [26], M. Ye and T. Wu showed that $\mathscr{C}(R)$ is connected and $diam(\mathscr{C}(R)) \leq 3$ and $girth(\mathscr{C}(R)) \leq 4$ if $\mathscr{C}(R)$ contains a cycle. They also studied the clique number and chromatic number of $\mathscr{C}(R)$ and the results proved in [26] on $\mathscr{C}(R)$ demonstrated the influence of certain graph parameters of $\mathscr{C}(R)$ on the ring structure of R. Interesting research work has been done on comaximal graph and comaximal ideal graph in [2, 11, 14, 15, 13, 16, 18, 20, 23] and on annihilating-ideal graphs as well as zero-divisor graphs in [1, 3, 4, 8, 9, 12, 17, 19, 22, 24]. A. Gaur and A. Sharma have studied the line graph associated to the maximal graph in [10, 21].

The graphs considered in this article are undirected. Let $G = (V, E)$ be a simple graph. Recall from [6] that the *complement of* G, denoted by G^c is a graph whose vertex set is V and two distinct $u, v \in V$ are joined by an edge in G^c if and only if there exists no edge in G joining u and v. Motivated by the results proved on $\mathscr{C}(R)$ in [25, 26], we have considered a super graph of $(\mathscr{C}(R))^c$ denoted by $G(R)$ whose vertex set is the set of all proper ideals of R and two distinct vertices I and J are adjacent in $G(R)$ if and only if $I + J \neq R$. So, $G(R)$ is a super-graph of $(\mathscr{C}(R))^c$. As any proper ideal of a ring is contained in at least one maximal ideal, it follows that I_1 and I_2 are adjacent in $G(R)$ if and only if there exists at least one maximal ideal $\mathfrak{m}$ of R such that $I_1 + I_2 \subseteq \mathfrak{m}$.

It is useful to recall the following definitions and results from graph theory. Let $a, b \in V$, $a \neq b$. Recall that the *distance between* a *and* b, denoted by $d(a, b)$ is defined as the length of a shortest path in G between a and b if such a path exists, otherwise $d(a, b) = \infty$. We define $d(a, a) = 0$. A graph G is said to be *connected* if for any distinct vertices $a, b \in V$, there exists a path in G between a and b. Recall from [6] that the *diameter* of a connected graph $G = (V, E)$ denoted by $diam(G)$ is defined as $diam(G) = sup\{d(a, b) | a, b \in V\}$. Let $G = (V, E)$ be a connected graph. Recall that G is a *split graph* if $V(G)$ is the disjoint union of two nonempty subsets K and S such that the subgraph of G induced on K is complete and S is an independent set of G. Let G be a simple undirected finite graph. Recall from [5] that line graph of G, denoted as $L(G)$ has its vertex set in 1-1 correspondence with the edge set of G and two vertices of $L(G)$ are joined by an edge if and only if the corresponding edges of G are adjacent in G. If $u - v$ is an edge in G, then we denote the vertex uv of $L(G)$ by $[u, v]$.

Let $G = (V, E)$ be a graph such that G contains a cycle. Recall from [6] that the *girth of* G, denoted by $girth(G)$ is defined as the length of a shortest cycle in G. If a graph G does not contain any cycle, then we define $girth(G) = \infty$. Let $n \in \mathbb{N}$. A complete graph on n vertices is denoted by K_n. Let $G = (V, E)$ be a graph. Then G is said to be *bipartite* if the vertex set V of G can be partitioned into two nonempty subsets V_1 and V_2 such that each edge of G has one end in V_1 and the other end in V_2. A bipartite graph with vertex partition V_1 and V_2 is said to be *complete*, if each element of V_1 is adjacent to every element of V_2. Let $m, n \in \mathbb{N}$. Let $G = (V, E)$ be a complete bipartite graph with $V = V_1 \cup V_2$. If $|V_1| = m$ and $|V_2| = n$, then G is denoted by $K_{m,n}$. A *star graph* is a complete bipartite graph of the form $K_{1,n}$. Recall from [6] that a subset V' of the vertex set $V(G)$ of a connected graph G is a *vertex cut* of G if $G \setminus V$ is disconnected; it is a *k-vertex cut* if $|V| = k$. A vertex v of G is a *cut vertex* of G if {v} is a vertex cut of G. A subset S of the vertex set V of a graph G is called *independent* if no two vertices of S are adjacent in G. $S \subseteq V$ is a *maximum independent set* of G if G has no independent set S_0 with $|S_0| > |S|$. Cardinality of maximum independent set of G is called *independence number*. Let $G = (V, E)$ be a graph. Recall from [3] that two distinct vertices u, v of G are said to be *orthogonal*, written as $u \perp v$ if u and v are adjacent in G and there is no vertex of G which is adjacent to both u and v in G; that is, the edge $u - v$ is not an edge of any triangle in G. Let $u \in V$. A vertex v of G is said to be a complement of u if $u \perp v$ [3]. Moreover, we recall from [3] that G is complemented if each vertex of G admits a complement in G.

A ring R is said to be *local* if R has a unique maximal ideal. Recall that a principal ideal ring R is said to be a *special principal ideal ring* (SPIR) if R admits only one prime ideal. If $\mathfrak{m}$ is the only prime ideal of R, then $\mathfrak{m}$ is necessarily nilpotent. If R is a special principal ideal ring with $\mathfrak{m}$ as its only prime ideal, then we describe it using the notation that $(R, \mathfrak{m})$ is a SPIR. Let $\mathfrak{m}$ be a nonzero maximal ideal of a ring R such that $\mathfrak{m}$ is principal and is nilpotent. Let $n \geq 2$ be the least positive integer with the property that $\mathfrak{m}^n = (0)$. Then it follows from [5] that $\{\mathfrak{m}^i | i \in \{1, \ldots, n-1\}\}$ is the set of all nonzero proper ideals of R. As each ideal of R is principal with $\mathfrak{m}$ as its only prime ideal, it follows that $(R, \mathfrak{m})$ is a SPIR.

Now, we give brief of the theorems proved in this article. In Theorem 2.1, for a ring R we have proved that $G(R)$ is connected and $diam(G(R)) \leq 2$. In Theorem 2.2, we have proved that if $|Max(R)| \geq 3$, then $G(R)$ is not a star graph. In Theorem 2.3 (resp. Theorem 2.4), we have classified rings R with $|Max(R)| = 2$ (resp. $|Max(R)| = 1$) such that $G(R)$ is a star graph. A necessary and sufficient condition for $G(R)$ to be a star graph is provided in

Theorem 2.5. In Theorem 2.6, a classification of rings R is provided for which

$$diam(L(G(R))) < diam(G(R)).$$

We have proved in Theorem 2.7 that if $|Max(R)| \geq 4$, then $diam(L(G(R))) = 3$. In Theorem 2.8, we have proved that for a ring R with $|Max(R)| = 3$, $diam(L(G(R))) = 3$ if and only if $R \ncong F_1 \times F_2 \times F_3$, where F_1, F_2, F_3 are fields. In Theorem 2.9, we have classified the rings R with $|Max(R)| = 2$ for which $diam(L(G(R))) = 2$. In Theorem 2.10, we have classified the rings for which $L(G(R))$ is complete. In Theorem 2.11, we have proved for a reduced ring $R = \prod_{i=1}^{n} R_i$, where R_i is a finite local ring, $diam(L(G(R))) = diam(G(R)) = 2$ if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

