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POLYMATROIDAL IDEALS AND LINEAR RESOLUTION

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Abstract. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and $I \subset S$ be a monomial ideal with a linear resolution. Let $\mathfrak{m} = (x_1, \dots, x_n)$ be the unique homogeneous maximal ideal and $I\mathfrak{m}$ be a polymatroidal ideal. We prove that if either $I\mathfrak{m}$ is polymatroidal with strong exchange property, or I is a monomial ideal in at most 4 variables, then I is polymatroidal. We also show that the first homological shift ideal of polymatroidal ideal is again polymatroidal.

1. Introduction

Throughout the paper, $S = K[x_1, \dots, x_n]$ denotes the polynomial ring in n indeterminates over an arbitrary field K with the unique homogeneous maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$ and $I \subset S$ is a monomial ideal of S. The unique minimal set of monomial generators of I will be denoted by $\mathrm{G}(I)$. The monomial localization of I with respect to a monomial prime ideal P is the monomial ideal $I(P)$ which is obtained from I by substituting the variables $x_i \notin P$ by 1. Observe that $I(P)$ is the unique monomial ideal with the property that $I(P)S_P = IS_P$. The monomial localization $I(P)$ can also be described as the saturation $I : (\prod_{x_i \notin P} x_i)^\infty$. When I is a squarefree monomial ideal, we see that $I(P) = I : u$ where $u = \prod_{x_i \notin P} x_i$. Note that $I(P)$ is a monomial ideal

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in $S(P)$, where $S(P)$ is the polynomial ring in the variables which generate P.

It has been observed that a monomial localization of a polymatroidal ideal is again polymatroidal ([7, Corollary 3.2]).

The author and Herzog conjectured that a monomial ideal I is polymatroidal if and only if $I(P)$ has a linear resolution for all monomial prime ideals P ([1, Conjecture 2.9]). They gave an affirmative answer to the conjecture in the following cases: 1) I is generated in degree 2; 2) I contains at least $n-1$ pure powers; 3) I is monomial ideal in at most three variables; 4) I has no embedded prime ideal and either $|\operatorname{Ass}(S/I)| \leq 3$ or height $(I) = n-1$.

Now, we consider the following statement: $(*)$ Let I be a monomial ideal with linear resolution such that $I\mathfrak{m}$ is polymatroidal. Then I is polymatroidal.

Observe that $(*)$ holds if Bandari-Herzog's conjecture is satisfied, because $I(P) = (I\mathfrak{m})(P)$ for all $P \neq \mathfrak{m}$.

In this paper, we prove the statement $(*)$ in the following cases: 1) $I\mathfrak{m}$ is polymatroidal with strong exchange property; 2) I is a monomial ideal in at most 4 variables.

Due to experimental evidence, the author, Bayati and Herzog conjectured that the homological shift ideals of a polymatroidal ideal are again polymatroidal. This conjecture is still open. There is a positive answer to the conjecture for matroidal ideals [2], and for polymatroidal ideals with strong exchange property [6]. In this paper, we prove that the first homological shift ideal of polymatroidal ideal is again polymatroidal.

2. Main results

Definition 2.1. Let $I \subset S$ be a monomial ideal. We say that I has a *d-linear resolution*, if I has the following minimal graded free resolution:

$$0 \to S^{m_t}(-(d+t)) \to \cdots \to S^{m_i}(-(d+i)) \to$$
$$S^{m_{i-1}}(-(d+(i-1)) \to \cdots \to S^{m_1}(-(d+1)) \to S^{m_0}(-d) \to I \to 0$$

Lemma 2.2. *Let $I \subset S$ be a monomial ideal with d-linear resolution and f be a homogeneous element of $I : \mathfrak{m} \setminus I$. Then* $\deg(f) = d-1$.

