Journal of Algebraic Systems
Vol. 11, No. 2, (2024), pp 155-177

ON THE DOMINATION NUMBER OF THE SUM ANNIHILATING IDEAL GRAPH OF A COMMUTATIVE RING AND ON THE DOMINATION NUMBER OF ITS COMPLEMENT

S. VISWESWARAN* AND P. SARMAN

Abstract. The rings considered in this article are commutative with identity which admit at least one non-zero annihilating ideal. Let R be a ring. Let $\mathbb{A}(R)$ denote the set of all annihilating ideals of R and let us denote $\mathbb{A}(R)\backslash\{(0)\}$ by $\mathbb{A}(R)^*$. Recall that the sum annihilating ideal graph of R, denoted by $\Omega(R)$ is an undirected graph whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I and J are adjacent in $\Omega(R)$ if and only if $I + J \in \mathbb{A}(R)$. The aim of this article is to discuss some results on the domination number of $\Omega(R)$ (respectively, $(\Omega(R))^c$), where $(\Omega(R))^c$ is the complement of $\Omega(R)$.

1. Introduction

The rings considered in this article are commutative with identity and unless otherwise specified, they are not integral domains. Beginning with the work of Beck on the coloring of a commutative ring in [7], several algebraists have introduced graphs with algebraic structures and investigated the interplay between the algebraic properties of the algebraic structures and the graph-theoretic properties of the graphs associated with them. The graphs considered in this article are undirected and simple. For a graph G, we denote

DOI: 10.22044/JAS.2022.12110.1630.
MSC(2010): Primary: 13A15; Secondary: 05C25.
Keywords: Reduced ring; Minimal prime ideal; Maximal N-prime of (0); B-prime of (0); Domination number of a graph.
Received: 14 July 2022, Accepted: 23 November 2022.
*Corresponding author.

the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. Let R be a ring. Let $Z(R)$ denote the set of all zero-divisors of R and let us denote $Z(R)\backslash\{0\}$ by $Z(R)^*$. We recall that the *zero-divisor graph* of R, denoted by $\Gamma(R)$ is an undirected graph with $V(\Gamma(R)) = Z(R)^*$ and distinct vertices x and y are adjacent in this graph if and only if $xy = 0$ [4]. For an excellent and inspiring survey on the zero-divisor graphs of commutative rings, one can refer the survey article [2].

In [3], with any commutative ring R, Anderson and Badawi have introduced and investigated an undirected graph called the *total graph* of R, denoted by $T(\Gamma(R))$ with $V(T(\Gamma(R))) = R$ and distinct vertices x and y are adjacent in this graph if and only if $x + y \in Z(R)$. Several interesting theorems illustrating the interplay between the ring-theoretic properties of R and the graph-theoretic properties of $T(\Gamma(R))$ have been proved in [3].

We recall that an ideal I of a ring R is said to be an *annihilating ideal* of R if there exists $r \in R\backslash\{0\}$ such that $Ir = (0)$ [8]. As in [8], we denote the set of all annihilating ideals of R by $\mathbb{A}(R)$ and $\mathbb{A}(R)\backslash\{(0)\}$ by $\mathbb{A}(R)^*$. It is useful to recall that the *annihilating-ideal graph* of R denoted by $\mathbb{AG}(R)$ is an undirected graph with $V(\mathbb{AG}(R)) = \mathbb{A}(R)^*$ and distinct vertices I and J are adjacent in this graph if and only if $IJ = (0)$ [8]. For several interesting and inspiring results on $\mathbb{AG}(R)$, the reader is referred to [8, 9].

Motivated by the research work done by Anderson and Badawi on the total graph of a commutative ring in [3] and the research work of Behboodi and Rakeei on the annihilating-ideal graph of a commutative ring in [8, 9], with any commutative ring R, in [19], we have introduced and investigated an undirected graph, denoted by $\Omega(R)$ such that $V(\Omega(R)) = \mathbb{A}(R)^*$ and distinct vertices I and J are adjacent in this graph if and only if $I + J \in \mathbb{A}(R)$. The graph $\Omega(R)$ is called as the *sum annihilating ideal graph* of R in [15]. Let $G = (V, E)$ be a simple graph. We recall that the *complement* of G, denoted by G^c is a graph whose vertex set is V and distinct vertices u and v are adjacent in G^c if and only if they are not adjacent in G [6, Definition 1.2.13]. We have studied some graph parameters of $(\Omega(R))^c$ in [20].

Let $G = (V, E)$ be a graph. We recall that a set $S \subseteq V$ is a *dominating set* of G if every vertex $u \in V\backslash S$ has a neighbor $v \in S$ [6, Definition 10.2.1]. We recall that a γ*-set* of G is a minimum dominating set of G, that is, a dominating set whose cardinality is minimum. A dominating set S of G is said to be *minimal* if S properly contains no dominating set of G [6, Definition 10.2.2]. We recall that the *domination number* of G is the cardinality of a minimum dominating set of G; it is denoted by $\gamma(G)$[6, Definition 10.2.3].

Let R be a ring. Several researchers have studied the dominating sets and the domination number of graphs associated with commutative rings (see, for example [1, 14, 17, 18]). Motivated by the above mentioned research work on the domination number of some well known graphs associated with commutative rings, in this article, we focus on determining the domination number of $\Omega(R)$ (respectively, the domination number of $(\Omega(R))^c$).

Before we give a brief account of results that are proved in this article, we recall some definitions, notation, and results from commutative ring theory that are often used in this article. Let R be a ring. We denote the set of all prime ideals of R by $Spec(R)$ and for an ideal I of R, we denote the set

$$\{\mathfrak{p} \in Spec(R) \mid \mathfrak{p} \supseteq I\}$$

by $V(I)$. We denote the set of all maximal ideals of R by $Max(R)$. We denote the nilradical of R by $nil(R)$. We recall that R is said to be *reduced* if $nil(R) = (0)$. Let $Min(R)$ denote the set of all minimal prime ideals of R. If $\mathfrak{p} \in Spec(R)$, then $\mathfrak{p} \supseteq \mathfrak{p}'$ for some $\mathfrak{p}' \in Min(R)$ by [16, Theorem 10]. Thus if R is a reduced ring, then it follows from [5, Proposition 1.8] that $\bigcap_{\mathfrak{p} \in Min(R)} \mathfrak{p} = (0)$. We denote the set of all proper ideals of R by $\mathbb{I}(R)$ and the set $\mathbb{I}(R)\backslash\{(0)\}$ by $\mathbb{I}(R)^*$. Let $I \in \mathbb{I}(R)$. We recall that $\mathfrak{p} \in Spec(R)$ is said to be a *maximal N-prime* of I if $\mathfrak{p}$ is maximal with respect to the property of being contained in

$$Z_R(\tfrac{R}{I}) = \{r \in R \mid rx \in I \text{ for some } x \in R\backslash I\}$$

[13]. Thus $\mathfrak{p} \in Spec(R)$ is a maximal N-prime of (0) if $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z(R)$. For convenience, let us denote the set of all maximal N-primes of (0) in R by $MNP(R)$. We denote the cardinality of a set A by $|A|$. Let $S = R\backslash Z(R)$. Then S is a multiplicatively closed subset of R. Let $x \in Z(R)$. Then $Rx \cap S = \emptyset$. Hence, we obtain from Zorn's lemma and [16, Theorem 1] that there exists $\mathfrak{p} \in MNP(R)$ such that $x \in \mathfrak{p}$. Thus if $MNP(R) = \{\mathfrak{p}_\alpha\}_{\alpha \in \Lambda}$, then it follows from the above argument that $Z(R) = \bigcup_{\alpha \in \Lambda} \mathfrak{p}_\alpha$. Hence, $Z(R)$ is an ideal of R if and only if $|MNP(R)| = 1$. Let $I \in \mathbb{I}(R)$. We recall that $\mathfrak{p} \in Spec(R)$ is said to be an *associated prime of I in the sense of Bourbaki* if $\mathfrak{p} = (I :_R x)$ for some $x \in R$ [12]. In such a case, we say that $\mathfrak{p}$ is a B-prime of I. Let $\mathfrak{p} \in MNP(R)$. As $((0) :_R x) \subseteq Z(R)$ for any $x \in R\backslash\{0\}$, it follows that $\mathfrak{p} \in \mathbb{A}(R)$ if and only if $\mathfrak{p}$ is a B-prime of (0) in R. For an ideal I of a ring R, the *annihilator* of I in R, denoted by $Ann_R(I)$ is defined as $Ann_R(I) = \{r \in R \mid Ir = (0)\}$. Note that $Ann_R(I) = ((0) :_R I)$.

A principal ideal ring R is said to be a *special principal ideal ring* (SPIR) if R has a unique prime ideal. If $\mathfrak{m}$ is the unique prime ideal of R, then it follows from [5, Proposition 1.8] that $\mathfrak{m} = nil(R)$ and as $\mathfrak{m}$ is principal, we get that $\mathfrak{m}$ is nilpotent. If R is an SPIR with $Spec(R) = \{\mathfrak{m}\}$, then we denote it by mentioning that $(R, \mathfrak{m})$ is an SPIR. Let R be a ring such that $Max(R) = \{\mathfrak{m}\}$. Suppose that $\mathfrak{m} = Rm$ is principal and nilpotent. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. Then it follows from the proof of $(iii) \Rightarrow (i)$ of [5, Proposition 8.8] that

$$\mathbb{I}(R)^* = \{\mathfrak{m}^i = Rm^i \mid i \in \{1, \ldots, n-1\}\}$$

and hence, $(R, \mathfrak{m})$ is an SPIR.

This article consists of three sections including the introduction. Let R be a ring. In Section 2 of this article, we discuss some results on the domination number of $\Omega(R)$. For a connected graph G, we denote the radius of G by $r(G)$. If $|\mathbb{A}(R)^*| \geq 2$, then it is proved in Proposition 2.2 that the following statements are equivalent: (1) $\gamma(\Omega(R)) = 1$; (2) R is not reduced; and (3) $\Omega(R)$ is connected and $r(\Omega(R)) = 1$. If R is a reduced ring, then it is shown in Proposition 2.4 that $\gamma(\Omega(R)) = 2$. Let R be a reduced ring such that $\Omega(R)$ is connected. Then it is proved in Proposition 2.6 that $\gamma(\Omega(R)) = diam(\Omega(R)) = r(\Omega(R)) = 2$.