In Theorem 3.1, we have proved that for a ring R, if $|Max(R)| \geq 4$, then $G(R)$ is not a split graph. Classification of rings R with $|Max(R)| = 3$ (resp. $|Max(R)| = 2$) for which $G(R)$ is a split graph is provided in Theorem 3.2 (resp. Theorem 3.3). In Theorem 3.4, we have proved that if (R, M) is a local ring which is not a field, then $G(R)$ is a split graph. In Theorem 3.5, we have proved that $G(R)$ admits a cut-vertex if and only if $R \cong F_1 \times F_2$; where F_1 and F_2 are fields. In Lemma 3.6, we have showed that if R is a ring which is not a field then $girth(G(R)) \in \{3, \infty\}$. Necessary and sufficient conditions for which $girth(G(R)) = 3$ $(resp. \infty)$ is provided in Theorem 3.7 (resp. Theorem 3.8). Independence number of $G(R)$ has been discussed in Theorem 3.9. In Theorem 3.10, we have proved that for a non-zero commutative ring R, $G(R)$ is complemented if and only if $R \cong F_1 \times F_2$; where F_1 and F_2 are fields or (R, M) is SPIR with $M \neq (0)$ but $M^2 = (0)$.

2. Diameter of $L(G(R))$

Theorem 2.1. *Let R be a ring which is not a field. Then $G(R)$ is connected and $diam(G(R)) \leq 2$.*

Proof. Let R be a ring which is not a field. Then for any two non-adjacent vertices I, J in $G(R)$, there is a path $I - (0) - J$ of length two between them. So, $G(R)$ is connected and $diam(G(R)) \leq 2$. □

Theorem 2.2. *Let R be a ring. If $|Max(R)| \geq 3$, then $G(R)$ is not a star graph.*

Proof. Let $M_1, M_2, M_3 \in Max(R)$. Note that $M_1M_2 \neq (0)$. Suppose that $M_1M_2 = (0)$. Then $(0) \subseteq M_3$. So, $M_1 \subseteq M_3$ or $M_2 \subseteq M_3$ which is not possible. So, $M_1M_2 \neq (0)$. Suppose that $M_1M_2 = M_1$. Then $M_1 \subseteq M_2$ which is again a contradiction. So, $M_1M_2 \neq M_1$. So, we have a cycle $M_1M_2 - (0) - M_1 - M_1M_2$. Hence, $G(R)$ is not a star graph. Hence, if $|Max(R)| \geq 3$, then $G(R)$ is not a star graph. □

Theorem 2.3. *Let R be a ring with $|Max(R)| = 2$. Then $G(R)$ is a star graph if and only if $R \cong F_1 \times F_2$; where F_1 and F_2 are fields. Indeed, in this case $G(R)$ is $K_{1,2}$.*

Proof. Let $Max(R) = \{M_1, M_2\}$. Suppose that $G(R)$ is a star graph. Note that $M_1M_2 \neq M_i$; for any $i \in \{1, 2\}$. Suppose that $M_1M_2 \neq (0)$. Then $(0) - M_1M_2 - M_1 - (0)$ is a cycle. So, $G(R)$ is not a star graph. Hence, we have $M_1M_2 = (0)$. Therefore, by the Chinese Remainder Theorem [5, Proposirion 1.10(ii),(iii)],

$$R \cong R/J(R) \cong R/M_1 \times R/M_2 \cong F_1 \times F_2;$$

where F_1 and F_2 are fields.

Conversely, suppose that $R \cong F_1 \times F_2$. Note that

$$V(G(R)) = \{F_1 \times (0), (0) \times F_2, (0) \times (0)\}.$$

Hence, $G(R)$ is the star graph $K_{1,2}$ given by

$$F_1 \times (0) - (0) \times (0) - (0) \times F_2.$$

□

Theorem 2.4. *Let R be a ring which is not a field with $|Max(R)| = 1$. Then $G(R)$ is a star graph if and only if R is SPIR with $M \neq (0)$ but $M^2 = (0)$. Indeed, in this case $G(R) = K_{1,1}$.*

Proof. Let $Max(R) = \{M\}$. Suppose that $G(R)$ is a star graph. Let $x \in M \setminus \{0\}$. Clearly, $Rx \neq (0)$. If $M \neq Rx$, then $(0) - Rx - M - (0)$ is a cycle. So, $G(R)$ is not a star graph which is a contradiction. Hence, $M = Rx$. Suppose that $M^2 = M$. Since, $M = J(R)$ and $M = Rx$, we have from the Nakayama's lemma [5, Proposition 2.6], $M = \{0\}$. Hence, R is a field which is a contradiction to the assumption. So, $M^2 \neq M$. If $M^2 \neq (0)$, then again $(0) - M - M^2 - (0)$ is a cycle which is not possible. So, $M^2 = (0)$. Let P be any prime ideal. Note that $M^2 = (0) \subseteq P$. So, $M \subseteq P$. Hence, $M = P$. So, $P = M$ is the only prime ideal of R. So, (R, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$.

Conversely suppose that (R, M) is SPIR with $M \neq (0)$ and $M^2 = (0)$. Note that $V(G(R)) = \{(0), M\}$. So, $G(R)$ is $K_{1,1}$ given by $M - (0)$. □

Theorem 2.5. *Let R be a ring which is not a field. Then $G(R)$ is a star graph if and only if R is isomorphic to one of the following rings.*

(i) (R, M) is SPIR with $M \neq (0)$ and $M^2 = (0)$.

(ii) $F_1 \times F_2$; *where* F_1 *and* F_2 *are fields.*
Indeed, if (i) or (ii) holds, then $G(R)$ *is either* $K_{1,1}$ *or* $K_{1,2}$.

Proof. Proof follows from Theorems 2.2, 2.3 and 2.4. $\square$

Theorem 2.6. *Let* R *be a ring. Then* $diam(L(G(R))) < diam(G(R))$ *if and only if one of the following holds.*
(i) $R \cong F_1 \times F_2$ *where* F_1 *and* F_2 *are fields.*
(ii) (R, M) *is SPIR with* M *as its unique maximal ideal such that* $M \neq (0)$ *but* $M^2 = (0)$.

Proof. Suppose that $diam(L(G(R))) < diam(G(R))$. By Theorem 2.1, $G(R)$ is connected and $diam(G(R)) \leq 2$. Since, R is not a field, it has at least one maximal ideal $M \neq (0)$. Hence, $diam(G(R)) = 1$ or 2. If $diam(G(R)) = 1$ then $G(R)$ is a complete graph. Since $diam(L(G(R))) < diam(G(R)) = 1$, we have $diam(L(G(R))) = 0$. Now, $G(R)$ is connected. So by [21, Proposition 2.2], we have $L(G(R))$ is also connected. Thus,$L(G(R)) = K_1$. Hence, $G(R) = K_{1,1}$. Hence, by Theorem 2.4, (R, M) is a SPIR with M as its unique maximal ideal such that $M \neq (0)$ but $M^2 = (0)$. If $diam(G(R)) = 2$, then $diam(L(G(R))) = 0$ or 1. If $diam(L(G(R))) = 0$, then $G(R) = K_{1,1}$. So, $diam(G(R)) = 1$ which is a contradiction. So, $diam(L(G(R))) = 1$. Therefore, $L(G(R)) = K_n$; $n \in \mathbb{N}$. Now, if $L(G(R)) = K_3$, then $G(R) = K_3$ or $K_{1,3}$. Note that from Theorem 2.5, $G(R) \neq K_{1,3}$. So, $G(R) = K_3$. Then $diam(G(R)) = 1$ which is not possible. Hence, $L(G(R)) \neq K_3$. So, $L(G(R)) = K_n$; $n \in \mathbb{N}, n \neq 3$. Hence, $G(R)$ is a star graph. By Theorem 2.5, $R \cong F_1 \times F_2$; where F_1 and F_2 are fields or (R, M) is SPIR with M as its unique maximal ideal such that $M \neq (0)$ but $M^2 = (0)$.

