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# ŁUKASIEWICZ FUZZY FILTERS IN HOOPS

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Abstract. By applying the concept of the Łukasiewicz fuzzy set to the filter in hoops, the Łukasiewicz fuzzy filter is introduced and its properties are investigated. The relationship between fuzzy filter and Łukasiewicz fuzzy filter is discussed. Conditions for the Łukasiewicz fuzzy set to be a Łukasiewicz fuzzy filter are provided, and characterizations of Łukasiewicz fuzzy filter are displayed. The conditions under which the three subsets, $\in$-set, $q$-set, and $O$-set, will be filter are explored.

## 1. Introduction

The hoop is introduced by Bosbach in [10, 11], and it is a naturally ordered pocrim, i.e., a partially ordered commutative residuated integral monoid. In the past few years, hoop theory has become enriched with deep structural arrangements. Filter theory and its fuzzy works have been studied, including studies of various properties in hoops (see [1, 2, 5, 4, 6, 7, 8, 9, 13, 18, 22, 23]). Łukasiewicz logic, which is the logic of the Łukasiewicz $t$-norm, is a non-classical and many-valued logic. Using the idea of Łukasiewicz $t$-norm, Jun [16] constructed the concept of Łukasiewicz fuzzy sets based on a given fuzzy set and applied it to BE-algebras, BCK-algebras and BCI-algebras (see [3, 14, 15, 17, 20, 21]).

In this paper, the concept of the Łukasiewicz fuzzy set is applied to the filter of hoops. We introduce the notion of Łukasiewicz fuzzy

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filters in hoops, and investigate several properties. We discuss the relationship between fuzzy filter and Łukasiewicz fuzzy filter, and consider characterizations of Łukasiewicz fuzzy filter. We provide conditions for the Łukasiewicz fuzzy set to be a Łukasiewicz fuzzy filter. We explore the conditions under which $\in$-set, $q$-set, and $O$-set of the Łukasiewicz fuzzy set can be filters.

## 2. Preliminaries

**Definition 2.1.** By a *hoop* we mean an algebra $(H, \odot, \rightarrow, 1)$ in which $(H, \odot, 1)$ is a commutative monoid and the following assertions are valid.

(H1) $(\forall a \in H)(a \rightarrow a = 1)$,
(H2) $(\forall a, b \in H)(a \odot (a \rightarrow b) = b \odot (b \rightarrow a))$,
(H3) $(\forall a, b, c \in H)(a \rightarrow (b \rightarrow c) = (a \odot b) \rightarrow c)$.

On hoop $H$, we define $a \leq b$ if and only if $a \rightarrow b = 1$. It is easy to see that $\leq$ is a partial order relation on $H$.

**Proposition 2.2** ([10, 11])**.** *Every hoop $H$ satisfies the following assertions:*

$$(\forall a, b \in H)(a \odot b \leq c \;\Leftrightarrow\; a \leq b \rightarrow c). \tag{2.1}$$

$$(\forall a, b \in H)(a \odot b \leq a, b). \tag{2.2}$$

$$(\forall a, b \in H)(a \leq b \rightarrow a). \tag{2.3}$$

$$(\forall a \in H)(a \rightarrow 1 = 1,\; 1 \rightarrow a = a). \tag{2.4}$$

$$(\forall a, b, c \in H)((a \rightarrow b) \odot (b \rightarrow c) \leq a \rightarrow c). \tag{2.5}$$

$$(\forall a, b, c \in H)(a \leq b \;\Rightarrow\; a \odot c \leq b \odot c). \tag{2.6}$$

$$(\forall a, b \in H)(a \odot (a \rightarrow b) \leq b). \tag{2.7}$$

**Definition 2.3.** By a *filter* of a hoop $H$ we mean a subset $F$ of $H$ which satisfies:

$$(\forall a, b \in H)(a, b \in F \;\Rightarrow\; a \odot b \in F), \tag{2.8}$$

$$(\forall a, b \in H)(a \leq b,\, a \in F \;\Rightarrow\; b \in F). \tag{2.9}$$

**Proposition 2.4** ([12])**.** *A subset $F$ of a hoop $H$ is a filter of $H$ if and only if it satisfies:*

$$1 \in F, \tag{2.10}$$

$$(\forall a, b \in H)(a,\, a \rightarrow b \in F \;\Rightarrow\; b \in F). \tag{2.11}$$

A fuzzy set $\lambda$ in a set $H$ of the form

$$\lambda(b) := \begin{cases} t \in (0, 1] & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

is said to be a *fuzzy point* with support $a$ and value $t$ and is denoted by $\langle a/t\rangle$.

For a fuzzy set $\lambda$ in a set $H$, we say that a fuzzy point $\langle a/t\rangle$ is

(i) *contained* in $\lambda$, denoted by $\langle a/t\rangle \in \lambda$, (see [19]) if $\lambda(a) \geq t$.
(ii) *quasi-coincident* with $\lambda$, denoted by $\langle a/t\rangle\, q\, \lambda$, (see [19]) if $\lambda(a) + t > 1$.