Let R be a ring. In Section 3 of this article, we discuss some results on the domination number of $(\Omega(R))^c$. If $|\mathbb{A}(R)^*| \geq 2$, then it is proved in Theorem 3.1 that the following statements are equivalent: (1) $\gamma((\Omega(R))^c) = 1$; (2) $R \cong R_1 \times R_2$ as rings, where R_i is an integral domain for each $i \in \{1, 2\}$ with R_i is a field for at least one $i \in \{1, 2\}$; and (3) $(\Omega(R))^c$ is a star graph. If R is not reduced with $|\mathbb{A}(R)^*| \geq 2$ and $Z(R)$ is an ideal of R, then it is shown in Proposition 3.6 that the following statements are equivalent: (1) $(\Omega(R))^c$ admits a finite dominating set; and (2) Either $(R, \mathfrak{m})$ is an SPIR with $\mathfrak{m}^n = (0)$ but $\mathfrak{m}^{n-1} \neq (0)$ for some $n \geq 3$ or $(R, \mathfrak{m})$ is a finite local ring such that $\mathfrak{m}$ is not principal. Let R be a non-reduced ring with $|MNP(R)| \geq 2$. We are not able to characterize R such that $(\Omega(R))^c$ admits a finite dominating set. For a reduced ring R, it is proved in Theorem 3.10 that the following statements are equivalent: (1) $\gamma((\Omega(R))^c) = 2$; (2) R has exactly two minimal prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ such that $\mathfrak{p}_i$ is not a simple R-module for each $i \in \{1, 2\}$; and (3) $(\Omega(R))^c$ is a complete bipartite graph with vertex partition V_1 and V_2 such that $|V_i| \geq 2$ for each $i \in \{1, 2\}$. Let R be a reduced ring such that $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$. If $\gamma((\Omega(R))^c) \geq 2$, then it is shown in Theorem 3.13 that $\gamma((\Omega(R))^c) = |Min(R)|$. It is remarked in the paragraph which appears just preceding the statement of Lemma 3.15 that for a von Neumann regular ring R, $\gamma((\Omega(R))^c) = 1$ if and only if $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$.

It is verified in Corollary 3.16 that for a von Neumann regular ring R, $\gamma((\Omega(R))^c) \neq 2$. Let $n \in \mathbb{N}$ be such that $n \geq 3$. For a von Neumann regular ring R, it is proved in Theorem 3.17 that the following statements are equivalent: (1) $\gamma((\Omega(R))^c) = n$; (2) $|Min(R)| = n$; and (3) $R \cong F_1 \times F_2 \times F_3 \times \cdots \times F_n$ as rings, where F_i is a field for each $i \in \{1, 2, 3, \ldots, n\}$.

Whenever A, B are sets with A is a subset of B and $A \neq B$, we denote it by either $A \subset B$ or $B \supset A$. The Krull dimension of a ring is referred to as the dimension of R and is denoted by $dimR$. A ring R is said to be *quasi-local* if $|Max(R)| = 1$. A Noetherian quasi-local ring is referred to as a *local* ring.

2. On the domination number of $\Omega(R)$

As mentioned in the introduction, unless otherwise specified, the rings considered in this article are commutative with identity which are not integral domains. Let R be a ring. The aim of this section is to discuss some results on the domination number of $\Omega(R)$.

Let $G = (V, E)$ be a graph. Let $a, b \in V$ with $a \neq b$. Suppose that there exists a path in G between a and b. We recall that the *distance between* a *and* b denoted by $d(a, b)$ is defined as the length of a shortest path in G between a and b [6, Definition 1.5.5]. We define $d(a, b) = \infty$ if there exists no path in G between a and b. We define $d(a, a) = 0$. We recall that G is said to be *connected* if for any distinct $a, b \in V$, there exists at least one path in G between a and b [6, Definition 1.5.4]. A simple graph G is said to be *complete* if every pair of distinct vertices of G are adjacent in G [6, Definition 1.2.11]. Let $G = (V, E)$ be a connected graph. Then the *diameter* of G, denoted by $diam(G)$ is defined as $diam(G) = max\{d(a, b) \mid a, b \in V\}$ [6, Definition 4.3.1(1)]. Let $v \in V$. We recall that the *eccentricity* of v, denoted by $e(v)$ is defined as $e(v) = max\{d(v, w) \mid w \in V\}$ [6, Definition 4.3.1(2)] and the *radius* of G, denoted by $r(G)$ is defined as $r(G) = min\{e(v) \mid v \in V\}$ [6, Definition 4.3.1(3)].

Lemma 2.1. *Let $G = (V, E)$ be a graph with $|V| \geq 2$. The following statements are equivalent:*

(1) *$\gamma(G) = 1$.*
(2) *G is connected and $r(G) = 1$.*

Proof. This proof of this lemma is quite easy and so, we omit its proof. □

Proposition 2.2. *Let R be a ring with $|\mathbb{A}(R)^*| \geq 2$. The following statements are equivalent:*

(1) $\gamma(\Omega(R)) = 1$.
(2) *R is not reduced.*
(3) $\Omega(R)$ *is connected and* $r(\Omega(R)) = 1$.

Proof. (1) $\Rightarrow$ (2) Assume that $\gamma(\Omega(R)) = 1$. Let $I \in \mathbb{A}(R)^*$ be such that $\{I\}$ is a dominating set of $\Omega(R)$. If $I^2 = (0)$, then it is clear that R is not reduced. Suppose that $I^2 \neq (0)$. Let J be given by $J = ((0) :_R I)$. Note that $J \in \mathbb{A}(R)^*$ and $J \neq I$ and so, J and I are adjacent in $\Omega(R)$. Therefore, $I + J \in \mathbb{A}(R)$. Let $r \in R\backslash\{0\}$ be such that $(I + J)r = (0)$. This implies that $r \in J = ((0) :_R I)$ and $Jr = (0)$. Hence, $r \in R\backslash\{0\}$ is such that $r^2 = 0$. This shows that R is not reduced.

(2) $\Rightarrow$ (1) Assume that R is not reduced. Let $a \in R\backslash\{0\}$ be such that $a^2 = 0$. Let us denote the ideal Ra by A. It is clear that $A \in \mathbb{A}(R)^*$. By hypothesis, $|\mathbb{A}(R)^*| \geq 2$. Let $B \in \mathbb{A}(R)^*$ be such that $B \neq A$. Since A is a nilpotent ideal of R and $B \in \mathbb{A}(R)^*$, we obtain from [9, Lemma 1.5] that $A + B \in \mathbb{A}(R)$ and so, A and B are adjacent in $\Omega(R)$. Hence, $\{A\}$ is a dominating set of $\Omega(R)$ and so, $\gamma(\Omega(R)) = 1$.

(1) $\Leftrightarrow$ (3) This follows from (1) $\Leftrightarrow$ (2) of Lemma 2.1. □

Lemma 2.3. *Let* $G_1, \ldots, G_k$ *be the connected components of a graph* G. *Then* $\gamma(G) = \sum_{i=1}^{k} \gamma(G_i)$.

Proof. This lemma is well known and hence, we omit its proof. □

Proposition 2.4. *Let* R *be a reduced ring. Then* $\gamma(\Omega(R)) = 2$.

Proof. Since R is a reduced ring, it follows that $\bigcap_{\mathfrak{p} \in Min(R)} \mathfrak{p} = (0)$. As R is not an integral domain, we get that $|Min(R)| \geq 2$. We consider the following cases.

Case(1): $|Min(R)| = 2$.

Let $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. Observe that $\bigcap_{i=1}^{2} \mathfrak{p}_i = (0)$, $Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$, and $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. Let $i \in \{1, 2\}$ and let $V_i = \{I \in \mathbb{A}(R)^* \mid I \subseteq \mathfrak{p}_i\}$. It follows from the proof of [19, Lemma 3.1] that $\Omega(R)$ is not connected and it has exactly two components G_1, G_2, where for each $i \in \{1, 2\}$, G_i is the subgraph of $\Omega(R)$ induced by V_i and it is complete. Therefore, $\gamma(G_i) = 1$ for each $i \in \{1, 2\}$ and so, we obtain from Lemma 2.3 that $\gamma(\Omega(R)) = \sum_{i=1}^{2} \gamma(G_i) = 2$.

Case(2): $|Min(R)| \geq 3$.

In this case, we know from [20, Proposition 2.18] that $diam((\Omega(R))^c) = 3$. Hence, there exists $I, J \in \mathbb{A}(R)^*$ such that $d(I, J) = 3$ in $(\Omega(R))^c$. Let $A \in \mathbb{A}(R)^*\backslash\{I, J\}$. It follows from $d(I, J) = 3$ in $(\Omega(R))^c$ that either I and A are not adjacent in $(\Omega(R))^c$ or J and A are not adjacent in $(\Omega(R))^c$. Hence, either A and I are adjacent in $\Omega(R)$ or A and J are adjacent in $\Omega(R)$. This shows that $\{I, J\}$ is a dominating set of $\Omega(R)$. Therefore, $\gamma(\Omega(R)) \leq 2$. Since R is reduced, it follows from $(1) \Rightarrow (2)$ of Proposition 2.2 that $\gamma(\Omega(R)) \geq 2$. Hence, $\gamma(\Omega(R)) = 2$.

This proves that $\gamma(\Omega(R)) = 2$. $\square$

Let R be a reduced ring. If $\Omega(R)$ is connected, then we prove in Proposition 2.6 that $\gamma(\Omega(R)) = diam(\Omega(R)) = r(\Omega(R)) = 2$. We use Lemma 2.5 in the proof of Proposition 2.6.

Lemma 2.5. *Let R be a reduced ring such that $\Omega(R)$ is connected. Then $e(I) \geq 2$ in $\Omega(R)$ for each $I \in \mathbb{A}(R)^*$.*

Proof. Let $I \in \mathbb{A}(R)^*$. Let $J = ((0) :_R I)$. It is clear that $J \in \mathbb{A}(R)^*$. Since R is reduced by hypothesis, $I^2 \neq (0)$ and so, $I \neq J$. Note that $I + J \notin \mathbb{A}(R)$ by [20, Lemma 2.3]. Therefore, $d(I, J) \geq 2$ in $\Omega(R)$ and so, $e(I) \geq 2$ in $\Omega(R)$. $\square$

Proposition 2.6. *Let R be a reduced ring such that $\Omega(R)$ is connected. Then $\gamma(\Omega(R)) = diam(\Omega(R)) = r(\Omega(R)) = 2$.*

Proof. Assume that R is reduced and $\Omega(R)$ is connected. Either $|MNP(R)| = 1$ or $|MNP(R)| \geq 2$. Suppose that $MNP(R) = \{\mathfrak{p}\}$. Then $Z(R) = \mathfrak{p}$ and as R is reduced, it follows that $\mathfrak{p}$ is not a B-prime of (0) in R. Hence, we obtain from [19, Lemma 2.2(ii)] that $diam(\Omega(R)) \leq 2$. Suppose that $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. Then it follows from [19, Lemma 3.1] that $\bigcap_{i=1}^{2} \mathfrak{p}_i \neq (0)$ and it follows from [19, Lemmas 3.3 and 3.4] that $diam(\Omega(R)) = 2$. If $|MNP(R)| \geq 3$, then $diam(\Omega(R)) = 2$ by [19, Lemma 4.1]. Note that $r(\Omega(R)) \geq 2$ by Lemma 2.5 and $\gamma(\Omega(R)) = 2$ by Proposition 2.4. Therefore,

$$\gamma(\Omega(R)) = diam(\Omega(R)) = r(\Omega(R)) = 2.$$

$\square$

3. On the domination number of $(\Omega(R))^c$

Let R be a ring which is not an integral domain. The aim of this section is to discuss some results on the domination number of $(\Omega(R))^c$. In Theorem 3.1, we characterize rings R with $|\mathbb{A}(R)^*| \geq 2$ such that $\gamma((\Omega(R))^c) = 1$.