Conversely, assume that $R \cong F_1 \times F_2$; where F_1 and F_2 are fields. Then by Theorem 2.3, $G(R) = K_{1,2}$ and so $L(G(R)) = K_{1,1}$. Therefore,

$$1 = diam(L(G(R))) < diam(G(R)) = 2.$$

Now, let (R, M) be SPIR with M as its unique maximal ideal such that $M \neq (0)$ but $M^2 = (0)$. Then by Theorem 2.4, $G(R) = K_{1,1}$. So, $L(G(R))$ is a null graph. Therefore,

$$0 = diam(L(G(R))) < diam(G(R)) = 1.$$

$\square$

Theorem 2.7. *Let* R *be a ring with* $|Max(R)| \geq 4$. *Then*

$$diam(L(G(R))) = 3.$$

Proof. Let $M_1, M_2, M_3, M_4 \in Max(R)$. Note that $[M_1, M_1M_2]$ and $[M_3, M_3M_4]$ are non-adjacent in $L(G(R))$. Suppose that there exists a path of length two between $[M_1, M_1M_2]$ and $[M_3, M_3M_4]$, say $[M_1, M_1M_2] - [I, J] - [M_3, M_3M_4]$; for some $[I, J] \in V(L(G(R)))$. Then $[I, J] = [M_1, M_3]$ or $[M_1, M_3M_4]$ or $[M_1M_2, M_3]$ or $[M_1M_2, M_3M_4]$. But, $M_1 + M_3 = R$, $M_1 + M_3M_4 = R$, $M_1M_2 + M_3 = R$,

$$M_1M_2 + M_3M_4 = R.$$

So, no such $[I, J]$ exists in $V(L(G(R)))$. Hence, the length of path between $[M_1, M_1M_2]$ and $[M_3, M_3M_4]$ is of atleast three. By Theorem 2.1 and [21, Proposition 2.2], $diam(L(G(R))) \leq 3$. Hence,

$$diam(L(G(R))) = 3.$$

□

Theorem 2.8. *Let R be a finite ring with $|Max(R)| = 3$. Then $diam(L(G(R))) = 3$ if and only if $R \ncong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields.*

Proof. Let R be a ring with $|Max(R)| = 3$. Let

$$Max(R) = \{M_1, M_2, M_3\}.$$

Assume that $diam(L(G(R))) = 3$. Let if possible

$$R \cong F_1 \times F_2 \times F_3;$$

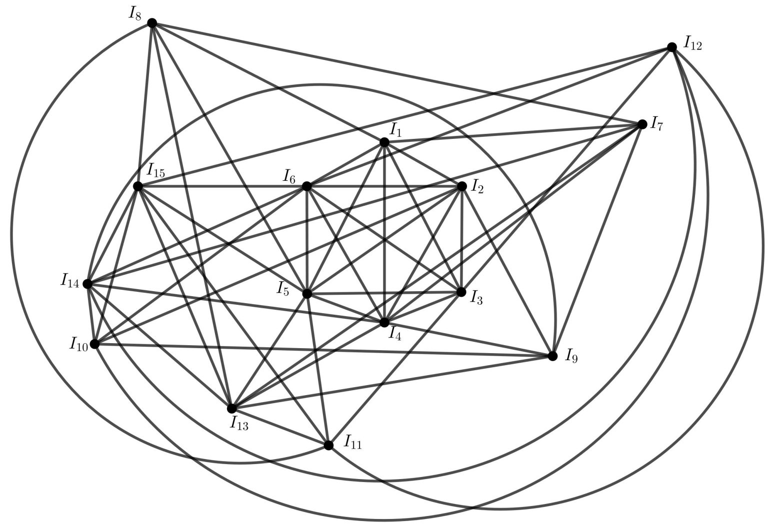
where F_1, F_2 and F_3 are fields. Note that

$$\begin{aligned} V(L(G(R))) = \{ & I_1 = [(0), M_1],\ I_2 = [(0), M_2],\ I_3 = [(0), M_3], \\ & I_4 = [(0), M_1M_2],\ I_5 = [(0), M_1M_3], \\ & I_6 = [(0), M_2M_3],\ I_7 = [M_1, M_1M_2], \\ & I_8 = [M_1, M_1M_3],\ I_9 = [M_2, M_1M_2], \\ & I_{10} = [M_2, M_2M_3],\ I_{11} = [M_3, M_1M_3], \\ & I_{12} = [M_3, M_2M_3], I_{13} = [M_1M_2, M_1M_3], \\ & I_{14} = [M_1M_2, M_2M_3],\ I_{15} = [M_1M_3, M_2M_3]\}. \end{aligned}$$

From the following figure and distance matrix of the graph $G(F_1 \times F_2 \times F_3)$, it is clear that

$$diam(L(G(F_1 \times F_2 \times F_3))) = 2.$$

Hence, $R \ncong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields.



$Figure.1: \; L(G(F_1 \times F_2 \times F_3))$

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 & 1 & 2 & 1 & 0 & 2 & 2 & 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 0 & 1 & 2 & 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 0 & 2 & 1 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 1 \\ 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 0 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 0 & 1 & 1 \\ 2 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Conversely, assume that $R \ncong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields. Let $x_i \in M_i \setminus (M_j \cup M_k)$; for distinct $i, j, k \in \{1, 2, 3\}$. If $Rx_i \neq M_i$ then $[Rx_i, M_i]$ and $[M_j, M_jM_k]$ are non-adjacent vertices in $L(G(R))$. Suppose that there exists a path of length two between them, say $[Rx_i, M_i] - [K, P] - [M_j, M_jM_k]$. Then $[K, P] = [Rx_i, M_j]$ or $[Rx_i, M_jM_k]$ or $[M_i, M_j]$ or $[M_i, M_jM_k]$. But, Rx_i and M_j, Rx_i and M_jM_k, M_i and M_j, M_i and MjM_k are non-adjacent in $G(R)$. So, no such $[K, P]$ exists in $V(L(G(R)))$. Thus,

$$diam(L(G(R))) \geq 3.$$

By Theorem 2.1 and [21, Proposition 2.2], $diam(L(G(R))) \leq 3$. Hence, $diam(L(G(R))) = 3$. Suppose that $Rx_i = M_i$ for each $i \in \{1, 2, 3\}$. Suppose that $M_i^2 \neq M_i$. Let

$$x \in M_i^2 \smallsetminus (M_j \cup M_k);$$

for distinct $i, j, k \in \{1, 2, 3\}$. Then, $Rx \subseteq M_i^2 \neq M_i$. But as $x \in M_i \smallsetminus (M_j \cup M_k)$, we have $Rx = M_i$ which is a contradiction. So, $M_i^2 = M_i$; for all $i \in \{1, 2, 3\}$. Then

$$J(R) = M_1M_2M_3 = Rx_1x_2x_3$$

and $(J(R))^2 = J(R)$. By the Nakayama's lemma [5, Proposition 2.6], $J(R) = (0)$. Thus, by the Chinese Remainder Theorem [5, Proposition 1.10(ii),(iii)],

$$R \cong \frac{R}{J(R)} \cong \frac{R}{M_1} \times \frac{R}{M_2} \times \frac{R}{M_3} \cong F_1 \times F_2 \times F_3;$$

where F_1, F_2 and F_3 are fields.