If $\langle a/t\rangle\, \alpha\, \lambda$ is not established for $\alpha \in \{\in, q\}$, it is denoted by $\langle a/t\rangle\, \overline{\alpha}\, \lambda$.

**Definition 2.5.** A fuzzy set $\lambda$ in a hoop $H$ is called a *fuzzy filter* of $H$ (see [7]) if it satisfies:

$$(\forall a \in H)(\lambda(1) \geq \lambda(a)), \tag{2.12}$$

$$(\forall a, b \in H)(\lambda(b) \geq \min\{\lambda(a), \lambda(a \to b)\}). \tag{2.13}$$

**Definition 2.6** ([16])**.** Let $\lambda$ be a fuzzy set in a set $H$ and let $\delta \in (0, 1)$. A function

$$Ł_\lambda^\delta : H \to [0, 1],\ x \mapsto \max\{0, \lambda(x) + \delta - 1\}$$

is called the *Łukasiewicz fuzzy set* of $\lambda$ in $H$.

Let $\lambda$ be a fuzzy set in a set $H$. For the Łukasiewicz fuzzy set $Ł_\lambda^\delta$ of $\lambda$ in $H$ and $t \in (0, 1]$, consider the sets

$$(Ł_\lambda^\delta, t)_\in := \{x \in H \mid \langle x/t\rangle \in Ł_\lambda^\delta\} \text{ and } (Ł_\lambda^\delta, t)_q := \{x \in H \mid \langle x/t\rangle\, q\, Ł_\lambda^\delta\},$$

which are called the $\in$-*set* and $q$-*set*, respectively, of $Ł_\lambda^\delta$ (with value $t$). Also, consider a set:

$$O(Ł_\lambda^\delta) := \{x \in H \mid Ł_\lambda^\delta(x) > 0\} \tag{2.14}$$

which is called the $O$-*set* of $Ł_\lambda^\delta$. It is observed that

$$O(Ł_\lambda^\delta) = \{x \in H \mid \lambda(x) + \delta - 1 > 0\}.$$

## 3. Łukasiewicz fuzzy filters

In this section, we introduce the notion of Łukasiewicz fuzzy filter of hoops and investigate some properties of it.

**Definition 3.1.** Let $\lambda$ be a fuzzy set in $H$. Then its Łukasiewicz fuzzy set $Ł_\lambda^\delta$ in $H$ is called a *Łukasiewicz fuzzy filter* of $H$ if it satisfies:

$$(\forall x, y \in H)(\forall t_a, t_b \in (0, 1]) \left( \begin{array}{c} \langle x/t_a\rangle \in Ł_\lambda^\delta,\ \langle y/t_b\rangle \in Ł_\lambda^\delta \\ \Rightarrow \langle (x \odot y)/\min\{t_a, t_b\}\rangle \in Ł_\lambda^\delta \end{array} \right), \tag{3.1}$$

$$(\forall x, y \in H)(\forall t \in (0, 1])\left(x \leq y,\ \langle x/t\rangle \in Ł_\lambda^\delta\ \Rightarrow \langle y/t\rangle \in Ł_\lambda^\delta\right). \tag{3.2}$$

TABLE 1. Cayley table for the binary operation "$\odot$"

| $\odot$ | 0 | $a_1$ | $a_2$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a_1$ | 0 | 0 | $a_1$ | $a_1$ |
| $a_2$ | 0 | $a_1$ | $a_2$ | $a_2$ |
| 1 | 0 | $a_1$ | $a_2$ | 1 |

TABLE 2. Cayley table for the binary operation "$\odot$"

| $\rightarrow$ | 0 | $a_1$ | $a_2$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| $a_1$ | $a_1$ | 1 | 1 | 1 |
| $a_2$ | 0 | $a_1$ | 1 | 1 |
| 1 | 0 | $a_1$ | $a_2$ | 1 |

**Example 3.2.** Let $H = \{0, a_1, a_2, 1\}$ be a set with binary operations "$\odot$" and "$\rightarrow$" in Table 1 and Table 2, respectively.
Then $H$ is a (bounded) hoop. Define a fuzzy set $\lambda$ in $H$ as follows:

$$\lambda : H \rightarrow [0,1],\ x \mapsto \begin{cases} 0.59 \text{ if } x = 0, \\ 0.59 \text{ if } x = a_1, \\ 0.78 \text{ if } x = a_2, \\ 0.89 \text{ if } x = 1 \end{cases}$$

Given $\delta := 0.63$, the Łukasiewicz fuzzy set $Ł_\lambda^\delta$ of $\lambda$ in $H$ is given as follows:

$$Ł_\lambda^\delta : H \rightarrow [0,1],\ x \mapsto \begin{cases} 0.22 & \text{if } x = 0, \\ 0.22 & \text{if } x = a_1, \\ 0.41 & \text{if } x = a_2, \\ 0.52 & \text{if } x = 1. \end{cases}$$

It is routine to verify that $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy filter of $H$.