We recall that a graph $G = (V, E)$ is said to be *bipartite* if the vertex set V can be partitioned into two non-empty subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X, Y) is called the *bipartition* of the bipartite graph G. We denote the bipartite graph G with bipartition (X, Y) by $G(X, Y)$. A simple bipartite graph $G(X, Y)$ is said to be *complete* if each vertex of X is adjacent to all the vertices of Y. A complete bipartite graph $G(X, Y)$ is said to be a *star* if either $|X| = 1$ or $|Y| = 1$ [6, Definition 1.2.12]. Let $G(X, Y)$ be a complete bipartite graph. If $G(X, Y)$ is a star, then $\gamma(G(X, Y)) = 1$. If $G(X, Y)$ is not a star, then $|X| \geq 2$ and $|Y| \geq 2$ and in such a case, $\gamma(G(X, Y)) = 2$.

Theorem 3.1. *Let R be a ring such that $|\mathbb{A}(R)^*| \geq 2$. The following statements are equivalent:*

(1) $\gamma((\Omega(R))^c) = 1$.
(2) $R \cong R_1 \times R_2$ *as rings, where R_i is an integral domain for each $i \in \{1, 2\}$ with R_i is a field for at least one $i \in \{1, 2\}$.*
(3) $(\Omega(R))^c$ *is a star graph.*

Proof. (1) $\Rightarrow$ (2) Assume that $\gamma((\Omega(R))^c) = 1$. It follows from (1) $\Rightarrow$ (2) of Lemma 2.1 that $(\Omega(R))^c$ is connected, $r((\Omega(R))^c) = 1$, and so, $diam((\Omega(R))^c) \leq 2$. As $|\mathbb{A}(R)^*| \geq 2$ by hypothesis and $(\Omega(R))^c$ is connected, it follows that R is reduced by [20, Lemma 2.1]. We claim that $|Min(R)| = 2$. From $diam((\Omega(R))^c) \leq 2$, it follows that $|Min(R)| \leq 2$ by [20, Proposition 2.18]. As R is not an integral domain and $\bigcap_{\mathfrak{p} \in Min(R)} \mathfrak{p} = (0)$, we obtain that $|Min(R)| \geq 2$. Therefore, $|Min(R)| = 2$. In such a case, we know from the proof of $(ii) \Rightarrow (i)$ of [20, Proposition 2.10] that $(\Omega(R))^c$ is a complete bipartite graph with vertex partition $\mathbb{A}(R)^* = V_1 \cup V_2$, where

$$V_i = \{A \in \mathbb{A}(R)^* \mid A \subseteq \mathfrak{p}_i\}$$

for each $i \in \{1, 2\}$. From the assumption $\gamma((\Omega(R))^c) = 1$, it follows that $(\Omega(R))^c$ is a star graph. If $|\mathbb{A}(R)^*| = 2$, then we get that $(\Omega(R))^c$ is complete. Hence, by $(i) \Rightarrow (ii)$ of [20, Proposition 2.11], we obtain that $R \cong K_1 \times K_2$ as rings, where K_i is a field for each $i \in \{1, 2\}$. If $|\mathbb{A}(R)^*| \geq 3$, then we obtain from $(i) \Rightarrow (ii)$ of [20, Proposition 2.12] that $R \cong D \times F$ as rings, where F is a field and D is an integral domain which is not a field. Therefore, $R \cong R_1 \times R_2$ as rings, where R_i is an integral domain for each $i \in \{1, 2\}$ with R_i is a field for at least one $i \in \{1, 2\}$.

(2) $\Rightarrow$ (3) Assume that $R \cong R_1 \times R_2$ as rings, where R_i is an integral domain for each $i \in \{1, 2\}$ with R_i is a field for at least one $i \in \{1, 2\}$. If both R_1 and R_2 are fields, then by $(ii) \Rightarrow (i)$ of [20, Proposition 2.11], we get that $(\Omega(R))^c$ is a complete graph with two vertices and hence, it is a star graph. Suppose that exactly one between R_1 and R_2 is a field. Then it is clear that $|\mathbb{A}(R)^*| \geq 3$ and by $(ii) \Rightarrow (i)$ of [20, Proposition 2.12], we obtain that $(\Omega(R))^c$ is a star graph.

(3) $\Rightarrow$ (1) This is clear. $\square$

Let $n \geq 3$. In Proposition 3.2, we verify that there are non-reduced rings R such that $\gamma((\Omega(R))^c) = n - 1$.

Proposition 3.2. *Let $(R, \mathfrak{m})$ be an SPIR and let $n \geq 3$ be least with the property that $\mathfrak{m}^n = (0)$. Then $\gamma((\Omega(R))^c) = n - 1$.*

Proof. It is noted in Section 1 that

$$\mathbb{I}(R)^* = \{\mathfrak{m}^i \mid i \in \{1, 2, \ldots, n-1\}\}.$$

It is clear that $\mathbb{I}(R)^* = \mathbb{A}(R)^*$. Note that $\Omega(R)$ is complete by [19, Example 2.4] and so, $(\Omega(R))^c$ has no edges. Therefore, $\mathbb{A}(R)^*$ is the only dominating set of $(\Omega(R))^c$. Hence, $\gamma((\Omega(R))^c) = |\mathbb{A}(R)^*| = n - 1$. It is clear that R is not reduced. $\square$

Let $T = K[X]$ be the polynomial ring in one variable X over a field K. Let $n \geq 3$. Let $I = TX^n$ and let $R = \frac{T}{I}$. Then $(R, \mathfrak{m} = \frac{TX}{I})$ is an SPIR and n is least with the property that $\mathfrak{m}^n = (0 + I)$. Hence, it follows from Proposition 3.2 that $\gamma((\Omega(R))^c) = n - 1$.

Let M be an unitary module over a ring R. A submodule N of M is said to be a *simple R-module* if $N \neq (0)$ and there is no non-zero submodule W of M such that $W \subset N$. If N is simple, then it is clear that $((0) :_R N) \in Max(R)$.

Let R be a non-reduced ring with $|\mathbb{A}(R)^*| \geq 2$ and $Z(R)$ is an ideal of R. In Proposition 3.6, we characterize R such that $(\Omega(R))^c$ admits a finite dominating set. We use Corollary 3.4 in the proof of Proposition 3.6 and Lemma 3.3 is used in the proof of Corollary 3.4.

Lemma 3.3. *Let R be a non-reduced ring with $|\mathbb{A}(R)^*| \geq 2$. If $(\Omega(R))^c$ admits a finite dominating set, then there exists $a \in nil(R) \backslash \{0\}$ such that $((0) :_R Ra) \in Max(R)$.*

Proof. Assume that R is not reduced, $|\mathbb{A}(R)^*| \geq 2$, and $(\Omega(R))^c$ admits a finite dominating set. Let $\gamma((\Omega(R))^c) = m$. Since R is not reduced, we obtain from $(1) \Rightarrow (2)$ of Theorem 3.1 that $m \geq 2$. Let $D \subseteq \mathbb{A}(R)^*$ be such that D is a dominating set of $(\Omega(R))^c$ with $|D| = m$. Let I be a non-zero nilpotent ideal of R. Then by [9, Lemma 1.5], $I + J \in \mathbb{A}(R)$ for any $J \in \mathbb{A}(R)^*$. If $I \notin D$, then as D being a dominating set of $(\Omega(R))^c$, we get that there exists $A \in D$ such that I and A are adjacent in $(\Omega(R))^c$. Hence, $I + A \notin \mathbb{A}(R)$. This is impossible and so, $I \in D$. This shows that $I \in D$ for any non-zero nilpotent ideal I of R. From $|D| < \infty$, it follows that R can admit only a finite number of nilpotent ideals. Hence, $nil(R)$ is necessarily finitely generated and so,

$$|\{A \mid A \text{ is an ideal of } R \text{ with } A \subseteq nil(R)\}| < \infty.$$

Therefore, it is possible to find an ideal I of R such that $I \subseteq nil(R)$ and I is a simple R-module. Let $a \in I\backslash\{0\}$. Then $I = Ra$. It is clear that $a \in nil(R)\backslash\{0\}$ and since Ra is a simple R-module, it follows that $((0) :_R Ra) \in Max(R)$. $\square$

Corollary 3.4. *Let R be a non-reduced ring with $|\mathbb{A}(R)^*| \geq 2$. Suppose that $Z(R)$ is an ideal of R. If $(\Omega(R))^c$ admits a finite dominating set, then $(\Omega(R))^c$ has no edges.*

Proof. By hypothesis, $nil(R) \neq (0)$, $|\mathbb{A}(R)^*| \geq 2$ and $Z(R)$ is an ideal of R. Assume that $(\Omega(R))^c$ admits a finite dominating set. Then there exists $a \in nil(R)\backslash\{0\}$ such that $((0) :_R Ra) \in Max(R)$ by Lemma 3.3. Let us denote $((0) :_R Ra)$ by $\mathfrak{m}$. From $\mathfrak{m}a = (0)$, we get that $\mathfrak{m} \subseteq Z(R)$. As $Z(R)$ is an ideal of R, it follows that $\mathfrak{m} = Z(R) = ((0) :_R a)$. Therefore, $MNP(R) = \{\mathfrak{m}\}$ and $\mathfrak{m}$ is a B-prime of (0) in R. In such a case, we know from the proof of [19, Lemma 2.3, Case 1] that $\Omega(R)$ is complete. Therefore, $(\Omega(R))^c$ has no edges. $\square$

We provide Example 3.5 to illustrate that the conclusion of Corollary 3.4 can fail to hold if the assumption $Z(R)$ is an ideal of R is omitted in the statement of Corollary 3.4.. For any $n \in \mathbb{N}\backslash\{1\}$, we denote the ring of integers modulo n by $\mathbb{Z}_n$.

Example 3.5. Let $R = \mathbb{Z}_4 \times F$, where F is a field. Then R is not reduced, $\gamma((\Omega(R))^c) = 2$ but $|E((\Omega(R))^c)| = 2$.