□

Theorem 2.9. *Let R be an Artinian ring with $|Max(R)| = 2$. Then $diam(L(G(R))) = 2$ if and only if $R \cong R_1 \times R_2$; where (R_1, m_1) and (R_2, m_2) are local rings, $R_ia = m_i$ for some $a \in R_i$ and $m_i^2 = m_i$; for atleast one $i \in \{1, 2\}$.*

Proof. Since $|Max(R)| = 2$ and R is an Artinian ring, $R \cong R_1 \times R_2$; where (R_i, m_i) is a local ring for all $i \in \{1, 2\}$. Note that

$$Max(R) = \{M_1 = m_1 \times R_2, M_2 = R_1 \times m_2\}.$$

Assume that $diam(L(G(R))) = 2$. Suppose that $m_1^2 \neq m_1$ and $m_2^2 \neq m_2$. So, $M_1^2 \neq M_1$ and $M_2^2 \neq M_2$. Note that

$$[M_1, M_1^2], [M_2, M_2^2] \in V(L(G(R)))$$

are non-adjacent. Let if possible, there exists a path of length two between $[M_1, M_1^2]$ and $[M_2, M_2^2]$, say $[M_1, M_1^2] - [I, J] - [M_2, M_2^2]$. Now, $[I, J] = [M_1, M_2]$ or $[M_1, M_2^2]$ or $[M_1^2, M_2]$ or $[M_1^2, M_2^2]$. But, M_1 and M_2, M_1 and M_2^2, M_1^2 and M_2, M_1^2 and M_2^2 are not adjacent in $G(R)$. So, no such $[I, J]$ exists in $V(L(G(R)))$. So, the length of the path between $[M_1, M_1^2]$ and $[M_2, M_2^2]$ is atleast three. Thus, $M_1^2 = M_1$ or $M_2^2 = M_2$. Without loss of generality, we may assume that $M_1^2 = M_1$. So, $m_1^2 = m_1$. Let $x_1 \in M_1 \smallsetminus (0)$. Suppose that $Rx_1 = M_1$. If $x_1 = (a, 1)$; for some $a \in R_1$ then $m_1 = R_1a$. Suppose that $Rx_1 \neq M_1$. Let $M_2^2 \neq M_2$. Suppose that there exists a path of length two between $[Rx_1, M_1]$ and $[M_2, M_2^2]$, say $[Rx_1, M_1] - [I, J] - [M_2, M_2^2]$.

Then $[I, J] = [Rx_1, M_2]$ or $[Rx_1, M_2^2]$ or $[M_1, M_2]$ or $[M_1, M_2^2]$. But, Rx_1 and M_2, Rx_1 and M_2^2, M_1 and M_2, M_1 and M_2^2 are not adjacent in $G(R)$. So, in any case such $[I, J]$ does not exist in $V(L(G(R)))$. Thus, the length of path between $[Rx_1, M_1]$ and $[M_2, M_2^2]$ is atleast three. Hence, $M_2^2 = M_2$. Let $x_2 \in M_2 \setminus (0)$. If $Rx_2 \neq M_2$, then again by similar argument, the length of the path between $[M_1, Rx_1]$ and $[M_2, Rx_2]$ is atleast three. So, $M_2 = Rx_2$. Hence, $M_2^2 = M_2$ and $M_2 = Rx_2$; for some $x_2 \in M_2$. If $x_2 = (1, b)$; for some $b \in R_2$, then $m_2 = R_2 b$.

Conversely, assume that $R \cong R_1 \times R_2$; where (R_1, m_1) and (R_2, m_2) are local rings, $R_i a = m_i$ for some $a \in R_i$ and $m_i^2 = m_i$; for atleast one $i \in \{1, 2\}$. Let $R_1 a = m_1$; for some $a \in R_1$ and $m_1^2 = m_1$. $V(L(G(R)))$ contains vertices of the form $[M_1, I]$ and $[K, P]$; where $I \subseteq J(R)$ and $K, P \subseteq M_2$. Non-adjacent vertices in $L(G(R))$ are either of the form $[M_1, I]$ and $[K, P]$; where $I \subseteq J(R)$, $K, P \subseteq M_2$ or of the form $[K_1, P_1]$ and $[K_2, P_2]$; where $K_1, P_1, K_2, P_2 \subseteq M_2$ are distinct vertices in $G(R)$. Let $[M_1, I]$ and $[K, P]$ be two non-adjacent vertices in $L(G(R))$; $I \subseteq J(R)$ and $K, P \subseteq M_2$. Then, $[M, I] - [I, K] - [K, P]$ is a path of length two between them as $I \subseteq J(R)$ implies $I \subseteq K$. Now, let $[K_1, P_1]$ and $[K_2, P_2]$ be non-adjacent vertices in $L(G(R))$; where $K_1, P_2 \subseteq M_2$. Thus, $[K_1, P_1] - [K_1, P_2] - [K_2, P_2]$ is a path of length two between $[K_1, P_1]$ and K_2, P_2 in $L(G(R))$. Hence, $diam(L(G(R))) = 2$. □

Theorem 2.10. *Let R be a ring. Then $L(G(R))$ is complete if and only if R is isomorphic to one of the following rings:*

(i) $F_1 \times F_2$; where F_1 and F_2 are fields.

(ii) (R, M) is SPIR with $M \neq (0)$ but $M^2 = (0)$.

(iii) (R, M) is SPIR with $M^2 \neq (0)$ but $M^3 = (0)$.