**Theorem 3.3.** *Let $\lambda$ be a fuzzy set in $H$. Then its Łukasiewicz fuzzy set $Ł_\lambda^\delta$ in $H$ is a Łukasiewicz fuzzy filter of $H$ if and only if the following conditions are valid.*

$$Ł_\lambda^\delta(1) \text{ is an upper bound of } \{Ł_\lambda^\delta(x) \mid x \in H\}, \tag{3.3}$$

$$(\forall x, y \in H)(\forall t_a, t_b \in (0,1]) \left( \begin{array}{c} \langle x/t_a \rangle \in Ł_\lambda^\delta,\ \langle (x \rightarrow y)/t_b \rangle \in Ł_\lambda^\delta \\ \Rightarrow \langle y/\min\{t_a, t_b\} \rangle \in Ł_\lambda^\delta \end{array} \right). \tag{3.4}$$

*Proof.* Suppose that $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy filter of $H$. If $Ł_\lambda^\delta(1)$ is not an upper bound of $\{Ł_\lambda^\delta(x) \mid x \in H\}$, then $Ł_\lambda^\delta(1) < Ł_\lambda^\delta(z)$ for some $z \in H$. Since $z \leq 1$ and $\langle z/Ł_\lambda^\delta(z)\rangle \in Ł_\lambda^\delta$, it follows from (3.2) that $\langle 1/Ł_\lambda^\delta(z)\rangle \in Ł_\lambda^\delta$, i.e., $Ł_\lambda^\delta(1) \geq Ł_\lambda^\delta(z)$. This is a contradiction, and thus $Ł_\lambda^\delta(1)$ is an upper bound of $\{Ł_\lambda^\delta(x) \mid x \in H\}$. Let $x, y \in H$ and $t_a, t_b \in (0, 1]$ be such that $\langle x/t_a\rangle \in Ł_\lambda^\delta$ and $\langle (x \to y)/t_b\rangle \in Ł_\lambda^\delta$. Then

$$\langle ((x \to y) \odot x)/\min\{t_a, t_b\}\rangle \in Ł_\lambda^\delta$$

by (3.1). Since $(x \to y) \odot x \leq y$, it follows from (3.2) that

$$\langle y/\min\{t_a, t_b\}\rangle \in Ł_\lambda^\delta.$$

Assume that $Ł_\lambda^\delta$ satisfies (3.3) and (3.4). Let $x, y \in H$ and $t \in (0, 1]$ be such that $x \leq y$ and $\langle x/t\rangle \in Ł_\lambda^\delta$. Then

$$Ł_\lambda^\delta(x \to y) = Ł_\lambda^\delta(1) \geq Ł_\lambda^\delta(x) \geq t$$

by (3.3), that is, $\langle (x \to y)/t\rangle \in Ł_\lambda^\delta$. Using (3.4) leads to $\langle y/t\rangle \in Ł_\lambda^\delta$, which proves (3.2). Let $x, y \in H$ and $t_a, t_b \in (0, 1]$ be such that $\langle x/t_a\rangle \in Ł_\lambda^\delta$ and $\langle y/t_b\rangle \in Ł_\lambda^\delta$. Since

$$x \to (y \to x \odot y) = x \odot y \to x \odot y = 1$$

by (H1) and (H3), we have $Ł_\lambda^\delta(x \to (y \to x \odot y)) = Ł_\lambda^\delta(1) \geq Ł_\lambda^\delta(y) \geq t_b$, that is, $\langle (x \to (y \to x \odot y))/t_b\rangle \in Ł_\lambda^\delta$. Hence

$$\langle (y \to x \odot y)/\min\{t_a, t_b\}\rangle \in Ł_\lambda^\delta$$

by (3.4). Using (3.4) again leads to $\langle (x \odot y)/\min\{t_a, t_b\}\rangle \in Ł_\lambda^\delta$, which shows (3.1). Therefore, $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy filter of $H$. □

**Theorem 3.4.** *Let $\lambda$ be a fuzzy set in $H$. Then its Łukasiewicz fuzzy set $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy filter of $H$ if and only if it satisfies:*

$$(\forall x \in H)(\forall t \in (0, 1])\left(\langle x/t\rangle \in Ł_\lambda^\delta \;\Rightarrow\; \langle 1/t\rangle \in Ł_\lambda^\delta\right), \tag{3.5}$$

$$(\forall x, y \in H)(Ł_\lambda^\delta(y) \geq \min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(x \to y)\}). \tag{3.6}$$

*Proof.* Assume that $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy filter of $H$. Let $x \in H$ and $t \in (0, 1]$ be such that $\langle x/t\rangle \in Ł_\lambda^\delta$. Using (3.3) leads to $Ł_\lambda^\delta(1) \geq Ł_\lambda^\delta(x) \geq t$, and so $\langle 1/t\rangle \in Ł_\lambda^\delta$. Note that $\langle x/Ł_\lambda^\delta(x)\rangle \in Ł_\lambda^\delta$ and

$$\langle (x \to y)/Ł_\lambda^\delta(x \to y)\rangle \in Ł_\lambda^\delta$$

for all $x, y \in H$. It follows from (3.4) that

$$\langle y/\min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(x \to y)\}\rangle \in Ł_\lambda^\delta,$$

and hence $Ł_\lambda^\delta(y) \geq \min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(x \to y)\}$ for all $x, y \in H$.