Proof. Since $(2, 0)^2 = (0, 0)$, it follows that R is not reduced. Observe that

$$\mathbb{I}(R)^* = \mathbb{A}(R)^* = \{(0) \times F, 2\mathbb{Z}_4 \times (0), 2\mathbb{Z}_4 \times F, \mathbb{Z}_4 \times (0)\}.$$

As R is not reduced, it follows from (1) $\Rightarrow$ (2) of Theorem 3.1 that $\gamma((\Omega(R))^c) \geq 2$. (This can be verified directly in this example.) Let $D = \{2\mathbb{Z}_4 \times (0), \mathbb{Z}_4 \times (0)\}$. Observe that

$$((0) \times F) + (\mathbb{Z}_4 \times (0)) = \mathbb{Z}_4 \times F \notin \mathbb{A}(R)$$

and so, $(0) \times F$ is adjacent to $\mathbb{Z}_4 \times (0)$ in $(\Omega(R))^c$. From

$$(2\mathbb{Z}_4 \times F) + (\mathbb{Z}_4 \times (0)) = \mathbb{Z}_4 \times F \notin \mathbb{A}(R),$$

it follows that $2\mathbb{Z}_4 \times F$ and $\mathbb{Z}_4 \times (0)$ are adjacent in $(\Omega(R))^c$. This shows that D is a dominating set of $(\Omega(R))^c$ and as $|D| = 2$, we obtain that $\gamma((\Omega(R))^c) \leq 2$. Hence, $\gamma((\Omega(R))^c) = 2$. It is not hard to verify that

$$E((\Omega(R))^c) = \{\mathbb{Z}_4 \times (0) - (0) \times F, \mathbb{Z}_4 \times (0) - 2\mathbb{Z}_4 \times F\}$$

and so, $|E((\Omega(R))^c)| = 2$. $\square$

Proposition 3.6. *Let R be a non-reduced ring with $|\mathbb{A}(R)^*| \geq 2$ and $Z(R)$ is an ideal of R. The following statements are equivalent:*

(1) *$(\Omega(R))^c$ admits a finite dominating set.*
(2) *Either $(R, \mathfrak{m})$ is an SPIR with $\mathfrak{m}^n = (0)$ but $\mathfrak{m}^{n-1} \neq (0)$ for some $n \geq 3$ or $(R, \mathfrak{m})$ is a finite local ring such that $\mathfrak{m}$ is not principal.*

Proof. By hypothesis, the ring R is not reduced, $|\mathbb{A}(R)^*| \geq 2$, and $Z(R)$ is an ideal of R.

(1) $\Rightarrow$ (2) Assume that $(\Omega(R))^c$ admits a finite dominating set. It follows from Corollary 3.4 that $(\Omega(R))^c$ has no edges. Hence, $\mathbb{A}(R)^*$ is the only dominating set of $(\Omega(R))^c$ and so, it follows that $|\mathbb{A}(R)^*| < \infty$. Therefore, we obtain from [8, Theorem 1.1] that R is Artinian. Hence, R is Noetherian and $dim R = 0$ by [5, Theorem 8.5]. It is well known that in an Artinian ring T, $\mathbb{I}(T)^* = \mathbb{A}(T)^*$. We claim that R is local. Suppose that $|Max(R)| \geq 2$. Let $\mathfrak{m}_1, \mathfrak{m}_2$ be distinct members of $Max(R)$. Then $\mathfrak{m}_i \in \mathbb{A}(R)^*$ for each $i \in \{1, 2\}$ and $\mathfrak{m}_1 + \mathfrak{m}_2 = R \notin \mathbb{A}(R)$. This implies that $\mathfrak{m}_1 - \mathfrak{m}_2$ is an edge of $(\Omega(R))^c$ and this contradicts the fact that $(\Omega(R))^c$ has no edges. Therefore, R is local. Let $\mathfrak{m}$ denote the unique maximal ideal of R. Note that $\mathfrak{m}$ is nilpotent by [5, Corollary 8.2 and Proposition 8.4]. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. Either $\mathfrak{m}$ is principal or $\mathfrak{m}$ is not principal. Suppose that $\mathfrak{m}$ is principal. In such a case, it is already noted in Section 1 that

$$\mathbb{I}(R)^* = \{\mathfrak{m}^i \mid i \in \{1, \ldots, n-1\}\}$$

and $(R, \mathfrak{m})$ is an SPIR. As $|\mathbb{A}(R)^*| \geq 2$, it follows that $n - 1 \geq 2$ and so, $n \geq 3$. Suppose that $\mathfrak{m}$ is not principal. Hence, it follows from [5, Proposition 2.8] that $dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{\mathfrak{m}^2}) \geq 2$. Let $m_1, m_2 \in \mathfrak{m}$ be such that $\{m_i + \mathfrak{m}^2 \mid i \in \{1, 2\}\}$ is linearly independent over $\frac{R}{\mathfrak{m}}$. If $r, s \in R \backslash \mathfrak{m}$ are such that $r + \mathfrak{m} \neq s + \mathfrak{m}$, then $R(m_1 + rm_2) \neq R(m_1 + sm_2)$. Since $|\mathbb{A}(R)^*| < \infty$, we get that $|\frac{R}{\mathfrak{m}}| < \infty$. Observe that for each $i \in \{1, \ldots, n-1\}$, $\frac{\mathfrak{m}^{i-1}}{\mathfrak{m}^i}$ (with $\mathfrak{m}^0 = R$) is a finite-dimensional vector space over the field $\frac{R}{\mathfrak{m}}$ and so, $|\frac{\mathfrak{m}^{i-1}}{\mathfrak{m}^i}| < \infty$. Hence, from $|R| = \prod_{i=1}^{n} |\frac{\mathfrak{m}^{i-1}}{\mathfrak{m}^i}|$, it follows that R is finite.

(2) $\Rightarrow$ (1) Suppose that $(R, \mathfrak{m})$ is an SPIR such that $\mathfrak{m}^n = (0)$ but $\mathfrak{m}^{n-1} \neq (0)$ for some $n \geq 3$. Then $\gamma((\Omega(R))^c) = n - 1$ by Proposition 3.2. Suppose that $(R, \mathfrak{m})$ is a finite local ring with $\mathfrak{m}$ is not principal. Then it is clear that $|\mathbb{A}(R)^*| < \infty$ and note that $\mathbb{A}(R)^*$ is the only dominating set of $(\Omega(R))^c$. $\square$

Let R be a non-reduced ring such that $Z(R)$ is not an ideal of R. We are not able to characterize R such that $(\Omega(R))^c$ admits a finite dominating set.

Let R be a reduced ring. In Theorem 3.10, we characterize R such that $\gamma((\Omega(R))^c) = 2$. First, we state and prove some results which are needed for the proof of Theorem 3.10.

Proposition 3.7. *Let $n \geq 3$ and let R_i be a reduced ring for each $i \in \{1, 2, 3, \ldots, n\}$. Let $R = R_1 \times R_2 \times R_3 \times \cdots \times R_n$. Then*

$$\gamma((\Omega(R))^c) \geq n.$$

Proof. Let $i \in \{1, 2, 3, \ldots, n\}$. Let us denote $\{1, 2, 3, \ldots, n\}\backslash\{i\}$ by W_i. Let the element of R whose i-th coordinate equals 1 and j-th coordinate equals 0 for all $j \in W_i$ by e_i. Let us denote $(1, 1, 1, \ldots, 1) - e_i$ by f_i. Let $A_i = \{Re_i\}$ and let

$$B_i = \{I_1 \times I_2 \times I_3 \times \cdots \times I_n \mid I_i \in \mathbb{A}(R_i), I_j \notin \mathbb{A}(R_j) \text{ for all } j \in W_i\}.$$

It is clear that $Rf_i \in B_i$ and so, $B_i \neq \emptyset$. Let D be any dominating set of $(\Omega(R))^c$. We claim that $D \cap (A_i \cup B_i) \neq \emptyset$. If $Re_i \in D$, then it is clear that $D \cap (A_i \cup B_i) \neq \emptyset$. Suppose that $Re_i \notin D$. Since D is a dominating set of $(\Omega(R))^c$, there exists $I \in D$ such that Re_i and I are adjacent in $(\Omega(R))^c$. Hence, $Re_i + I \notin \mathbb{A}(R)$. Let $I = I_1 \times I_2 \times I_3 \times \cdots \times I_n$, where I_k is an ideal of R_k for each $k \in \{1, 2, 3, \ldots, n\}$. From $Re_i + I \notin \mathbb{A}(R)$, we get that $I_j \notin \mathbb{A}(R_j)$ for all $j \in W_i$. As $I \in \mathbb{A}(R)$, it follows that $I_i \in \mathbb{A}(R_i)$. Thus $I \in B_i$ and so, $I \in D \cap B_i$. This proves that $D \cap (A_i \cup B_i) \neq \emptyset$.

Let $i, j \in \{1, 2, 3, \ldots, n\}$ with $i \neq j$. We next verify that

$$(A_i \cup B_i) \cap (A_j \cup B_j) = \emptyset.$$

As $Re_i \neq Re_j$, it follows that $A_i \cap A_j = \emptyset$. Since $n \geq 3$, it is possible to find $k \in \{1, 2, 3, \ldots, n\}\backslash\{i, j\}$. If $J = J_1 \times J_2 \times J_3 \times \cdots \times J_n \in B_j$, then $J_k \notin \mathbb{A}(R_k)$. Hence, $Re_i \notin B_j$. Therefore, $A_i \cap B_j = \emptyset$. If $I = I_1 \times I_2 \times I_3 \times \cdots \times I_n \in B_i$, then $I_k \notin \mathbb{A}(R_k)$ and so, $B_i \cap A_j = \emptyset$. If $I = I_1 \times I_2 \times I_3 \times \cdots \times I_n \in B_i$, then $I_j \notin \mathbb{A}(R_j)$, whereas if

$$J = J_1 \times J_2 \times J_3 \times \cdots \times J_n \in B_j,$$

then $J_j \in \mathbb{A}(R_j)$. Hence, $B_i \cap B_j = \emptyset$. This shows that

$$(A_i \cup B_i) \cap (A_j \cup B_j) = \emptyset$$

and so, $(D \cap (A_i \cup B_i)) \cap (D \cap (A_j \cup B_j)) = \emptyset$ for all distinct $i, j \in \{1, 2, 3, \ldots, n\}$. As $|D \cap (A_i \cup B_i)| \geq 1$ for each $i \in \{1, 2, 3, \ldots, n\}$, it follows that

$$|D| \geq |\textstyle\bigcup_{i=1}^{n}(D \cap (A_i \cup B_i))| = \sum_{i=1}^{n} |D \cap (A_i \cup B_i)| \geq n.$$

Hence, $\gamma((\Omega(R))^c) \geq n$. $\square$

Lemma 3.8. *Let $I_1, I_2 \in \mathbb{A}(R)^*$ be such that $I_1 \neq I_2$. If I_1 and I_2 are comparable under inclusion, then I_1 and I_2 are not adjacent in $(\Omega(R))^c$.*

Proof. Let $I_1, I_2 \in \mathbb{A}(R)^*$ be distinct. Assume that I_1 and I_2 are comparable under inclusion. As $I_1 + I_2$ is either I_1 or I_2, we get that $I_1 + I_2 \in \mathbb{A}(R)$ and so, I_1 and I_2 are not adjacent in $(\Omega(R))^c$. $\square$

An element e of a ring R is said to be *idempotent* if $e = e^2$. An idempotent element e of R is said to be *non-trivial* if $e \notin \{0, 1\}$.

Proposition 3.9. *Let R be a reduced ring. Suppose that $\gamma((\Omega(R))^c) = 2$. Let $D = \{I_1, I_2\}$ be a dominating set of $(\Omega(R))^c$. Then the following statements hold:*

(1) *I_1 and I_2 are not comparable under inclusion.*
(2) *$I_1 + I_2 \notin \mathbb{A}(R)$.*
(3) *$I_1 \cap I_2 = (0)$.*

Proof. Let R be a reduced ring. Assume that $\gamma((\Omega(R))^c) = 2$ and $D = \{I_1, I_2\}$ is a dominating set of $(\Omega(R))^c$.