Proof. Suppose $L(G(R))$ is complete. Let $L(G(R)) = K_n$; $n \in \mathbb{N}$. If $n \neq 3$, then $G(R)$ is a star graph. Hence, by Theorem 2.5, $R \cong F_1 \times F_2$; where F_1 and F_2 are fields or (R, M) is SPIR with $M \neq (0)$ but $M^2 = (0)$. If $L(G(R)) = K_3$, then $G(R) = K_3$ or $K_{1,3}$. But, by Theorem 2.5, $G(R) \neq K_{1,3}$. So, $G(R) = K_3$. Suppose $|Max(R)| \geq 2$. Let $M_1, M_2 \in Max(R)$. Then M_1 and M_2 are not adjacent in $G(R)$. So, $G(R) \neq K_3$ which is a contradiction to the assumption. So, $|Max(R)| = 1$. Let $Max(R) = \{M\}$. As R is not a field, $M \neq (0)$. Let $x \in M \setminus (0)$. Suppose that $M \neq Rx$. Let $y \in M \setminus Rx$. Then it is clear that $Ry \neq Rx$. Also, $Ry \neq (0)$ as $y \neq 0$. Now, if $M \neq Ry$, then $M, Rx, Ry, (0) \in V(G(R))$ forms K_4 which is not possible. Hence, $M = Ry$. Suppose that $M^2 = (0)$. Now, let P be any prime ideal. Then $M^2 = (0) \subseteq P$. So, $M \subseteq P$. Thus $M = P$. Hence, (R, M) is SPIR with $M \neq (0)$ but $M^2 = (0)$. Then $G(R) = K_2$

which is not possible. So, $M^2 \neq (0)$. Let if possible, $M^2 = M$. Since $M = Ry$ and $M^2 = M$, by the Nakayama's lemma [5, Proposition 2.6], we have $M = (0)$. This is not possible. So, $M^2 \neq M$. Hence, $M^2 \neq (0)$ and $M^2 \neq M$. As $M = Ry$, we have $M^2 = Ry^2$. Now, if $M^3 = M^2$, then by the Nakayama's lemma [5, Proposition 2.6], $M = (0)$. So, $M^3 \neq M^2$. If $M^3 \neq (0)$, then $M, M^2, M^3, (0) \in V(G(R))$ which is also not possible. So, $M^3 = (0)$. Let P be any prime ideal of R. Then $M^3 = (0) \subseteq P$. So, $M \subseteq P$. So, $M = P$. Thus, (R, M) is SPIR with $M^2 \neq (0)$ but $M^3 = (0)$.

Conversely, assume that $R \cong F_1 \times F_2$; where F_1 and F_2 are fields. Then by Theorem 2.3, $G(R)) = K_{1,2}$. So, $L(G(R))) = K_2$. Now, we assume that (R, M) is SPIR with $M \neq (0)$ but $M^2 = (0)$. Then by Theorem 2.4, $G(R) = K_{1,1}$. So, $L(G(R)) = K_1$. If (R, M) is SPIR with $M^2 \neq (0)$ but $M^3 = (0)$ then $G(R)$ is K_3 given by

$$(0) - M - M^2 - (0)$$

and so $L(G(R)) = K_3$. □

Theorem 2.11. *Let $R = \prod_{i=1}^{n} R_i$ be a reduced ring with maximal ideals $M_1, M_2, ..., M_n$; for some $n \in \mathbb{N}$ where R_i is a finite local ring with maximal ideals $\mathfrak{n}_1, \mathfrak{n}_2, ..., \mathfrak{n}_n$. Then $diam(L(G(R))) = diam(G(R))$ $= 2$ if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Assume that $diam(L(G(R))) = diam(G(R)) = 2$. As R is a finite reduced ring, it is a direct product of finitely many fields. Let $R \cong F_1 \times F_2 \times ... \times F_n$; where F_i is a field for each $n \in \mathbb{N}$. Let if possible,

$$\left| M_i \setminus \bigcup{}_{\substack{j=1\\ j\neq i}}^{\,n} M_j \right| \geq 2;$$

for some $i \in \{1, 2, ..., n\}$. Recall from [21, Remark 2.9],

$$\left| M_i \setminus \bigcup{}_{\substack{j=1\\ j\neq i}}^{\,n} M_j \right| \geq 2;$$

for atleast (n-1) i's. Choose $s, t \in M_1 \setminus \bigcup_{j=2}^{n} M_j$ and

$$u, v \in M_2 \setminus \bigcup{}_{\substack{j=1\\ j\neq 2}}^{\,n} M_j.$$

Take $V_1 = [Rs, Rt]$ and $V_2 = [Ru, Rv]$. Then $V_1, V_2 \in V(L(G(R)))$. Clearly, V_1 and V_2 are not adjacent in $L(G(R))$. Let if possible,

$$V_1 = [Rs, Rt] - [I, J] - [Ru, Rv] = V_2$$

be a path between V_1 and V_2. Without loss of generality, we may assume that $I = Rs$ and $J = Rv$. So, $[Rs, Rv] \in V(L(G(R)))$. So, Rs and Rv are adjacent in $G(R)$. Thus, there exists a maximal ideal, say M that contains both Rs and Rv which is not possible. So, the path between V_1 and V_2 is atleast of length three which contradicts the hypothesis. So, we have $\left|M_i \setminus \bigcup_{j\neq i}^{j=1\,n} M_j\right| = 1$; for all i. Hence, $F_i = \mathbb{Z}_2$; for each $i \in \{1, 2, ..., n\}$. Let if possible, $n \geq 4$. Choose $s \in M_1 \setminus \bigcup_{j=2}^{n} M_j$, $t \in (M_1 \cap M_2) \setminus \bigcup_{j=3}^{n} M_j$, $u \in M_3 \setminus \bigcup_{\substack{j=1\\ j\neq 3}}^{n} M_j$ and $v \in (M_3 \cap M_4) \setminus \bigcup_{\substack{j=1\\ j\neq 3,4}}^{n} M_j$. Then $[Rs, Rt]$ and $[Ru, Rv]$ are not adjacent in $L(G(R))$. Let if possible, $[Rs, Rt] - [I, J] - [Ru, Rv]$ be a path between $[Rs, Rt]$ and $[Ru, Rv]$ in $L(G(R))$. Without loss of generality, we may assume that $I = Rs$ and $J = Rv$. So, Rs and Rv are adjacent in $G(R)$ which is not true by the choice of s and v. So, the path between $[Rs, Rt]$ and $[Ru, Rv]$ is atleast of length three which again contradicts the hypothesis. So, $|Max(R)| \leq 3$. Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then

$$V(R) = \{\mathbb{Z}_2 \times (0), (0) \times \mathbb{Z}_2, (0) \times (0)\}.$$

Note that $G(\mathbb{Z}_2 \times \mathbb{Z}_2) = K_{1,2}$ and so $L(G(\mathbb{Z}_2 \times \mathbb{Z}_2)) = K_2$. Hence, $diam(L(G(\mathbb{Z}_2 \times \mathbb{Z}_2))) = 1$ which contradicts the hypothesis. So, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Conversely, assume that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that

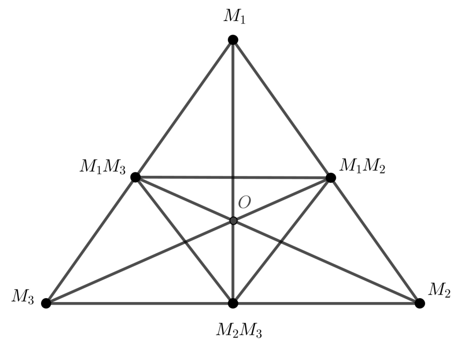
$$\begin{aligned} V(G(R)) = \big\{ & O = (0) \times (0) \times (0), M_1 = (0) \times \mathbb{Z}_2 \times \mathbb{Z}_2, \\ & M_2 = \mathbb{Z}_2 \times (0) \times \mathbb{Z}_2, M_3 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times (0), \\ & M_1M_2 = (0) \times (0) \times \mathbb{Z}_2, M_1M_3 = (0) \times \mathbb{Z}_2 \times (0), \\ & M_2M_3 = \mathbb{Z}_2 \times (0) \times (0)\big\}. \end{aligned}$$