Conversely, suppose that $Ł_\lambda^\delta$ satisfies (3.5) and (3.6). Since

$$\langle x/Ł_\lambda^\delta(x)\rangle \in Ł_\lambda^\delta$$

for all $x \in H$, we have $\langle 1/Ł_\lambda^\delta(x)\rangle \in Ł_\lambda^\delta$ and so $Ł_\lambda^\delta(1) \geq Ł_\lambda^\delta(x)$ for all $x \in H$ by (3.5). Hence $Ł_\lambda^\delta(1)$ is an upper bound of $\{Ł_\lambda^\delta(x) \mid x \in H\}$. Let $x, y \in H$ and $t_a, t_b \in (0, 1]$ be such that $\langle x/t_a\rangle \in Ł_\lambda^\delta$ and $\langle (x \to y)/t_b\rangle \in Ł_\lambda^\delta$. Then $Ł_\lambda^\delta(x) \geq t_a$ and $Ł_\lambda^\delta(x \to y) \geq t_b$, which imply from (3.6) that

$$Ł_\lambda^\delta(y) \geq \min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(x \to y)\} \geq \min\{t_a, t_b\}.$$

Thus $\langle y/\min\{t_a, t_b\}\rangle \in Ł_\lambda^\delta$. Therefore $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy filter of $H$ by Theorem 3.3. □

**Proposition 3.5.** *Every Łukasiewicz fuzzy filter $Ł_\lambda^\delta$ of $H$ satisfies:*

$$(\forall x, y, z \in H)(\forall t_a, t_c \in (0, 1]) \left( \begin{array}{l} z \leq x \to y,\ \langle x/t_a\rangle \in Ł_\lambda^\delta,\ \langle z/t_c\rangle \in Ł_\lambda^\delta \\ \Rightarrow\ \langle y/\min\{t_a, t_c\}\rangle \in Ł_\lambda^\delta \end{array} \right). \tag{3.7}$$

$$(\forall x, y, z \in H)(\forall t_a, t_b \in (0, 1]) \left( \begin{array}{l} x \leq y \to z,\ \langle x/t_a\rangle \in Ł_\lambda^\delta,\ \langle y/t_b\rangle \in Ł_\lambda^\delta \\ \Rightarrow \langle z/\min\{t_a, t_b\}\rangle \in Ł_\lambda^\delta \end{array} \right). \tag{3.8}$$

$$(\forall x, y, z \in H)(\forall t_a, t_b \in (0, 1]) \left( \begin{array}{l} \langle (x \to y)/t_a\rangle \in Ł_\lambda^\delta,\ \langle (y \to z)/t_b\rangle \in Ł_\lambda^\delta \\ \Rightarrow \langle (x \to z)/\min\{t_a, t_b\}\rangle \in Ł_\lambda^\delta \end{array} \right). \tag{3.9}$$

$$(\forall x, y, z \in H)(\forall t_a, t_c \in (0, 1]) \left( \begin{array}{l} \langle (x \to y)/t_a\rangle \in Ł_\lambda^\delta,\ \langle (x \odot z)/t_c\rangle \in Ł_\lambda^\delta \\ \Rightarrow \langle (y \odot z)/\min\{t_a, t_c\}\rangle \in Ł_\lambda^\delta \end{array} \right). \tag{3.10}$$

*Proof.* Let $x, y, z \in H$ and $t_a, t_c \in (0, 1]$ be such that $z \leq x \to y$, $\langle x/t_a\rangle \in Ł_\lambda^\delta$ and $\langle z/t_c\rangle \in Ł_\lambda^\delta$. Then $z \to (x \to y) = 1$, $Ł_\lambda^\delta(x) \geq t_a$ and $Ł_\lambda^\delta(z) \geq t_c$. Hence

$$\begin{aligned} Ł_\lambda^\delta(y) &\geq \min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(x \to y)\} \\ &\geq \min\{Ł_\lambda^\delta(x), \min\{Ł_\lambda^\delta(z \to (x \to y)), Ł_\lambda^\delta(z)\}\} \\ &= \min\{Ł_\lambda^\delta(x), \min\{Ł_\lambda^\delta(1), Ł_\lambda^\delta(z)\}\} \\ &= \min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(z)\} \\ &\geq \min\{t_a, t_c\}, \end{aligned}$$