Proof. Let $0 \neq f \in I : \mathfrak{m} \setminus I$ be a homogeneous element of degree r. We want to show that $r = d-1$. We have the homogeneous isomorphism of degree n,

$$\begin{array}{rccl} \varphi : & (0 :_{S/I} \mathfrak{m}) & \to & H_n(x_1, \ldots, x_n; S/I) \\ & g & \mapsto & g e_1 \wedge \cdots \wedge e_n \end{array}$$

where $H_n(x_1, \dots, x_n; S/I)$ is the nth Koszul homology module of $x_1, \dots, x_n$ (see ([5, page 268]). Hence, there exists K-module isomorphism

$$(0 :_{S/I} \mathfrak{m})_r \cong H_n(x_1, \dots, x_n; S/I)_{r+n}.$$

Now, since $0 \neq f + I \in (0 :_{S/I} \mathfrak{m})_r$, we have that

$$H_n(x_1, \dots, x_n; S/I)_{r+n} \neq 0.$$

Hence, it follows by [5, Corollary A.3.5] that

$$\beta_{n,r+n}(S/I) = \dim_K H_n(x_1, \dots, x_n; S/I)_{r+n} \neq 0.$$

Therefore $\beta_{n,r+n+1}(I) = \beta_{n,r+n}(S/I) \neq 0$. Now, since I has a d-linear resolution, it follows that $r + n + 1 = n + d$, and so $r = d - 1$. □

The next result has been proven in [1, Page 760]. We provide more explanations of the proof by using Lemma 2.2.

Lemma 2.3. *Let $I \subset S$ be a monomial ideal with linear resolution. Then $I = I\mathfrak{m} : \mathfrak{m}$.*

Proof. Obviously we have $I \subseteq I\mathfrak{m} : \mathfrak{m}$. Assume that the inclusion is strict. Then there exists a homogeneous element $f \in I\mathfrak{m} : \mathfrak{m} \setminus I$ and so f is a homogeneous element of $I : \mathfrak{m} \setminus I$. Let I have a d-linear resolution. it follows by Lemma 2.2 that $\deg(f) = d - 1$. On the other hand, since $I\mathfrak{m}$ has $(d+1)$-linear resolution and $f \in I\mathfrak{m} : \mathfrak{m} \setminus I\mathfrak{m}$, it follows again by Lemma 2.2 that $\deg(f) = d$, which is a contradiction. □

Definition 2.4. Let $I \subset S$ be a monomial ideal. We say that I has *linear quotients*, if there exists an order $u_1, \dots, u_r$ of $\mathrm{G}(I)$ such that for $j = 2, \dots, r$, the minimal monomial generators of the colon ideal $(u_1, \dots, u_{j-1}) : u_j$ are variables.

Definition 2.5. Let $I \subset S$ be a monomial ideal generated in a single degree. The ideal I is *polymatroidal* if for any two elements $u, v \in \mathrm{G}(I)$ such that

$$\deg_{x_i}(u) > \deg_{x_i}(v)$$

there exists an index j with $\deg_{x_j}(u) < \deg_{x_j}(v)$ such that $x_j(u/x_i) \in I$.

In the case that the polymatroidal ideal I is squarefree, it is called matroidal.

Any polymatroidal ideal I has linear quotients ([8, Lemma 1.3]), which implies that I has a linear resolution ([3, Lemma 4.1]). We have also the product of polymatroidal ideals is again polymatroidal ([3, Theorem 5.3]). In particular, if I is a polymatroidal ideal, then $I\mathfrak{m}$ is polymatroidal.

The author and Herzog conjectured that a monomial ideal I is polymatroidal if and only if all monomial localizations of I have a linear resolution. If the conjecture is satisfied, then the following statement holds:

$(*)$ Let I be a monomial ideal with linear resolution such that $I\mathfrak{m}$ is polymatroidal. Then I is polymatroidal.

The following example shows that the linear resolution condition of the statement $(*)$ cannot be weakened.

Example 2.6. The ideal $I = (x_1^2, x_1x_2, x_3^2, x_2x_3) \subset S = K[x_1, x_2, x_3]$ is generated in a single degree, but it does not have a linear resolution. On the other hand $I\mathfrak{m}$ is polymatroidal, but I is not.