From the following Figure.2, it is clear that $diam(G(R)) = 2$. Note that

$$\begin{aligned} V(L(G(R))) = \big\{ & I_1 = [(0),\ M_1],\ I_2 = [(0), M_2], \\ & I_3 = [(0), M_3],\ I_4 = [(0), M_1M_2], \\ & I_5 = [(0), M_1M_3],\ I_6 = [(0), M_2M_3], \\ & I_7 = [M_1, M_1M_2],\ I_8 = [M_1, M_1M_3], \\ & I_9 = [M_2, M_1M_2],\ I_{10} = [M_2, M_2M_3], \\ & I_{11} = [M_3, M_1M_3],\ I_{12} = [M_3, M_2M_3]\big\} \end{aligned}$$

From Figure.1, it is clear that $diam(V(L(G(R)))) = 2$. Hence,

$$diam(G(R)) = 2 = diam(L(G(R))).$$



$Figure.2: \ G(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

□

3. Some more Results on $G(R)$

Theorem 3.1. *Let R be a ring with $|Max(R)| \geq 4$. Then $G(R)$ is not a split graph.*

Proof. Let $G(R)$ be a split graph. Let $V(G(R)) = K \cup S$; where the subgraph of $G(R)$ induced on K is complete, S is an independent set of $G(R)$ and $K \cap S = \emptyset$. Let

$$M_1, M_2, M_3, M_4 \in Max(R).$$

Suppose that $(0) \in S$. Then $M_i \notin S$; for $i \in \{1, 2, 3, 4\}$ as (0) is adjacent to each M_i; for $i \in \{1, 2, 3, 4\}$. So, $M_i \in K$; for all $i \in \{1, 2, 3, 4\}$ which is not possible as any two distinct maximal ideals are not adjacent in $G(R)$. So, $(0) \in K$. Note that at most one of the maximal ideal can be in K. Without loss of generality, we may assume that $M_1 \in K$. Hence, $M_2, M_3, M_4 \in S$. If $M_2M_3 = (0)$, then $M_2M_3 = (0) \subseteq M_1$. So, $M_2 = M_1$ or $M_3 = M_1$ which is not possible. Hence, $M_2M_3 \neq (0)$. Also, $M_2M_3 \neq M_i$; for any $i \in \{1, 2, 3, 4\}$ as if $M_2M_3 = M_i$; for some $i \in \{1, 2, 3, 4\}$, then $M_2 = M_i$ or $M_3 = M_i$ which is not possible. Since $M_2 \in S$ and M_2M_3 is adjacent to M_2 in $G(R)$, we have $M_2M_3 \notin S$. Also, $M_1 + M_2M_3 = R$. So, $M_2M_3 \notin K$. Thus, $Max(R) \subseteq S$. Note that M_iM_j is adjacent with M_i; for any distinct $i, j \in \{1, 2, 3, 4\}$. Hence, $M_iM_j \in K$; for distinct $i, j \in \{1, 2, 3, 4\}$. So, we have M_1M_2, $M_3M_4 \in K$. But, then $M_1M_2 + M_3M_4 = R$. So, M_1M_2 and M_3M_4 are not adjacent in $G(R)$. Hence, $G(R)$ is not a split graph. This is a contradiction. So, $|Max(R)| \leq 3$. □

Theorem 3.2. *Let R be a ring with $|Max(R)| = 3$. Then $G(R)$ is a split graph if and only if $R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields.*

Proof. Suppose that $R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields. Note that

$$\begin{aligned} V(G(R)) &= \{(0) \times (0) \times (0), M_1 = (0) \times F_2 \times F_3, \\ &\quad M_2 = F_1 \times (0) \times F_3, M_3 = F_1 \times F_2 \times (0), \\ &\quad M_1M_2, M_2M_3, M_1M_3\}. \end{aligned}$$

Let $K = \{M_1M_2, M_2M_3, M_1M_3, (0)\}$ and $S = \{M_1, M_2, M_3\}$. Then we have $V(G(R)) = K \cup S$, where the subgraph of $G(R)$ induced on K is complete, S is an independent set of $G(R)$ and $K \cap S = \emptyset$. Therefore, $G(R)$ is a split graph.

Conversely, assume that $G(R)$ is a split graph. Let

$$V(G(R)) = K \cup S;$$

where the subgraph of $G(R)$ induced on K is complete, S is an independent set of $G(R)$ and $K \cap S = \emptyset$. Let

$$Max(R) = \{M_1, M_2, M_3\}.$$

Note that $M_iM_j \neq (0)$; for any $i, j \in \{1, 2, 3\}$. As $M_i + M_j = R$; for $i \neq j \text{ and } i, j \in \{1, 2, 3\}$, we have at most one $M_i \in K$, for $i \in \{1, 2, 3\}$. Let $M_1 \in K$ and $M_2, M_3 \in S$. Now, M_2 and M_2M_3 are adjacent in $G(R)$. So, $M_2M_3 \notin S$. Also, $M_1 + M_2M_3 = R$. So, $M_2M_3 \notin K$. So, $Max(R) \subseteq S$. Since,(0) is adjacent to all other vertices, we have $(0) \in K$. Note that $M_2M_3 \neq (0)$. Observe that M_2M_3 and M_2 are adjacent in $G(R)$. As $M_2 \in S$, we have $M_2M_3 \in K$. Let

$$x \in M_1 \setminus (M_2 \cup M_3).$$

Let if possible, $Rx \neq M_1$. As Rx is adjacent to M_1 in $G(R)$, $Rx \notin S$. Also, $Rx + M_2M_3 = R$. So, $Rx \notin K$. Hence, $Rx = M_1$. Similarly, $M_2 = Ry$; for some $y \in M_2 \setminus (M_1 \cup M_3)$ and $M_3 = Rz$; for some $z \in M_3 \setminus (M_1 \cup M_2)$. Thus, $J(R) = Rxyz$. Let if possible, $M_1^2 = (0)$. Then $M_1^2 = (0) \subseteq M_2$. This implies that $M_1 = M_2$ which is not possible. So, $M_1^2 \neq (0)$. If $M_1^2 \neq M_1$, then $M_1^2 + M_2M_3 = R$. So, M_1^2 cannot be in K. Also, M_1^2 and M_1 are adjacent in $G(R)$. So, $M_1^2 \notin S$. Hence, $M_1^2 = M_1$. By similar argument, $M_2^2 = M_2$ and $M_3^2 = M_3$. So, $(J(R))^2 = J(R)$. Since, $J(R)$ is principal, by the Nakayama's lemma [5, Proposition 2.6], we have $J(R) = (0)$. So, by the Chinese Remainder Theorem [5, Proposition 1.10(ii),(iii)],

$$R \cong \tfrac{R}{J(R)} \cong \tfrac{R}{M_1} \times \tfrac{R}{M_2} \times \tfrac{R}{M_3} \cong F_1 \times F_2 \times F_3;$$

where F_1, F_2 and F_3 are fields. $\square$

Theorem 3.3. *Let R be an Artinian ring with $|Max(R)| = 2$. Then $G(R)$ is a split graph if and only if $R \cong R_1 \times R_2$; where (R_1, m_1) and (R_2, m_2) are local rings, $R_i a = m_i$ for some $a \in R_i$ and $m_i^2 = m_i$; for at least one $i \in \{1, 2\}$.*