and so $\langle y/\min\{t_a, t_c\}\rangle \in Ł_\lambda^\delta$. Therefore (3.7) is valid. Let $x, y, z \in H$ and $t_a, t_b \in (0, 1]$ be such that $x \leq y \to z$, $\langle x/t_a\rangle \in Ł_\lambda^\delta$, and $\langle y/t_b\rangle \in Ł_\lambda^\delta$. Then $x \odot y \leq z$ by (2.1) and $\langle (x \odot y)/\min\{t_a, t_b\}\rangle \in Ł_\lambda^\delta$ by (3.1), and hence $\langle z/\min\{t_a, t_b\}\rangle \in Ł_\lambda^\delta$ by (3.2). This proves (3.8). Let $x, y, z \in H$

and $t_a, t_b \in (0,1]$ be such that $\langle (x \to y)/t_a \rangle \in Ł_\lambda^\delta$ and $\langle (y \to z)/t_b \rangle \in Ł_\lambda^\delta$. Then

$$\langle ((x \to y) \odot (y \to z))/\min\{t_a, t_b\} \rangle \in Ł_\lambda^\delta$$

by (3.1). Since $(x \to y) \odot (y \to z) \leq x \to z$, it follows from (3.2) that

$$\langle (x \to z)/\min\{t_a, t_b\} \rangle \in Ł_\lambda^\delta.$$

Hence (3.9) is valid. Let $x, y, z \in H$ and $t_a, t_c \in (0,1]$ be such that $\langle (x \to y)/t_a \rangle \in Ł_\lambda^\delta$ and $\langle (x \odot z)/t_c \rangle \in Ł_\lambda^\delta$. Using (3.1) leads to

$$\langle (z \odot x \odot (x \to y))/\min\{t_a, t_c\} \rangle \in Ł_\lambda^\delta.$$

Since $x \odot (x \to y) \leq y$, we have $z \odot x \odot (x \to y) \leq z \odot y = y \odot z$ by (2.6) and the commutativity of $\odot$. Hence

$$\langle (y \odot z)/\min\{t_a, t_c\} \rangle \in Ł_\lambda^\delta$$

by (3.2), and so (3.10) is valid. □

**Theorem 3.6.** *If $\lambda$ is a fuzzy filter of $H$, then its Łukasiewicz fuzzy set $Ł_\lambda^\delta$ in $H$ is a Łukasiewicz fuzzy filter of $H$.*

*Proof.* Let $Ł_\lambda^\delta$ be a Łukasiewicz fuzzy set of a fuzzy filter $\lambda$ in $H$. Then

$$Ł_\lambda^\delta(1) = \max\{0, \lambda(1) + \delta - 1\} \geq \max\{0, \lambda(x) + \delta - 1\} = Ł_\lambda^\delta(x)$$

for all $x \in H$. Hence $Ł_\lambda^\delta(1)$ is an upper bound of $\{Ł_\lambda^\delta(x) \mid x \in H\}$. Let $x, y \in H$ and $t_a, t_b \in (0,1]$ be such that $\langle x/t_a \rangle \in Ł_\lambda^\delta$ and $\langle (x \to y)/t_b \rangle \in Ł_\lambda^\delta$. Then $Ł_\lambda^\delta(x) \geq t_a$ and $Ł_\lambda^\delta(x \to y) \geq t_b$, which imply that

$$\begin{aligned}
Ł_\lambda^\delta(y) &= \max\big\{0, \lambda(y) + \delta - 1\big\} \\
&\geq \max\big\{0, \min\{\lambda(x), \lambda(x \to y)\} + \delta - 1\big\} \\
&= \max\big\{0, \min\{\lambda(x) + \delta - 1, \lambda(x \to y) + \delta - 1\}\big\} \\
&= \min\big\{\max\{0, \lambda(x) + \delta - 1\}, \max\{0, \lambda(x \to y) + \delta - 1\}\big\} \\
&= \min\big\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(x \to y)\big\} \\
&\geq \min\big\{t_a, t_b\big\}.
\end{aligned}$$

Hence $\langle y/\min\{t_a, t_b\} \rangle \in Ł_\lambda^\delta$, and therefore $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy filter of $H$ by Theorem 3.3. □

From the perspective of Theorem 3.6, the Łukasiewicz fuzzy filter can be said to be a generalization of fuzzy filters.

The converse of Theorem 3.6 may not be true as seen in the example below.