Definition 2.7. Let $I \subset S$ be a monomial ideal. We say that I satisfies the *strong exchange property* if I is generated in a single degree, and for all $u, v \in \mathrm{G}(I)$ and for all i, j with $\deg_{x_i}(u) > \deg_{x_i}(v)$ and $\deg_{x_j}(u) < \deg_{x_j}(v)$, one has $x_j(u/x_i) \in I$.

Now, we show that $(*)$ holds if $I\mathfrak{m}$ is a polymatroidal with strong exchange property.

Proposition 2.8. *Let $I \subset S$ be a monomial ideal with a linear resolution and $I\mathfrak{m}$ be polymatroidal with strong exchange property. Then I is polymatroidal with strong exchange property.*

Proof. Let $u, v \in \mathrm{G}(I)$ with $\deg_{x_i}(u) > \deg_{x_i}(v)$ and

$$\deg_{x_j}(u) < \deg_{x_j}(v).$$

So $ux_k, vx_k \in I\mathfrak{m}$ for each $k = 1, \dots, n$. Now, since

$$\deg_{x_i}(ux_k) > \deg_{x_i}(vx_k)$$

and $\deg_{x_j}(ux_k) < \deg_{x_j}(vx_k)$, it follows that $x_j(ux_k/x_i) \in I\mathfrak{m}$ for each $k = 1, \dots, n$. Hence $x_j(u/x_i)\mathfrak{m} \subseteq I\mathfrak{m}$. Since I has a linear resolution, it follows by Lemma 2.3, $x_j(u/x_i) \in I$. ☐

Lemma 2.9. *([4, Lemma 3.1]) Let $I \subset S$ be a polymatroidal ideal. Then for any monomials $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x^{b_1} \cdots x_n^{b_n}$ in $\mathrm{G}(I)$ and for each i with $a_i < b_i$, one has j with $a_j > b_j$ such that $x_i(u/x_j) \in \mathrm{G}(I)$.*

Lemma 2.10. *Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal with assumption $I = I\mathfrak{m} : \mathfrak{m}$. Let $u \in \mathrm{G}(I)$ and $I\mathfrak{m}$ be a polymatroidal ideal. If for $1 \leq i \neq j \leq n$, $(u/x_j)x_i^2 \in I\mathfrak{m}$, then $(u/x_j)x_i \in I$.*

Proof. Since $I = I\mathfrak{m} : \mathfrak{m}$, it is enough to show that $(ux_i/x_j)\mathfrak{m} \subseteq I\mathfrak{m}$. We have $(ux_i/x_j)x_j = ux_i \in I\mathfrak{m}$ and $(u/x_j)x_i^2 \in I\mathfrak{m}$. Now, let $k \neq i, j$. Then with considering Lemma 2.9 for monomials $(u/x_j)x_i^2 \in I\mathfrak{m}$ and $ux_k \in I\mathfrak{m}$, we have $(ux_i/x_j)x_k \in I\mathfrak{m}$. ☐

Finally, we are ready to prove that $(*)$ holds for monomial ideals in at most 4 variables.

Proposition 2.11. *Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal with $n \leq 4$. Let I have a linear resolution and $I\mathfrak{m}$ be polymatroidal. Then I is polymatroidal.*

Proof. We have already noted that the claim is true for $n \leq 3$. Now, let $n = 4$. Since I has a linear resolution, it follows by Lemma 2.3 that $I = I\mathfrak{m} : \mathfrak{m}$. Let $\deg_{x_1}(u) > \deg_{x_1}(v)$, so there exists an index j with $\deg_{x_j}(u) < \deg_{x_j}(v)$. For convenience, we assume that $j = 2$. So $\deg_{x_2}(u) < \deg_{x_2}(v)$. Now, we consider the following cases:

Case 1: $\deg_{x_3}(u) < \deg_{x_3}(v)$ and $\deg_{x_4}(u) < \deg_{x_4}(v)$. With considering Lemma 2.9 for ux_2 and vx_2, we have $(ux_2/x_1)x_2 \in I\mathfrak{m}$. So by Lemma 2.10, it follows that $(u/x_1)x_2 \in I$.