Proof. Since R is an Artinian ring, $R \cong R_1 \times R_2$; where (R_1, m_1) and (R_2, m_2) are local rings [5, Proposition 8.7]. Note that

$$Max(R) = \{M_1 = m_1 \times R_2, M_2 = R_1 \times m_2\}.$$

Assume that $G(R)$ is a split graph. Let $V(G(R)) = K \cup S$; where the subgraph induced on K is complete, S is an independent set of $G(R)$ and $K \cap S = \emptyset$. If $(0) \in S$, then $M_1, M_2 \in K$ which is not possible as $M_1 + M_2 = R$. So, $(0) \in K$. Note that any two distinct maximal ideals are not adjacent in $G(R)$. So, atmost one of M_1 or M_2 can be placed in K. Without loss of generality, we may assume that $M_2 \in K$. Thus, $M_1 \in S$. Let $x_1 \in M_1 \setminus M_2$. Let if possible, $Rx_1 \neq M_1$. Then, $Rx_1 + M_2 = R$. So, Rx_1 is not adjacent to M_2. So, $Rx_1 \notin K$. Also, Rx_1 is adjacent to M_1 in $G(R)$. So, $Rx_1 \notin S$. Thus, $Rx_1 = M_1$. Note that $M_1^2 \neq (0)$. Let if possible, $M_1^2 \neq M_1$. Then $M_1^2 \notin K$ as $M_1^2 + M_2 = R$. Also, M_1^2 is adjacent to M_1 in $G(R)$. So, $M_1^2 \notin S$. Thus, $M_1^2 = M_1$. Let $x_1 = (a, 1)$; for some $a \in R_1$. Then $R_1 a = m_1$ and $m_1^2 = m_1$. Suppose that $M_1, M_2 \in S$. Let $x_1 \in M_1 \setminus M_2$. Let if possible $Rx_1 \neq M_1$, then Rx_1 is adjacent to M_1. So, $Rx_1 \notin S$. So, $Rx_1 \in K$. Let $x_2 \in M_2 \setminus M_1$. If $M_2 \neq Rx_2$, then $Rx_2 \notin S$ as Rx_2 is adjacent to M_2. Also, $Rx_1 + Rx_2 = R$. So, $Rx_2 \notin K$. Thus, $Rx_2 = M_2$. If $M_2^2 \neq M_2$, then $M_2^2 \notin S$ as M_2^2 and M_2 are adjacent. Also, $Rx_1 + M_2^2 = R$. So, $M_2^2 \notin K$. Thus, $M_2^2 = M_2$. Let $x_2 = (1, b)$; for some $b \in R_2$. Then $R_2 b = m_2$ and $m_2^2 = m_2$. Suppose $Rx_1 = M_1$ and if $M_1^2 \neq M_1$, then $M_1^2 \in K$. By similar argument as above, $m_2^2 = m_2$ and $Rb = m_2$; for some $b \in R_2$.

Conversely, assume that $R \cong R_1 \times R_2$; where (R_1, m_1) and (R_2, m_2) are local rings, $R_i a = m_i$ for some $a \in R_i$ and $m_i^2 = m_i$; for atleast one $i \in \{1, 2\}$. Suppose that $R_1 a = m_1$; for some $a \in R_1$ and $m_1^2 = m_1$. Let

$$K = \{(0), M_2\} \cup \{I \in I(R) : I \subseteq M_2\}$$

and $S = \{M_1\}$. Hence, $G(R)$ is a split graph. $\square$

Theorem 3.4. *Let (R, M) be a local ring which is not a field. Then $G(R)$ is a split graph.*

Proof. Proof is clear. $\square$

Theorem 3.5. *$G(R)$ admits a cut-vertex if and only if $R \cong F_1 \times F_2$; where F_1 and F_2 are fields.*

Proof. Suppose that $G(R)$ has a cut-vertex, say K. Now, $G(R)$ admits a cut-vertex K if and only if there exists I and $J \in V(G(R))$ such that there is exactly one path $I - K - J$ between I and J in $G(R)$. Note that $I, J \neq (0)$ as (0) is adjacent to all the vertices in $G(R)$. If $K \neq (0)$, then there is another path $I-(0)-J$ between I and J in $G(R)$. Hence, $K = (0)$. Let $M, M' \in Max(R)$ such that $I \subseteq M$ and $J \subseteq M'$. As I and J are not adjacent in $G(R)$, $M \neq M'$. Let $x \in M \cap M'$; where $x \neq 0$. Then $I - Rx - J$ is another path between I and J which is not possible. So, $M \cap M' = (0)$. Let if possible, M'' be a maximal ideal distinct from M and M'. Then $M \cap M' = (0) \subseteq M''$. But then either $M = M''$ or $M' = M''$ which is also not possible. So, $|Max(R)| \leq 2$. If $|Max(R)| = 1$, then $G(R)$ is a complete graph. So, it will never admit a cut-vertex. So, $|Max(R)| = 2$. Let

$$Max(R) = \{M_1, M_2\}.$$

As $M_1 \cap M_2 = (0)$, by the Chinese Remainder Theorem [5, Proposition 1.10(ii),(iii)],

$$R \cong \tfrac{R}{J(R)} \cong \tfrac{R}{M_1} \times \tfrac{R}{M_2} \cong F_1 \times F_2;$$

where F_1 and F_2 are fields.

Conversely, assume that $R \cong F_1 \times F_2$; for some fields F_1 and F_2. Then $R \cong F_1 \times F_2$ is a path graph $F_1 \times (0) - (0) \times (0) - (0) \times F_2$ and clearly $(0) \times (0)$ is a cut-vertex. □

Lemma 3.6. *Let R be a ring which is not a field. Then*

$$girth(G(R)) \in \{3, \infty\}.$$

Proof. Assume that $|Max(R)| \geq 3$. Let $M_1, M_2, M_3 \in Max(R)$. Suppose that $M_1M_2 = (0)$. Then, $M_1M_2 = (0) \subseteq M_3$. So, $M_1 = M_3$ or $M_2 = M_3$ which is not possible. So, $M_1M_2 \neq (0)$. Suppose that $M_1M_2 = M_1$. Then $M_1 \subseteq M_2$ which is not possible. So, $M_1M_2 \neq M_1$. Thus, we have a cycle $M_1M_2 - (0) - M_1 - M_1M_2$ in $G(R)$ of length three. Hence, $girth(G(R)) = 3$.

Let $|Max(R)| = 2$. Let $Max(R) = \{M_1, M_2\}$. If

$$M_1M_2 = J(R) = (0),$$

then by the Chinese Remainder Theorem [5, Proposition 1.10(ii),(iii)],

$$R \cong R/J(R) \cong R/M_1 \times R/M_2 = F_1 \times F_2;$$

where F_1 and F_2 are fields. Then $G(R)$ is $K_{1,2}$ and so $girth(G(R)) = \infty$. If $J(R) = M_1M_2 \neq (0)$, then $M_1 - M_1M_2 - (0) - M_1$ forms a cycle of length 3. Hence, $girth(G(R)) = 3$.