Table 3. Cayley table for the binary operation "$\odot$"

| $\odot$ | $0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $1$ |
|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $a_1$ | $0$ | $a_1$ | $a_4$ | $0$ | $a_4$ | $a_1$ |
| $a_2$ | $0$ | $a_4$ | $a_3$ | $a_3$ | $0$ | $a_2$ |
| $a_3$ | $0$ | $0$ | $a_3$ | $a_3$ | $0$ | $a_3$ |
| $a_4$ | $0$ | $a_4$ | $0$ | $0$ | $0$ | $a_4$ |
| $1$ | $0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $1$ |

Table 4. Cayley table for the binary operation "$\to$"

| $\to$ | $0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $1$ |
|---|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a_1$ | $a_3$ | $1$ | $a_2$ | $a_3$ | $a_2$ | $1$ |
| $a_2$ | $a_4$ | $a_1$ | $1$ | $a_2$ | $a_1$ | $1$ |
| $a_3$ | $a_1$ | $a_1$ | $1$ | $1$ | $a_1$ | $1$ |
| $a_4$ | $a_2$ | $1$ | $1$ | $a_2$ | $1$ | $1$ |
| $1$ | $0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $1$ |

**Example 3.7.** Let $H = \{0, a_1, a_2, a_3, a_4, 1\}$ be a set with binary operations " $\odot$ " and " $\to$ " in Table 3 and Table 4, respectively.

Then $(H, \odot, \to, 1)$ is a hoop (see [6]). Define a fuzzy set $\lambda$ in $H$ as follows:

$$\lambda : H \to [0,1],\ x \mapsto \begin{cases} 0.38 \text{ if } x = 0, \\ 0.42 \text{ if } x = a_1, \\ 0.93 \text{ if } x = a_2, \\ 0.93 \text{ if } x = a_3, \\ 0.37 \text{ if } x = a_4, \\ 0.99 \text{ if } x = 1. \end{cases}$$

Given $\delta := 0.54$, the Łukasiewicz fuzzy set $Ł_\lambda^\delta$ of $\lambda$ in $H$ is given as follows:

$$Ł_\lambda^\delta : H \to [0,1],\ x \mapsto \begin{cases} 0.53 & \text{if } x = 1, \\ 0.47 & \text{if } x \in \{a_2, a_3\}, \\ 0.00 & \text{if } x \in \{0, a_1, a_4\}. \end{cases}$$

It is routine to check that $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy filter of $H$. But $\lambda$ is not a fuzzy filter of $H$ since

$$\lambda(a_4) = 0.37 \ngeq 0.42 = \min\{\lambda(a_1), \lambda(a_1 \to a_4)\}.$$

**Theorem 3.8.** *If a Łukasiewicz fuzzy set* $Ł_\lambda^\delta$ *in* $H$ *satisfies* (3.3) *and* (3.8)*, then* $Ł_\lambda^\delta$ *is a Łukasiewicz fuzzy filter of* $H$*.*

*Proof.* Let $x, y \in H$ and $t_a, t_b \in (0, 1]$ be such that $\langle x/t_a \rangle \in Ł_\lambda^\delta$ and $\langle (x \to y)/t_b \rangle \in Ł_\lambda^\delta$. Since $x \to y \le x \to y$, we have $\langle y/\min\{t_a, t_b\} \rangle \in Ł_\lambda^\delta$ by (3.8). Hence $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy filter of $H$ by Theorem 3.3. □

**Theorem 3.9.** *If a Łukasiewicz fuzzy set* $Ł_\lambda^\delta$ *in* $H$ *satisfies* (3.3) *and* (3.9)*, then* $Ł_\lambda^\delta$ *is a Łukasiewicz fuzzy filter of* $H$*.*

*Proof.* Let $y, z \in H$ and $t_b, t_c \in (0, 1]$ be such that $\langle y/t_b \rangle \in Ł_\lambda^\delta$ and $\langle (y \to z)/t_c \rangle \in Ł_\lambda^\delta$. Since $\langle (1 \to y)/t_b \rangle = \langle y/t_b \rangle \in Ł_\lambda^\delta$, it follows from (2.4) and (3.9) that

$$\langle z/\min\{t_b, t_c\} \rangle = \langle (1 \to z)/\min\{t_b, t_c\} \rangle \in Ł_\lambda^\delta.$$

Hence $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy filter of $H$ by Theorem 3.3. □

**Theorem 3.10.** *If a Łukasiewicz fuzzy set* $Ł_\lambda^\delta$ *in* $H$ *satisfies* (3.3) *and* (3.10)*, then* $Ł_\lambda^\delta$ *is a Łukasiewicz fuzzy filter of* $H$*.*

*Proof.* Let $y, z \in H$ and $t_b, t_c \in (0, 1]$ be such that $\langle y/t_b \rangle \in Ł_\lambda^\delta$ and $\langle z/t_c \rangle \in Ł_\lambda^\delta$. Then $\langle (1 \to y)/t_b \rangle = \langle y/t_b \rangle \in Ł_\lambda^\delta$ and

$$\langle (1 \odot z)/t_c \rangle = \langle z/t_c \rangle \in Ł_\lambda^\delta.$$