(1) Suppose that I_1 and I_2 are comparable under inclusion. Without loss of generality, we can assume that $I_1 \subset I_2$.

Let $x \in I_1 \backslash \{0\}$. It is clear that $I_2 \neq Rx$. We claim that $I_1 = Rx$. Suppose that $I_1 \neq Rx$. Note that $Rx \in \mathbb{A}(R)^* \backslash D$. As $Rx \subset I_1 \subset I_2$, it follows from Lemma 3.8 that Rx is not adjacent to I_i in $(\Omega(R))^c$ for each $i \in \{1, 2\}$. This is impossible, since $D = \{I_1, I_2\}$ is a dominating set of $(\Omega(R))^c$. Thus $I_1 = Rx$ for any non-zero $x \in I_1$ and hence, I_1 is a simple R-module. Since R is reduced, we obtain that $x^2 \neq 0$ and so, $I_1 = Rx = Rx^2$. It follows from $x = ax^2$ for some $a \in R$ that $e = ax$ is a non-trivial idempotent element of R and $I_1 = Rx = Re$. It is clear that the mapping $f : R \to Re \times R(1 - e)$ defined by $f(r) = (re, r(1 - e))$ is an isomorphism of rings. Since $I_1 = Re$ is a simple R-module, it follows that the ring Re is a field. Let us denote the ring Re by F_1. Let us denote the ring $R(1 - e)$ by R_2.

We next verify that $\frac{I_2}{I_1}$ is a simple R-module. Let $y \in I_2 \backslash I_1$. We claim that $I_1 + Ry = I_2$. Suppose that $I_1 + Ry \neq I_2$. Observe that $I_1 \subset I_1 + Ry \subset I_2$. Hence, $I_1 + Ry \in \mathbb{A}(R)^* \backslash D$. It follows from Lemma 3.8 that $I_1 + Ry$ is not adjacent to I_i in $(\Omega(R))^c$ for each $i \in \{1, 2\}$. This contradicts the assumption $\{I_1, I_2\}$ is a dominating set of $(\Omega(R))^c$. Therefore, $I_1 + Ry = I_2$ for any $y \in I_2 \backslash I_1$. This shows that $\frac{I_2}{I_1}$ is a simple R-module.

Let us denote the ring $F_1 \times R_2$ by T. Observe that the isomorphism f maps R onto T and under f, D is mapped onto $\{F_1 \times (0), F_1 \times J\}$, where

J is a simple R_2-module with $J \neq R_2$. Note that $\gamma((\Omega(T))^c) = 2$ and $\{F_1 \times (0), F_1 \times J\}$ is a dominating set of $(\Omega(T))^c$. Since R_2 is reduced and J is a proper ideal of R_2 and it is a simple R_2-module, it follows that $J = R_2e'$ for some non-trivial idempotent element e' of R_2. It is clear that $R_2 \cong R_2e' \times R_2(1-e-e')$ as rings. From $J = R_2e'$ is a simple R_2-module, we obtain that the ring R_2e' is a field. Let us denote the ring R_2e' by F_2 and the ring $R_2(1-e-e')$ by R_3. It is now clear that there is a ring isomorphism g from R onto $F_1 \times F_2 \times R_3$ and under g, D is mapped onto $\{F_1 \times (0) \times (0), F_1 \times F_2 \times (0)\}$. Let us denote the ring $F_1 \times F_2 \times R_3$ by T_1. Observe that $\gamma((\Omega(T_1))^c) = 2$ and $D_1 = \{F_1 \times (0) \times (0), F_1 \times F_2 \times (0)\}$ is a dominating set of $(\Omega(T_1))^c$. We claim that R_3 is an integral domain. Let $a \in R_3, a \neq 0$. Let $W = (0) \times (0) \times R_3a$. As $W \in \mathbb{A}(T_1)^* \backslash D_1$ and D_1 is a dominating set of $(\Omega(T_1))^c$, it follows that W must be adjacent to $F_1 \times F_2 \times (0)$ in $(\Omega(T_1))^c$. This implies that $F_1 \times F_2 \times R_3a \notin \mathbb{A}(T_1)$. Hence, $Ann_{R_3}(R_3a) = (0)$. This proves that R_3 is an integral domain. Note that we obtain from Proposition 3.7 that $\gamma((\Omega(T_1))^c) \geq 3$. This is a contradiction. Therefore, I_1 and I_2 are not comparable under inclusion.

(2) Suppose that $I_1 + I_2 \in \mathbb{A}(R)$. As I_1 and I_2 are not comparable under inclusion by (1), we obtain that $I_1 + I_2 \notin D$. Thus $I_i \subset I_1 + I_2$ for each $i \in \{1, 2\}$. Therefore, we obtain from Lemma 3.8 that $I_1 + I_2$ is not adjacent to I_i in $(\Omega(R))^c$ for each $i \in \{1, 2\}$. This contradicts the assumption $\{I_1, I_2\}$ is a dominating set of $(\Omega(R))^c$. Therefore, we get that $I_1 + I_2 \notin \mathbb{A}(R)$.

(3) Note that $I_1 \cap I_2 \in \mathbb{A}(R)$. Suppose that $I_1 \cap I_2 \neq (0)$. As I_1 and I_2 are not comparable under inclusion by (1), it follows that $I_1 \cap I_2 \notin D$. Observe that $I_1 \cap I_2 \subset I_i$ for each $i \in \{1, 2\}$. Therefore, $I_1 \cap I_2$ is not adjacent to I_i in $(\Omega(R))^c$ for each $i \in \{1, 2\}$ by Lemma 3.8. This contradicts the assumption $\{I_1, I_2\}$ is a dominating set of $(\Omega(R))^c$. Therefore, $I_1 \cap I_2 = (0)$. □

Theorem 3.10. *Let R be a reduced ring. The following statements are equivalent:*

(1) $\gamma((\Omega(R))^c) = 2$.
(2) *R has exactly two minimal prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ such that $\mathfrak{p}_i$ is not a simple R-module for each $i \in \{1, 2\}$.*
(3) *$(\Omega(R))^c$ is a complete bipartite graph with vertex partition V_1 and V_2 such that $|V_i| \geq 2$ for each $i \in \{1, 2\}$.*

Proof. (1) ⇒ (2) Assume that $\gamma((\Omega(R))^c) = 2$. Hence, it is possible to find a subset D of $\mathbb{A}(R)^*$ with $|D| = 2$ such that D is a dominating set of $(\Omega(R))^c$. Let $D = \{I_1, I_2\}$. Let $i \in \{1, 2\}$. We claim that there exists $\mathfrak{p}_i \in Spec(R) \cap \mathbb{A}(R)$ such that $I_i \subseteq \mathfrak{p}_i$. First, we prove that there exists $\mathfrak{p}_1 \in Spec(R) \cap \mathbb{A}(R)$ such that $I_1 \subseteq \mathfrak{p}_1$. Let

$$\mathcal{C} = \{W \mid W \in \mathbb{A}(R), W \supseteq I_1\}.$$

It is clear that $I_1 \in \mathcal{C}$ and hence, $\mathcal{C} \neq \emptyset$. Let $W \in \mathcal{C}$. Let $b \in I_2\backslash\{0\}$. We assert that $Wb = (0)$. Suppose that $Wb \neq (0)$. It is clear that $Wb \in \mathbb{A}(R)$. As $Wb \subseteq I_2$ and $Wb \neq (0)$ by assumption, it follows from Proposition 3.9(3) that $Wb \neq I_1$. If $Wb = I_2$, then as $I_1 \subseteq W$, we get that $I_1+I_2 \subseteq W$. Since $W \in \mathbb{A}(R)$, we obtain that $I_1+I_2 \in \mathbb{A}(R)$. This contradicts Proposition 3.9(2). Hence, $Wb \neq I_2$. Thus $Wb \in \mathbb{A}(R)^*\backslash D$. It is clear that $Wb + I_1 \subseteq W \in \mathbb{A}(R)$ and $Wb + I_2 \subseteq I_2 \in \mathbb{A}(R)$. Therefore, Wb is not adjacent to I_i in $(\Omega(R))^c$ for each $i \in \{1, 2\}$. This is a contradiction. Hence, $Wb = (0)$.