Case 2: $\deg_{x_3}(u) > \deg_{x_3}(v)$ and $\deg_{x_4}(u) > \deg_{x_4}(v)$. With considering exchange property between ux_2 and vx_2, we have

$$(ux_2/x_1)x_2 \in I\mathfrak{m}.$$

So Lemma 2.10, implies that $(u/x_1)x_2 \in I$.

Case 3: $\deg_{x_3}(u) < \deg_{x_3}(v)$ and $\deg_{x_4}(u) > \deg_{x_4}(v)$. With considering exchange property between ux_4 and vx_4, it follows that either $(ux_4/x_1)x_2 \in I\mathfrak{m}$ or $(ux_4/x_1)x_3 \in I\mathfrak{m}$.

- Assume $(ux_4/x_1)x_2 \in I\mathfrak{m}$. With considering Lemma 2.9 for ux_2 and vx_2, we have either $ux_2^2/x_1 \in I\mathfrak{m}$, so there is nothing to prove, or $ux_2^2/x_4 \in I\mathfrak{m}$. Now with comparing $(ux_4/x_1)x_2$ and ux_2^2/x_4, we have $ux_2^2/x_1 \in I\mathfrak{m}$, which implies that $(u/x_1)x_2 \in I$.

- Assume $(ux_4/x_1)x_3 \in I\mathfrak{m}$. With considering Lemma 2.9 for ux_3 and vx_3, we have either $ux_3^2/x_1 \in I\mathfrak{m}$, so there is nothing to prove, or $ux_3^2/x_4 \in I\mathfrak{m}$. Now with comparing $(ux_4/x_1)x_3$ and ux_3^2/x_4, we have $ux_3^2/x_1 \in I\mathfrak{m}$, which implies that $(u/x_1)x_3 \in I$.

Case 4: $\deg_{x_3}(u) > \deg_{x_3}(v)$ and $\deg_{x_4}(u) < \deg_{x_4}(v)$. This follows by a similar argument of case (3). $\square$

In the sequel, we want to show that the first homological shift ideal of polymatroidal ideal is again polymatroidal. Let $\mathbf{a} = (a_1, \dots, a_n)$ be an integer vector with $a_i \geq 0$. For a monomial ideal I, we set

$$I^{\leq \mathbf{a}} = (u \in \mathrm{G}(I) \mid \deg_{x_i}(u) \leq a_i \text{ for } i = 1, \dots, n).$$

Obviously, if I is polymatroidal, then $I^{\leq \mathbf{a}}$ is again polymatroidal.

A monomial $x_1^{a_1} \cdots x_n^{a_n}$ will be denoted by $\mathbf{x}^{\mathbf{a}}$.

Definition 2.12. Let $I \subset S$ be a monomial ideal with minimal multigraded free S-resolution

$$\mathcal{F} : 0 \to F_t \to F_{t-1} \to \cdots \to F_1 \to F_0 \to I \to 0,$$

where $F_i = \bigoplus_{j=1}^{b_i} S(-\mathbf{a}_{ij})$ for $i = 0, \ldots, t$. The vectors $\mathbf{a}_{ij}$ are called the multigraded shifts of the resolution $\mathcal{F}$. The monomial ideal

$$\mathrm{HS}_{\ i}(I) = (\mathbf{x}^{\mathbf{a}_{ij}} \mid j = 1, \ldots, b_i)$$

is called the *ith homological shift ideal* of I.

Proposition 2.13. *Let $I \subset S$ be a polymatroidal ideal. Then*

$$\mathrm{HS}_{\ 1}(I) = (I\mathfrak{m})^{\leq \mathbf{a}},$$

where $\mathbf{a} = (a_1, \ldots, a_n)$ and

$$a_i = \max\{\deg_{x_i}(u) \mid u \in \mathrm{G}\,(I)\}.$$

In particular, $\mathrm{HS}_{\ 1}(I)$ *is polymatroidal.*

Proof. Let $ux_i \in (I\mathfrak{m})^{\leq \mathbf{a}}$ such that $u \in \mathrm{G}\,(I)$. So $\deg_{x_i}(u) < a_i$. Hence there exists $v \in \mathrm{G}\,(I)$ such $\deg_{x_i}(v) > \deg_{x_i}(u)$. Now, since I is polymatroidal it follows by Lemma 2.9 that there exists an index j such that $\deg_{x_j}(v) < \deg_{x_j}(u)$ and $w = x_i(u/x_j) \in \mathrm{G}\,(I)$. Hence $x_i u - x_j w = 0$. Now, let