It follows from (3.10) that $\langle (y \odot z)/\min\{t_b, t_c\} \rangle \in Ł_\lambda^\delta$. Let $x, y \in H$ and $t \in (0, 1]$ be such that $x \le y$ and $\langle x/t \rangle \in Ł_\lambda^\delta$. The condition (3.3) induces

$$Ł_\lambda^\delta(x \to y) = Ł_\lambda^\delta(1) \ge Ł_\lambda^\delta(x) \ge t,$$

that is, $\langle (x \to y)/t \rangle \in Ł_\lambda^\delta$. Since $\langle (x \odot 1)/t \rangle = \langle x/t \rangle \in Ł_\lambda^\delta$, we have $\langle y/t \rangle = \langle (y \odot 1)/t \rangle \in Ł_\lambda^\delta$ by (3.10). Therefore $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy filter of $H$. □

**Theorem 3.11.** *Let* $Ł_\lambda^\delta$ *be a Łukasiewicz fuzzy set in* $H$ *satisfying the condition* (3.3)*. If it satisfies:*

$$(\forall x, y, z \in H)(\forall t_a, t_b \in (0, 0.5]) \left( \begin{array}{l} x \le y \to z,\ \langle x/t_a \rangle \in Ł_\lambda^\delta,\ \langle y/t_b \rangle \in Ł_\lambda^\delta \\ \Rightarrow \langle z/\min\{t_a, t_b\} \rangle \, q \, Ł_\lambda^\delta \end{array} \right), \tag{3.11}$$

*or*

$$(\forall x, y, z \in H)(\forall t_a, t_b \in (0.5, 1]) \left( \begin{array}{l} x \le y \to z,\ \langle x/t_a \rangle \, q \, Ł_\lambda^\delta,\ \langle y/t_b \rangle \, q \, Ł_\lambda^\delta \\ \Rightarrow \langle z/\min\{t_a, t_b\} \rangle \in Ł_\lambda^\delta \end{array} \right), \tag{3.12}$$

*then* ${}_\lambda^\delta$ *is a Łukasiewicz fuzzy filter of* $H$*.*

*Proof.* Let $Ł_\lambda^\delta$ be a Łukasiewicz fuzzy set in $H$ satisfying the condition (3.3). Assume that $Ł_\lambda^\delta$ satisfies the condition (3.11). Let $x, y \in H$ and $t_a, t_b \in (0, 0.5] \subseteq (0, 1]$ be such that $\langle x/t_a\rangle \in Ł_\lambda^\delta$ and $\langle (x \to y)/t_b\rangle \in Ł_\lambda^\delta$. Since $x \to y \leq x \to y$, we have $\langle y/\min\{t_a, t_b\}\rangle \, q \, Ł_\lambda^\delta$ by (3.11). Since $\min\{t_a, t_b\} \leq 0.5$, it follows that

$$Ł_\lambda^\delta(y) > 1 - \min\{t_a, t_b\} \geq \min\{t_a, t_b\},$$

i.e., $\langle y/\min\{t_a, t_b\}\rangle \in Ł_\lambda^\delta$. Therefore, $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy filter of $H$ by Theorem 3.3. Suppose that $Ł_\lambda^\delta$ satisfies the condition (3.12), and let $x, y \in H$ and $t_a, t_b \in (0.5, 1] \subseteq (0, 1]$ be such that $\langle x/t_a\rangle \in Ł_\lambda^\delta$ and $\langle (x \to y)/t_b\rangle \in Ł_\lambda^\delta$. Then

$$Ł_\lambda^\delta(x) \geq t_a > 1 - t_a \text{ and } Ł_\lambda^\delta(x \to y) \geq t_b > 1 - t_b$$

since $t_a > 0.5$ and $t_b > 0.5$. Thus $\langle x/t_a\rangle \, q \, Ł_\lambda^\delta$ and $\langle (x \to y)/t_b\rangle \, q \, Ł_\lambda^\delta$. Since $x \to y \leq x \to y$, we have $\langle y/\min\{t_a, t_b\}\rangle \in Ł_\lambda^\delta$ by (3.12). Therefore, $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy filter of $H$ by Theorem 3.3. □

**Theorem 3.12.** *Let $Ł_\lambda^\delta$ be a Łukasiewicz fuzzy set in $H$ satisfying the condition* (3.3). *If it satisfies:*

$$(\forall x, y, z \in H)(\forall t_a, t_b \in (0, 0.5]) \left( \begin{array}{l} \langle (x \to y)/t_a\rangle \in Ł_\lambda^\delta, \langle (y \to z)/t_b\rangle \in Ł_\lambda^\delta \\ \Rightarrow \langle (x \to z)/\min\{t_a, t_b\}\rangle \, q \, Ł_\lambda^\delta \end{array} \right) \tag{3.13}$$