It is clear that $(\mathcal{C}, \subseteq)$ is a partially ordered set. We claim that any chain in $(\mathcal{C}, \subseteq)$ has an upper bound in $(\mathcal{C}, \subseteq)$. Let $\{W_\alpha\}_{\alpha\in\Lambda}$ be a chain in $(\mathcal{C}, \subseteq)$. Let $W = \bigcup_{\alpha\in\Lambda} W_\alpha$. Since $\{W_\alpha\}_{\alpha\in\Lambda}$ is a chain in $(\mathcal{C}, \subseteq)$, we obtain that W is an ideal of R. Let $b \in I_2\backslash\{0\}$. As $W_\alpha b = (0)$ for each $\alpha \in \Lambda$, we get that $Wb = (0)$. Hence, $W \in \mathbb{A}(R)$. As $I_1 \subseteq W_\alpha$ for each $\alpha \in \Lambda$, it follows that $I_1 \subseteq \bigcap_{\alpha\in\Lambda} W_\alpha \subseteq W$. This shows that $W = \bigcup_{\alpha\in\Lambda} W_\alpha \in \mathcal{C}$. From $W_\alpha \subseteq W$ for each $\alpha \in \Lambda$, we obtain that $W \in \mathcal{C}$ is an upper bound of the chain $\{W_\alpha\}_{\alpha\in\Lambda}$. This proves that any chain in $(\mathcal{C}, \subseteq)$ has an upper bound in $(\mathcal{C}, \subseteq)$. Therefore, it follows from Zorn's lemma that $(\mathcal{C}, \subseteq)$ admits a maximal element. Let $\mathfrak{p}$ be a maximal element of $(\mathcal{C}, \subseteq)$. Observe that $\mathfrak{p} \in \mathbb{A}(R)$ and $\mathfrak{p} \supseteq I_1$. We verify that $\mathfrak{p} \in Spec(R)$. Let $x, y \in R$ be such that $xy \in \mathfrak{p}$. Suppose that $x \notin \mathfrak{p}$. Since $\mathfrak{p}$ is a maximal element of $(\mathcal{C}, \subseteq)$ and $\mathfrak{p} \subset \mathfrak{p} + Rx$, we obtain that $\mathfrak{p} + Rx \notin \mathbb{A}(R)$. Let $b \in I_2\backslash\{0\}$. Note that $\mathfrak{p}b = (0)$ and so, $xyb = 0$. This implies that $(\mathfrak{p} + Rx)yb = (0)$. As $\mathfrak{p} + Rx \notin \mathbb{A}(R)$, it follows that $yb = 0$. Observe that $(\mathfrak{p} + Ry)b = (0)$. Now, $\mathfrak{p} \subseteq \mathfrak{p} + Ry$ and $\mathfrak{p} + Ry \in \mathbb{A}(R)$. Hence, $\mathfrak{p} + Ry \in \mathcal{C}$. Since $\mathfrak{p}$ is a maximal element of $(\mathcal{C}, \subseteq)$, we obtain that $\mathfrak{p} + Ry = \mathfrak{p}$ and hence, $y \in \mathfrak{p}$. This shows that $\mathfrak{p} \in Spec(R)$. It is convenient to denote $\mathfrak{p}$ by $\mathfrak{p}_1$. Thus there exists $\mathfrak{p}_1 \in Spec(R)$ such that $\mathfrak{p}_1 \in \mathbb{A}(R)$ and $I_1 \subseteq \mathfrak{p}_1$. It follows using a similar argument that there exists $\mathfrak{p}_2 \in Spec(R)$ such that $\mathfrak{p}_2 \in \mathbb{A}(R)$ and $I_2 \subseteq \mathfrak{p}_2$. We next claim that $\bigcap_{k=1}^2 \mathfrak{p}_k = (0)$. Suppose that $\bigcap_{k=1}^2 \mathfrak{p}_k \neq (0)$. Then $\bigcap_{k=1}^2 \mathfrak{p}_k \in \mathbb{A}(R)^*$. If $\bigcap_{k=1}^2 \mathfrak{p}_k = I_1$, then $I_1 + I_2 \subseteq \mathfrak{p}_2 \in \mathbb{A}(R)$. This is impossible, since $I_1 + I_2 \notin \mathbb{A}(R)$ by Proposition 3.9(2). Therefore, $\bigcap_{k=1}^2 \mathfrak{p}_k \neq I_1$. Similarly, if $\bigcap_{k=1}^2 \mathfrak{p}_k = I_2$, then $I_1 + I_2 \subseteq \mathfrak{p}_1 \in \mathbb{A}(R)$. This is impossible, since $I_1 + I_2 \notin \mathbb{A}(R)$ by Proposition 3.9(2). Therefore, $\bigcap_{k=1}^2 \mathfrak{p}_k \neq I_2$ and so, $\bigcap_{k=1}^2 \mathfrak{p}_k \notin D$. Let $i \in \{1, 2\}$. As $(\bigcap_{k=1}^2 \mathfrak{p}_k) + I_i \subseteq \mathfrak{p}_i \in \mathbb{A}(R)$, it follows that $\bigcap_{k=1}^2 \mathfrak{p}_k$ is not adjacent to I_i in $(\Omega(R))^c$. This contradicts the assumption D is a dominating set of $(\Omega(R))^c$. Hence, $\bigcap_{k=1}^{2} \mathfrak{p}_k = (0)$.

From $\bigcap_{k=1}^{2} \mathfrak{p}_k = (0)$, we obtain that $|Min(R)| = 2$ and indeed,

$$Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}.$$

We next verify that $\mathfrak{p}_i$ is not a simple R-module for each $i \in \{1, 2\}$. Note that $((0) :_R \mathfrak{p}_1) = \mathfrak{p}_2$ and $((0) :_R \mathfrak{p}_2) = \mathfrak{p}_1$. Suppose that $\mathfrak{p}_1$ is a simple R-module. Then $((0) :_R \mathfrak{p}_1) = \mathfrak{p}_2 \in Max(R)$. In such a case, we obtain that $\mathfrak{p}_1 + \mathfrak{p}_2 = R$. As $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$, it follows from [5, Proposition 1.10(ii) and (iii)] that the mapping $f : R \to \frac{R}{\mathfrak{p}_1} \times \frac{R}{\mathfrak{p}_2}$ defined by $f(r) = (r + \mathfrak{p}_1, r + \mathfrak{p}_2)$ is an isomorphism of rings. Observe that $\frac{R}{\mathfrak{p}_1}$ is an integral domain and $\frac{R}{\mathfrak{p}_2}$ is a field. Let us denote the ring $\frac{R}{\mathfrak{p}_1} \times \frac{R}{\mathfrak{p}_2}$ by T. Note that $R \cong T$ and hence, we obtain from $(2) \Rightarrow (1)$ of Theorem 3.1 that $\gamma((\Omega(R))^c) = 1$. This contradicts the assumption $\gamma((\Omega(R))^c) = 2$. Therefore„ $\mathfrak{p}_1$ is not a simple R-module. Similarly, it follows that $\mathfrak{p}_2$ is not a simple R-module.

$(2) \Rightarrow (3)$ Assume that R is a reduced ring with

$$Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$$

such that $\mathfrak{p}_i$ is not a simple R-module for each $i \in \{1, 2\}$. The proof of $(ii) \Rightarrow (i)$ of [20, Proposition 2.10] implies that $(\Omega(R))^c$ is a complete bipartite graph with vertex partition $\mathbb{A}(R)^* = V_1 \cup V_2$, where

$$V_i = \{I \in \mathbb{A}(R)^* \mid I \subseteq \mathfrak{p}_i\}$$

for each $i \in \{1, 2\}$. Let $i \in \{1, 2\}$. From $\bigcap_{i=1}^{2} \mathfrak{p}_i = (0)$, it follows that $\mathfrak{p}_i \in \mathbb{A}(R)^*$. Since $\mathfrak{p}_i$ is not a simple R-module, there exists at least one $I_i \in \mathbb{I}(R)^*$ such that $I_i \subset \mathfrak{p}_i$. Thus $\{\mathfrak{p}_i, I_i\} \subseteq V_i$ and so, $|V_i| \geq 2$.

$(3) \Rightarrow (1)$ Assume that $(\Omega(R))^c$ is a complete bipartite graph with vertex partition V_1 and V_2 such that $|V_i| \geq 2$ for each $i \in \{1, 2\}$. In such a case, it follows from the observation noted in the paragraph which appears just preceding the statement of Theorem 3.1 that $\gamma((\Omega(R))^c) = 2$. □

Let R be a reduced ring. Let $n \in \mathbb{N}$ be such that $n \geq 3$. If $\gamma((\Omega(R))^c) = n$, then we do not know whether $|Min(R)| < \infty$. However, with the assumption that $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$ and $\gamma((\Omega(R))^c) \geq 2$, we prove in Theorem 3.13 that

$$\gamma((\Omega(R))^c) = |Min(R)|.$$

We use Lemmas 3.11 and 3.12 in the proof of Theorem 3.13.

Lemma 3.11. *Let R be a reduced ring. Suppose that $|Min(R)| \geq 3$ and $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$. Then $|D| \geq |Min(R)|$ for each dominating set D of $(\Omega(R))^c$.*

Proof. By hypothesis, R is a reduced ring with $|Min(R)| \geq 3$ and $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$. Let $Min(R) = \{\mathfrak{p}_\alpha\}_{\alpha \in \Lambda}$. Let $\alpha \in \Lambda$. As $\mathfrak{p}_\alpha \in \mathbb{A}(R)$, there exists $r_\alpha \in R\backslash\{0\}$ such that $\mathfrak{p}_\alpha r_\alpha = (0)$. Hence, $\mathfrak{p}_\alpha \subseteq ((0) :_R r_\alpha)$. Since R is reduced and $r_\alpha \neq 0$, it follows that $r_\alpha^2 \neq 0$ and so, $r_\alpha \notin \mathfrak{p}_\alpha$. From $r_\alpha((0) :_R r_\alpha) = (0) \subset \mathfrak{p}_\alpha$, we get that $((0) :_R r_\alpha) \subseteq \mathfrak{p}_\alpha$ and so, $\mathfrak{p}_\alpha = ((0) :_R r_\alpha)$ is a B-prime of (0) in R. Let $\beta \in \Lambda\backslash\{\alpha\}$. Now, $\mathfrak{p}_\alpha = ((0) :_R r_\alpha)$, $\mathfrak{p}_\beta = ((0) :_R r_\beta)$, and $\mathfrak{p}_\alpha \neq \mathfrak{p}_\beta$. Hence, we obtain from [7, Lemma 3.6] that $r_\alpha r_\beta = 0$. Since $\mathfrak{p}_\alpha \neq \mathfrak{p}_\beta$, it follows that $Rr_\alpha \neq Rr_\beta$ and $r_\alpha r_\beta = 0$. It is clear that $Rr_\alpha \in \mathbb{A}(R)^*$ for each $\alpha \in \Lambda$.

Let D be any dominating set of $(\Omega(R))^c$. Let $\alpha \in \Lambda$. Let $A_\alpha = \{Rr_\alpha\}$ and let

$$B_\alpha = \{I \in \mathbb{A}(R)^* \mid V(I) \cap Min(R) = \{\mathfrak{p}_\alpha\}\}.$$

Since $\mathfrak{p}_\alpha \in B_\alpha$, it follows that $B_\alpha \neq \emptyset$. We claim that

$$|D \cap (A_\alpha \cup B_\alpha)| \geq 1.$$

If $Rr_\alpha \in D$, then it is clear that $|D \cap (A_\alpha \cup B_\alpha)| \geq 1$. Suppose that $Rr_\alpha \notin D$. Since D is a dominating set of $(\Omega(R))^c$, there must be an element $I \in D$ such that Rr_α and I are adjacent in $(\Omega(R))^c$. Hence, $I + Rr_\alpha \notin \mathbb{A}(R)$. We know from [20, Lemma 2.14] that $I \subseteq \mathfrak{p}_{\alpha'}$ for some $\alpha' \in \Lambda$. We claim that $\mathfrak{p}_\alpha = \mathfrak{p}_{\alpha'}$. Suppose that $\mathfrak{p}_\alpha \neq \mathfrak{p}_{\alpha'}$. From $\mathfrak{p}_\alpha = ((0) :_R r_\alpha)$ and $\mathfrak{p}_{\alpha'} = ((0) :_R r_{\alpha'})$, we obtain from [7, Lemma 3.6] that $r_\alpha r_{\alpha'} = 0$. As $I \subseteq \mathfrak{p}_{\alpha'}$, it follows that $Ir_{\alpha'} = (0)$ and so, $(I+Rr_\alpha)r_{\alpha'} = (0)$. This is impossible, since $I+Rr_\alpha \notin \mathbb{A}(R)$. Therefore, $\mathfrak{p}_{\alpha'} = \mathfrak{p}_\alpha$. Hence, $V(I) \cap Min(R) = \{\mathfrak{p}_\alpha\}$ and so, $I \in B_\alpha$. The above arguments imply that $|D \cap (A_\alpha \cup B_\alpha)| \geq 1$. Let $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$. We verify that

$$(A_\alpha \cup B_\alpha) \cap (A_\beta \cup B_\beta) = \emptyset.$$

Since $A_\alpha = \{Rr_\alpha\}$, $A_\beta = \{Rr_\beta\}$, and $Rr_\alpha \neq Rr_\beta$, it follows that $A_\alpha \cap A_\beta = \emptyset$. Observe that $r_\alpha \in \mathfrak{p}_{\alpha'}$ for all $\alpha' \in \Lambda\backslash\{\alpha\}$ and as $|\Lambda| \geq 3$, we get that $|V(Rr_\alpha) \cap Min(R)| \geq 2$. For any $J \in B_\beta$, $|V(J) \cap Min(R)| = 1$. Therefore, $Rr_\alpha \notin B_\beta$ and so, $A_\alpha \cap B_\beta = \emptyset$. Let $I \in B_\alpha$. From $|V(I) \cap Min(R)| = 1$, $|V(Rr_\beta) \cap Min(R)| \geq 2$, it follows that $I \notin A_\beta = \{Rr_\beta\}$ and so, $B_\alpha \cap A_\beta = \emptyset$. As $V(I) \cap Min(R) = \{\mathfrak{p}_\alpha\}$, whereas $V(J) \cap Min(R) = \{\mathfrak{p}_\beta\}$ for any $J \in B_\beta$, we obtain that $I \notin B_\beta$ and so, $B_\alpha \cap B_\beta = \emptyset$. The above arguments show that

$$(A_\alpha \cup B_\alpha) \cap (A_\beta \cup B_\beta) = \emptyset$$

and so, for all distinct $\alpha, \beta \in \Lambda$,

$$(D \cap (A_\alpha \cup B_\alpha)) \cap (D \cap (A_\beta \cup B_\beta)) = \emptyset.$$