$$\mathrm{G}\,(I) = \{u_1, \ldots, u_t\}$$

and F be the free S-module with basis $e_1, \ldots, e_t$. Let $\varphi : F \to I$ be the S-module homomorphism with $\varphi(e_i) = u_i$ for $i = 1, \ldots, t$. Then the multi-degree of e_i is the same as that of u_i. We assume that $u = u_r$ and $w = u_s$ for $r, s \in \{1, \ldots, t\}$. So

$$\varphi(x_i e_r - x_j e_s) = x_i u - x_j w = 0,$$

hence $x_i e_r - x_j e_s \in \mathrm{Ker}\,(\varphi)$. Therefore, $x_i u \in \mathrm{HS}_{\ 1}(I)$.

Conversely, By [6, Proposition 1.3], $\mathrm{HS}_{\ 1}(I)$ is generated by all monomials of the form $x_i u$ with $u \in \mathrm{G}\,(I)$ for which there exists $j \neq i$ and $v \in \mathrm{G}\,(I)$ such that $x_i u = x_j v$. Therefore $\mathrm{HS}_{\ 1}(I) \subseteq (I\mathfrak{m})^{\leq \mathbf{a}}$. $\square$

References

1. S. Bandari and J. Herzog, Monomial localizations and polymatroidal ideals, *European J. Combin.*, **34**(4) (2013), 752–763.
2. S. Bayati, Multigraded shifts of matroidal ideals, *Arch. Math.*, **111** (2018), 239–246.
3. A. Conca and J. Herzog, Castelnuovo-Mumford regularity of products of ideals, *Collect. Math.*, **54**(2) (2003), 137–152.
4. J. Herzog and T. Hibi, Cohen-Macaulay polymatroidal ideals, *European J. Combin.*, **27**(4) (2006), 513–517.

5. J. Herzog and T. Hibi, *Monomial Ideals*, Graduate Texts in Mathematics, **260**, Springer-Verlag, 2011.
6. J. Herzog, S. Moradi, M. Rahimbeigi and G. Zhu, Homological shift ideals, *Collect. Math.*, **72** (2021), 157–174.
7. J. Herzog, A. Rauf and M. Vladoiu, The stable set of associated prime ideals of a polymatroidal ideal, *Journal of J. Algebraic Combin.*, **37**(2) (2013), 289–312.
8. J. Herzog and Y. Takayama, Resolutions by mapping cones, *Homology Homotopy Appl.*, **4**(2) (2002), 277–294.

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ایده‌آل‌های پلی‌ماترویدال و تحلیل خطی

سمیه بندری

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فرض کنید $S = K[x_1, \dots, x_n]$ یک حلقه چندجمله‌ای روی میدان K و $I \subset S$ یک ایده‌آل یک‌جمله‌ای با تحلیل خطی باشد. همچنین فرض کنید $\mathfrak{m} = (x_1, \dots, x_n)$ ایده‌آل یکتای ماکزیمال همگن و $I\mathfrak{m}$ یک ایده‌آل یک‌جمله‌ای پلی‌ماترویدال باشد. ثابت می‌کنیم که اگر $I\mathfrak{m}$ یک ایده‌آل پلی‌ماترویدال با ویژگی معاوضه‌ای قوی و یا I یک ایده‌آل یک‌جمله‌ای با حداکثر ۴ متغیر باشد، آنگاه I پلی ماترویدال است. همچنین نشان می‌دهیم که ایده‌آل شیفت اول همولوژیکی یک ایده‌آل پلی‌ماترویدال، پلی‌ماترویدال است.

کلمات کلیدی: ایده‌آل‌های پلی‌ماترویدال، موضعی سازی یک جمله‌ای، خارج قسمت‌های خطی، تحلیل خطی، ایده‌آل شیفت همولوژیکی.