Assume that (R, M) is a local ring. Let $x \in M \setminus (0)$. If $Rx \neq M$, then we have a cycle $(0) - Rx - M - (0)$ in $G(R)$. So, $girth(G(R)) = 3$. Let $Rx = M$. Let if possible, $M^2 = (0)$. Let P be any prime ideal of R. Then as $M^2 = (0)$, we have $M \subseteq P$. Thus, $M = P$. So, $P = M$ is the only prime ideal of R. So, (R, M) is SPIR with $M \neq (0)$ but $M^2 = (0)$. Hence, $girth(G(R)) = \infty$. Let $M^2 \neq (0)$. Also, let if possible $M^2 = M$. Then $M = J(R)$ and $M = Rx$. So, by the Nakayama's lemma [5, Proposition 2.6], we have $M = \{0\}$, which is not possible as R is not a field. Hence, R is a field which is contradiction to the assumption. So, $M^2 \neq M$. Hence, $(0) - M^2 - M - (0)$ is a cycle of length three in $G(R)$. So, $girth(G(R)) = 3$. □

Theorem 3.7. *Let R be a ring which is not a field. Then*

$$girth(G(R)) = 3$$

if and only if one of the following conditions hold.

(i) $|Max(R)| \geq 3$

(ii) $|Max(R)| = 2$ *and* $J(R) \neq (0)$.

(iii) (R, M) *is a local ring which is not isomorphic to SPIR* (S, M)*; where* $M \neq (0)$ *but* $M^2 = (0)$.

Proof. Proof follows from Theorem 3.6. □

Theorem 3.8. *Let R be a ring which is not a field. Then*

$$girth(G(R)) = \infty$$

if and only if one of the following conditions hold.

(i) $R \cong F_1 \times F_2$*; where* F_1 *and* F_2 *are fields.*

(ii) (R, M) *is SPIR with* $M \neq (0)$ *but* $M^2 = (0)$.

Proof. Proof follows from Theorem 3.6. □

Theorem 3.9. *Let R be a ring. Then $\alpha(G(R)) = n$ if and only if $|Max(R)| = n$; for some $n \in \mathbb{N}$.*

Proof. Assume that $\alpha(G(R)) = n$; for some $n \in \mathbb{N}$. Let

$$Max(R) = \{M_1, M_2, ..., M_m\};$$

for some $m \in \mathbb{N}$. It is clear that $Max(R)$ is an independent set in $G(R)$. So, $m \leq n$. Let $W = \{I_1, I_2, ..., I_n\}$ be an independent set of $G(R)$ with $|W| = n$. Since W is an independent set, I_i is not adjacent to any of the $I_j's$; for $i \neq j$ and $i, j \in \{1, 2, ..., n\}$. Let $I_i \subseteq M_i$; for some $i \in \{1, 2, ..., n\}$. Let if possible, $M_i = M_j$ for some $i \neq j$ and

$i, j \in \{1, 2, ..., n\}$. Then I_i and I_j will be adjacent in $G(R)$ which is not possible. Hence, $M_i \neq M_j$; for $\forall i, j \in \{1, 2, ..., n\}$. Therefore, $m \geq n$. Hence, $m = n$.

Conversely, asssume that $|Max(R)| = n$. Since, $Max(R)$ forms an independent set in $G(R)$, $\alpha(G(R)) \geq n$. Let if possible, $\alpha(G(R)) > n$. Then there exists an independent set, say $W = \{I_1, I_2, ..., I_n, ..., I_t\}$; where $t > n, t \in \mathbb{N}$. Let $I_i \subseteq M_i$; for some $i \in \{1, 2, ..., t\}$. Now, $t > n$. So, by the Pigeon-hole principle, there exists a maximal ideal M_r; for some $r \in \{1, 2, ..., n\}$ and $i, j \in \{1, 2, ..., t\}$ such that $I_i, I_j \subseteq M_r$ which is not possible. So, $\alpha(G(R)) = n$. □

Theorem 3.10. *Let R be a ring. Then $G(R)$ is complemented if and only if one of the following conditions hold.*
(i) $R \cong F_1 \times F_2$; where F_1 and F_2 are fields.
(ii) (R, M) is SPIR with $M \neq (0)$ but $M^2 = (0)$.

Proof. Suppose that $G(R)$ is complemented. Let $I \neq (0)$ be any vertex of $G(R)$. Since $G(R)$ is complemented, there exists a vertex J in $G(R)$ such that $I \perp J$. If $J \neq (0)$, then $I - J - (0) - I$ forms a triangle which is not possible. So, $J = (0)$. Now, $I \subseteq M$; for some $M \in Max(R)$. If $I \subsetneq M$, then $I - (0) - M - I$ forms a triangle which is not possible. So, $I = M$. Suppose that, $|Max(R)| \geq 3$. Let $M_1, M_2, M_3 \in Max(R)$. Let $I = M_1$. Note that $M_1M_2 \neq (0)$ and $M_1M_2 \neq M_i$; for $i \in \{1, 2\}$. Observe that $(0) - M_1 - M_1M_2 - (0)$ forms a triangle which is not possible. So, $|Max(R)| \leq 2$. Suppose that $|Max(R)| = 2$. Let $J(R) \neq (0)$. Then $(0) - J(R) - M - (0)$ forms a triangle which is not possible. So, $J(R) = (0)$. Hence, by the Chinese Remainder Theorem [5, Proposition 1.10(ii),(iii)],

$$R \cong \frac{R}{J(R)} \cong \frac{R}{M_1} \times \frac{R}{M_2} \cong F_1 \times F_2;$$

where F_1 and F_2 are fields. Let (R, M) be a local ring which is not a field. Let $x \in M \setminus (0)$. If $Rx \neq M$, then $(0) - Rx - M - (0)$ forms a triangle which is not possible. So, $Rx = M$. If $M^2 \neq (0)$ and $M^2 \neq M$, then $(0) - M - M^2 - (0)$ forms a triangle which is not possible. So, $M^2 = (0)$ or $M^2 = M$. If $M^2 = M$, then by the Nakayama's lemma [5, Proposition 2.6], $M = (0)$. So, $M^2 \neq M$. Hence, $M^2 = (0)$. Let P be any prime ideal of R. Now, $M^2 = (0) \subseteq P$ which implies that $M \subseteq P$. As M is maximal, $P = M = Rx$. Thus, (R, M) is SPIR with $M \neq (0)$ but $M^2 = (0)$. □

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SOME PROPERTIES OF SUPER-GRAPH OF $(\mathscr{C}(R))^c$ AND ITS LINE GRAPH

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برخی خواص ابرگراف $(\mathscr{C}(R))^c$ و گراف خطی آن

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فرض می‌کنیم R حلقه‌ای یکدار با $0 \neq 1$ باشد. گراف ایده‌آل هم‌بیشین R گرافی ساده و غیرجهتی است که رئوس آن مجموعه‌ی همه‌ی ایده‌آل‌های سره‌ی حلقه‌ی R هستند به‌طوری که مشمول در جیکوبسون رادیکال R نیستند و دو رأس I و J در این گراف مجاورند اگر و تنها اگر $I + J = R$. ما در این مقاله، گراف $G(R)$ که مجموعه‌ی رئوس آن برابر با مجموعه‌ی تمام ایده‌آل‌های سره‌ی R است و دو رأس I و J در این گراف مجاورند اگر و تنها اگر $I + J \neq R$، مطالعه می‌کنیم. همچنین، ما در این مقاله، برخی نتایج جالب در مورد $G(R)$ و گراف خطی آن را مورد بررسی قرار می‌دهیم.

کلمات کلیدی: $G(R)$، $SPIR$، گراف خطی.