For each $\alpha \in \Lambda$, let $I_\alpha \in D \cap (A_\alpha \cup B_\alpha)$. Note that $\{I_\alpha \mid \alpha \in \Lambda\} \subseteq D$ and so,

$$|Min(R)| = |\Lambda| = |\{I_\alpha \mid \alpha \in \Lambda\}| \leq |D|.$$

This proves that for any dominating set D of $(\Omega(R))^c$, $|D| \geq |Min(R)|$. $\square$

Lemma 3.12. *Let R be a reduced ring. Suppose that $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$. Then $Min(R)$ is a dominating set of $(\Omega(R))^c$.*

Proof. Assume that R is a reduced ring and $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$. Let

$$Min(R) = \{\mathfrak{p}_\alpha \mid \alpha \in \Lambda\}.$$

Since R is not an integral domain, it follows that $|Min(R)| \geq 2$. Note that $Min(R) \subseteq \mathbb{A}(R)^*$. We claim that $Min(R)$ is a dominating set of $(\Omega(R))^c$. Let $I \in \mathbb{A}(R)^* \backslash Min(R)$. Since $\bigcap_{\alpha \in \Lambda} \mathfrak{p}_\alpha = (0)$ and $I \neq (0)$, we obtain that there exists $\alpha \in \Lambda$ such that $I \not\subseteq \mathfrak{p}_\alpha$. We assert that $I + \mathfrak{p}_\alpha \notin \mathbb{A}(R)$. Suppose that $I + \mathfrak{p}_\alpha \in \mathbb{A}(R)$. Then by [20, Lemma 2.14], there exists $\beta \in \Lambda$ such that $I + \mathfrak{p}_\alpha \subseteq \mathfrak{p}_\beta$.. As distinct minimal prime ideals of a ring are not comparable under inclusion, it follows that $\mathfrak{p}_\alpha = \mathfrak{p}_\beta$. This is impossible, since $I \not\subseteq \mathfrak{p}_\alpha$. Hence, $I + \mathfrak{p}_\alpha \notin \mathbb{A}(R)$ and so, I and $\mathfrak{p}_\alpha$ are adjacent in $(\Omega(R))^c$. This shows that $Min(R)$ is a dominating set of $(\Omega(R))^c$. $\square$

Theorem 3.13. *Let R be a reduced ring. Suppose that $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$. If $\gamma((\Omega(R))^c) \geq 2$, then $\gamma((\Omega(R))^c) = |Min(R)|$.*

Proof. By hypothesis, R is a reduced ring such that $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$. Assume that $\gamma((\Omega(R))^c) \geq 2$. If $|Min(R)| = 2$, then it follows from Lemma 3.12 that $\gamma((\Omega(R))^c) \leq 2$ and so,

$$\gamma((\Omega(R))^c) = |Min(R)|.$$

Suppose that $|Min(R)| \geq 3$. Then it follows from Lemmas 3.11 and 3.12 that $Min(R)$ is a dominating set of minimum cardinality. Therefore, $\gamma((\Omega(R))^c) = |Min(R)|$. $\square$

Let R be a reduced ring. Let $|Min(R)| = n$ for some $n \in \mathbb{N} \backslash \{1\}$. Let $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2, \ldots, n\}\}$. Note that $\bigcap_{i=1}^{n} \mathfrak{p}_i = (0)$. Let $i \in \{1, 2, \ldots, n\}$. Let us denote the set $\{1, 2, \ldots, n\} \backslash \{i\}$ by A_i.

As distinct minimal prime ideals of a ring are not comparable under inclusion, it follows from [5, Proposition 1.11(ii)] that $\mathfrak{p}_i \not\supseteq \bigcap_{j\in A_i} \mathfrak{p}_j$. Let $x_i \in (\bigcap_{j\in A_i} \mathfrak{p}_j)\backslash\mathfrak{p}_i$. Then $x_i \neq 0$ and $\mathfrak{p}_i x_i = (0)$. Hence, $\mathfrak{p}_i \in \mathbb{A}(R)$. In Example 3.14, we provide an example of a reduced ring R with $Min(R)$ is infinite and $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$.

Let $T = K[X_1, X_2, \ldots, X_n]$ $(n \geq 2)$ be the polynomial ring in n variables $X_1, X_2, \ldots, X_n$ over a field K. Let $I = T(\prod_{i=1}^n X_i)$. Let $R = \frac{T}{I}$. Let $i \in \{1, 2, \ldots, n\}$. It is convenient to denote $X_i + I$ by x_i. Since $TX_i \in Spec(T)$, it follows that $\mathfrak{p}_i = Rx_i \in Spec(R)$. Observe that $(0 + I) = \prod_{i=1}^n \mathfrak{p}_i = \bigcap_{i=1}^n \mathfrak{p}_i$. Hence, R is a reduced ring and as TX_i and TX_j are not comparable under inclusion for all distinct $i, j \in \{1, 2, \ldots, n\}$, we get that $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are not comparable under inclusion. Hence,

$$Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2, \ldots, n\}\}.$$

Therefore, $|Min(R)| = n$. Hence, we obtain from Lemma 3.12 that $\gamma((\Omega(R))^c) \leq n$. Observe that for each $i \in \{1, 2, \ldots, n\}$, $x_i^2 \neq 0 + I$ and $Rx_i^2 \subset \mathfrak{p}_i$ and so, $\mathfrak{p}_i$ is not a simple R-module. Thus if $n = 2$, then it follows from (2) $\Rightarrow$ (1) of Theorem 3.10 that $\gamma((\Omega(R))^c) = 2$. If $n \geq 3$, then it follows from Lemma 3.11 that $\gamma((\Omega(R))^c) \geq n$. Therefore, we obtain that $\gamma((\Omega(R))^c) = n$.

In Example 3.14, we mention a reduced ring R due to Gilmer and Heinzer [11, Example, page 16] such that $(\Omega(R))^c$ does not admit any finite dominating set.

Example 3.14. Let $\{X_i\}_{i=1}^\infty$ be a set of indeterminates. Let

$$D = \bigcup_{n=1}^\infty K[[X_1, \ldots, X_n]],$$

where $K[[X_1, \ldots, X_n]]$ is the power series ring in $X_1, \ldots, X_n$ over a field K. Let I be the ideal of D generated by $\{X_iX_j \mid i, j \in \mathbb{N}, i \neq j\}$. Let $R = \frac{D}{I}$. Then R is a reduced ring, $Min(R)$ is infinite, $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$, and $\gamma((\Omega(R))^c) = |Min(R)|$.

Proof. For each $i \in \mathbb{N}$, it is convenient to denote $X_i + I$ by x_i. It follows from the proof of [11, Example, page 16] that R is reduced, $|Max(R)| = 1$, $\mathfrak{m} = \sum_{i=1}^\infty Rx_i$ is the unique maximal ideal of R, and $Min(R)$ is infinite with $Min(R) = \{\mathfrak{p}_i \mid i \in \mathbb{N}\}$, where for each $i \in \mathbb{N}$, $\mathfrak{p}_i$ is the ideal of R generated by $\{x_j \mid j \in \mathbb{N}, j \neq i\}$. Observe that for each $i \in \mathbb{N}$, $x_i \neq 0 + I$ and $\mathfrak{p}_i x_i = (0 + I)$. Hence, $\mathfrak{p}_i \in \mathbb{A}(R)$ for each $i \in \mathbb{N}$. It is clear from Lemma 3.11 that $\gamma((\Omega(R))^c) > n$ for each $n \in \mathbb{N}$ and it follows from Theorem 3.13 that $\gamma((\Omega(R))^c) = |Min(R)|$ and so, $(\Omega(R))^c$ does not admit any finite dominating set. $\square$

We recall that a ring R is said to be *von Neumann regular* if given $a \in R$, there exists $b \in R$ such that $a = a^2b$ [10, Exercise 16, page 111]. A ring R is von Neumann regular if and only if $dim R = 0$ and R is reduced by $(a) \Leftrightarrow (d)$ of [10, Exercise 16, page 111]. Thus if R is von Neumann regular, then $Spec(R) = Max(R) = Min(R)$. Let R be a von Neumann regular ring which is not a field. Hence, R is not an integral domain. Since R is reduced, we know from [20, Lemma 2.2] that $|\mathbb{A}(R)^*| \geq 2$. It is clear that a homomorphic image of a von Neumann regular ring is von Neumann regular and an integral domain is von Neumann regular if and only if it is a field. Hence, we obtain from $(1) \Leftrightarrow (2)$ of Theorem 3.1 that for a von Neumann regular ring R, $\gamma((\Omega(R))^c) = 1$ if and only if $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. We verify in Corollary 3.16 that there exists no von Neumann regular ring R such that $\gamma((\Omega(R))^c) = 2$. Let $n \in \mathbb{N}$ be such that $n \geq 3$. In Theorem 3.17,, we determine necessary and sufficient conditions in order that $\gamma((\Omega(R))^c)$ to be equal to n. We use Lemma 3.15 in the verification of Corollary 3.16.

Lemma 3.15. *Let R be a von Neumann regular ring with $|Min(R)| = 2$. Let $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. Then $\mathfrak{p}_i$ is a simple R-module for each $i \in \{1, 2\}$.*

Proof. By hypothesis,, R is von Neumann regular, $|Min(R)| = 2$, and $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. As $Min(R) = Max(R)$, we get that $\frac{R}{\mathfrak{p}_i}$ is a field for each $i \in \{1, 2\}$ and so, $\frac{R}{\mathfrak{p}_i}$ is a simple R-module for each $i \in \{1, 2\}$. Since R is reduced, it follows that $\bigcap_{i=1}^{2} \mathfrak{p}_i = (0)$ and it is clear that $\mathfrak{p}_1 + \mathfrak{p}_2 = R$. It is not hard to verify that the mapping $f_1 : \mathfrak{p}_1 \to \frac{R}{\mathfrak{p}_2}$ defined by $f_1(x) = x + \mathfrak{p}_2$ and the mapping $f_2 : \mathfrak{p}_2 \to \frac{R}{\mathfrak{p}_1}$ given by $f_2(y) = y + \mathfrak{p}_1$ are isomorphisms of R-modules. Hence, $\mathfrak{p}_i$ is a simple R-module for each $i \in \{1, 2\}$. $\square$

Corollary 3.16. *Let R be a von Neumann regular ring. Then*

$$\gamma((\Omega(R))^c) \neq 2.$$

Proof. By hypothesis, R is a von Neumann regular ring. Since R is reduced, it follows from $(1) \Rightarrow (2)$ of Theorem 3.10 and Lemma 3.15 that $\gamma((\Omega(R))^c) \neq 2$. $\square$

Theorem 3.17. *Let R be a von Neumann regular ring. Let $n \in \mathbb{N}$ be such that $n \geq 3$. The following statements are equivalent:*

(1) $\gamma((\Omega(R))^c) = n$.
(2) $|Min(R)| = n$.
(3) $R \cong F_1 \times F_2 \times F_3 \times \cdots \times F_n$ *as rings, where F_i is a field for each $i \in \{1, 2, 3, \ldots, n\}$.*

Proof. As R is von Neumann regular by hypothesis, we get that R is reduced and $Spec(R) = Max(R) = Min(R)$. By hypothesis, $n \in \mathbb{N}$ is such that $n \geq 3$.

(1) $\Rightarrow$ (2) Assume that $\gamma((\Omega(R))^c) = n$. Suppose that

$$|Min(R)| \geq n + 1.$$

As $dim R = 0$, we obtain from [21, Lemma 2.2] that there exist zero-dimensional rings $R_1, R_2, R_3, \ldots, R_n, R_{n+1}$ such that

$$R \cong R_1 \times R_2 \times R_3 \times \cdots \times R_n \times R_{n+1}$$

as rings. Since R is reduced, it follows that R_i is reduced for each $i \in \{1, 2, 3, \ldots, n, n + 1\}$. Let us denote the ring

$$R_1 \times R_2 \times R_3 \times \cdots \times R_n \times R_{n+1}$$

by T. Note that $\gamma((\Omega(T))^c) \geq n + 1$ by Proposition 3.7. Hence, $\gamma((\Omega(R))^c) \geq n + 1$. This contradicts the assumption $\gamma((\Omega(R))^c) = n$. Therefore, $|Min(R)| \leq n$. Hence, $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$. It follows from Lemma 3.12 that

$$\gamma((\Omega(R))^c) \leq |Min(R)|.$$

Thus $|Min(R)| \geq n$ and so, $|Min(R)| = n$.

(2) $\Rightarrow$ (3) Let

$$Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2, 3, \ldots, n\}\}.$$

Since R is von Neumann regular, it follows that R is reduced and $Min(R) = Max(R)$. Therefore, $\bigcap_{i=1}^n \mathfrak{p}_i = (0)$ and $\mathfrak{p}_i + \mathfrak{p}_j = R$ for all distinct $i, j \in \{1, 2, 3, \ldots, n\}$. Hence, we obtain from [5, Proposition 1.10(ii) and (iii)] that the mapping

$$f : R \to \tfrac{R}{\mathfrak{p}_1} \times \tfrac{R}{\mathfrak{p}_2} \times \tfrac{R}{\mathfrak{p}_3} \times \cdots \times \tfrac{R}{\mathfrak{p}_n}$$

defined by $f(r) = (r + \mathfrak{p}_1, r + \mathfrak{p}_2, r + \mathfrak{p}_3, \ldots, r + \mathfrak{p}_n)$ is an isomorphism of rings. Let $i \in \{1, 2, 3, \ldots, n\}$ and let us denote $\frac{R}{\mathfrak{p}_i}$ by F_i. Then F_i is a field and $R \cong F_1 \times F_2 \times F_3 \times \cdots \times F_n$ as rings.

(3) $\Rightarrow$ (1) Let us denote the ring $F_1 \times F_2 \times F_3 \times \cdots \times F_n$ by T, where F_i is a field for each $i \in \{1, 2, 3, \ldots, n\}$. Since $n \geq 3$, it follows from Proposition 3.7 that $\gamma((\Omega(T))^c) \geq n$. Since $R \cong T$ as rings by

assumption, it follows that $\gamma((\Omega(R))^c) \geq n$. Observe that

$$\begin{aligned} Spec(T) = Max(T) = Min(T) = \{\mathfrak{m}_1 &= (0) \times F_2 \times F_3 \times \cdots \times F_n, \\ \mathfrak{m}_2 &= F_1 \times (0) \times F_3 \times \cdots \times F_n, \\ \mathfrak{m}_3 &= F_1 \times F_2 \times (0) \times \cdots \times F_n, \\ &\ldots, \\ \mathfrak{m}_n &= F_1 \times F_2 \times F_3 \times \cdots \times (0)\} \end{aligned}$$

Thus $|Min(T)| = n$ and so, $|Min(R)| = n$. Hence, we obtain from Theorem 3.13 that $\gamma((\Omega(R))^c) = |Min(R)| = n$. $\square$

Acknowledgments

We are very much thankful to the referee for many useful and helpful suggestions. We are also very much thankful to Professor Ebrahim Hashemi for his support.

References

1. A. Alilou and J. Amjadi, The sum-annihilating essential ideal graph of a commutative ring, *Commun. comb. optim.*, **1**(2) (2016), 117–135.
2. D. F. Anderson, M. C. Axtell and J. A. Stickles Jr., Zero-divisor graphs in commutative rings, In *Commutative Algebra, Noetherian and non-Noetherian perspectives*, M. Fontana, S. E. Kabbaj, B. Olberding and I. Swanson (Editors), Springer-Verlag, New York (2011), 23–46.
3. D. F. Anderson and A. Badawi, The total graph of a commutative ring, *J. Algebra*, **320**(7) (2008), 2706–2719.
4. D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217**(2) (1999), 434–447.
5. M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Massachusetts, 1969.
6. R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Universitext, Second Edition, Springer, 2012.
7. I. Beck, Coloring of commutative rings, *J. Algebra*, **116**(1) (1988), 208–226.
8. M. Behboodi and Z. Rakeei, The annihilating-ideal graph of commutative rings I, *J. Algebra Appl.*, **10**(4) (2011), 727–739.
9. M. Behboodi and Z. Rakeei, The annihilating-ideal graph of commutative rings II, *J. Algebra Appl.*, **10**(4) (2011), 741–753.
10. R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.
11. R. Gilmer and W. Heinzer, The Laskerian property, power series rings and Noetherian spectra, *Proc. Amer. Math. Soc.*, **79**(1) (1980), 13–16.
12. W. Heinzer and J. Ohm, Locally Noetherian commutative rings, *Trans. Amer. Math. Soc.*, **158**(2) (1971), 273–284.
13. W. Heinzer and J. Ohm, On the Noetherian-like rings of E.G. Evans, *Proc. Amer. Math. Soc.*, **34**(1) (1972), 73–74.

14. N. Jaffari Rad, S. Heider Jaffari and D. A. Mojdeh, On domination in zero-divisor graphs, *Canad. Math. Bull.*, **56**(2) (2013), 407–411.
15. R. Kala and S. Kavitha, Sum annihilating ideal graph of a commutative ring, *Palest. J. Math.*, **5**(2) (2016), 282–290.
16. I. Kaplansky, *Commutative Rings*, The University of Chicago Press, Chicago, 1974.
17. D. A. Mojdeh and A. M. Rahimi, Dominating sets of some graphs associated to commutative rings, *Comm. Algebra*, **40**(9) (2012), 3389–3396.
18. R. Nikandish and H. R. Maimani, Dominating sets of the annihilating-ideal graphs, *Electron. Notes Discrete Math.*, **45** (2014), 17–22.
19. S. Visweswaran and H. D. Patel, A graph associated with the set of all nonzero annihilating ideals of a commutative ring, *Discrete Math. Algorithm Appl.*, **6**(4) (2014), Article ID: 1450047, 22 pp.
20. S. Visweswaran and P. Sarman, On the complement of a graph associated with the set of all nonzero annihilating ideals of a commutative ring, *Discrete Math. Algorithm Appl.*, **8**(3) (2016), Article ID: 1650043, 22 pp.
21. S. Visweswaran and P. T. Lalchandani, The exact zero-divisor graph of a reduced ring, *Indian J. Pure Appl. Math.*, **52**(4) (2021), 1123–1144.

Subramanian Visweswaran
Department of Mathematics, Saurashtra University, P.O. Box 360005, Rajkot, India.
Email: `s_visweswaran2006@yahoo.co.in`

Patat Sarman
Department of Mathematics, Government Polytechnic, P.O. Box 362263, Junagadh, India.
Email: `patat.sarman03@gmail.com`

Journal of Algebraic Systems

ON THE DOMINATION NUMBER OF THE SUM ANNIHILATING IDEAL GRAPH OF A COMMUTATIVE RING AND ON THE DOMINATION NUMBER OF ITS COMPLEMENT

S. VISWESWARAN AND P. SARMAN

بررسی عدد احاطه‌گری گراف ایده‌آل پوچ‌ساز جمعی از یک حلقه‌ی جابه‌جایی و بررسی عدد احاطه‌گری مکمل آن

سوبرامانی ویسوسواران[1] و پتات سارمان[2]

[1]گروه ریاضی، دانشگاه ساوراشترا، راجکوت، هند

[2]گروه ریاضی، پلی تکنیک دولتی، جوناگاد، هند

در این مقاله، همه‌ی حلقه‌ها جابه‌جایی و یکدار هستند و حداقل دارای یک ایده‌آل پوچ‌ساز ناصفر می‌باشند. فرض می‌کنیم R یک حلقه باشد. همچنین، فرض می‌کنیم $\mathbb{A}(R)$ نشان‌دهنده‌ی همه‌ی ایده‌آل‌های پوچ‌ساز R و $\mathbb{A}(R)^*$ نمایانگر $\mathbb{A}(R)\backslash\{(0)\}$ باشد. یادآوری می‌کنیم که گراف ایده‌آل پوچ‌ساز جمعی که با $\Omega(R)$ نشان داده می‌شود، گرافی غیرجهتی است که $\mathbb{A}(R)^*$ مجموعه‌ی رئوس آن می‌باشد و دو رأس متمایز I و J در $\Omega(R)$ مجاورند اگر و تنها اگر $I + J \in \mathbb{A}(R)$. هدف این مقاله بررسی برخی نتایج روی عدد احاطه‌گری $\Omega(R)$ (به‌ترتیب، $(\Omega(R))^c$) می‌باشد که $(\Omega(R))^c$ مکمل $\Omega(R)$ است.

کلمات کلیدی: حلقه‌ی کاهشی، ایده‌آل اول مینیمال، N–اول ماکسیمال (0)، B–اول (0)، عدد احاطه‌گری یک گراف.