*or*

$$(\forall x, y, z \in H)(\forall t_a, t_b \in (0.5, 1]) \left( \begin{array}{l} \langle (x \to y)/t_a\rangle \, q \, Ł_\lambda^\delta, \langle (y \to z)/t_b\rangle \, q \, Ł_\lambda^\delta \\ \Rightarrow \langle (x \to z)/\min\{t_a, t_b\}\rangle \in Ł_\lambda^\delta \end{array} \right), \tag{3.14}$$

*then $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy filter of $H$.*

*Proof.* Assume that $Ł_\lambda^\delta$ satisfies (3.13). Let $y, z \in H$ and

$$t_b, t_c \in (0, 0.5] \subseteq (0, 1]$$

be such that $\langle y/t_b\rangle \in Ł_\lambda^\delta$ and $\langle (y \to z)/t_c\rangle \in Ł_\lambda^\delta$. Then

$$\langle (1 \to y)/t_b\rangle = \langle y/t_b\rangle \in Ł_\lambda^\delta,$$

which implies from (2.4) and (3.13) that

$$\langle z/\min\{t_b, t_c\}\rangle = \langle (1 \to z)/\min\{t_b, t_c\}\rangle \, q \, Ł_\lambda^\delta.$$

Hence

$$Ł_\lambda^\delta(z) > 1 - \min\{t_b, t_c\} \geq \min\{t_b, t_c\}$$

since $\min\{t_b, t_c\} < 0.5$. Thus $\langle z/\min\{t_b, t_c\}\rangle \in Ł^\delta_\lambda$, and therefore $Ł^\delta_\lambda$ is a Łukasiewicz fuzzy filter of $H$. Now, suppose that $Ł^\delta_\lambda$ satisfies (3.14). Let $y, z \in H$ and $t_b, t_c \in (0.5, 1] \subseteq (0, 1]$ be such that $\langle y/t_b\rangle \in Ł^\delta_\lambda$ and $\langle (y \to z)/t_c\rangle \in Ł^\delta_\lambda$. Since $t_b > 0.5$ and $t_c > 0.5$, it follows that

$$Ł^\delta_\lambda(1 \to y) = Ł^\delta_\lambda(y) \geq t_b > 1 - t_b$$

and $Ł^\delta_\lambda(y \to z) \geq t_c > 1 - t_c$, i.e., $\langle (1 \to y)/t_b\rangle \, q \, Ł^\delta_\lambda$ and $\langle (y \to z)/t_c\rangle \, q \, Ł^\delta_\lambda$. Hence

$$\langle z/\min\{t_b, t_c\}\rangle = \langle (1 \to z)/\min\{t_b, t_c\}\rangle \in Ł^\delta_\lambda$$

by (2.4) and (3.14). Consequently, $Ł^\delta_\lambda$ is a Łukasiewicz fuzzy filter of $H$. □

**Theorem 3.13.** *Let $Ł^\delta_\lambda$ be a Łukasiewicz fuzzy set in $H$ satisfying the condition (3.3). If it satisfies:*

$$(\forall x, y, z \in H)(\forall t_a, t_c \in (0, 0.5]) \left( \begin{array}{l} \langle (x \to y)/t_a\rangle \in Ł^\delta_\lambda,\ \langle (x \odot z)/t_c\rangle \in Ł^\delta_\lambda \\ \Rightarrow \langle (y \odot z)/\min\{t_a, t_c\}\rangle \, q \, Ł^\delta_\lambda \end{array} \right) \tag{3.15}$$

*or*

$$(\forall x, y, z \in H)(\forall t_a, t_c \in (0.5, 1]) \left( \begin{array}{l} \langle (x \to y)/t_a\rangle \, q \, Ł^\delta_\lambda,\ \langle (x \odot z)/t_c\rangle \, q \, Ł^\delta_\lambda \\ \Rightarrow \langle (y \odot z)/\min\{t_a, t_c\}\rangle \in {}^\delta_\lambda \end{array} \right), \tag{3.16}$$

*then $Ł^\delta_\lambda$ is a Łukasiewicz fuzzy filter of $H$.*

*Proof.* If $Ł^\delta_\lambda$ satisfies (3.15), let $t_b, t_c \in (0, 0.5] \subseteq (0, 1]$ and $y, z \in H$ be such that $\langle y/t_b\rangle \in Ł^\delta_\lambda$ and $\langle z/t_c\rangle \in Ł^\delta_\lambda$. Then $\langle (1 \to y)/t_b\rangle = \langle y/t_b\rangle \in Ł^\delta_\lambda$ and $\langle (1 \odot z)/t_c\rangle = \langle z/t_c\rangle \in Ł^\delta_\lambda$. Hence $\langle (y \odot z)/\min\{t_b, t_c\}\rangle \, q \, Ł^\delta_\lambda$ by (3.15). Since $\min\{t_b, t_c\} \leq 0.5$, it follows that

$$Ł^\delta_\lambda(y \odot z) > 1 - \min\{t_b, t_c\} \geq \min\{t_b, t_c\},$$

that is, $\langle (y \odot z)/\min\{t_b, t_c\}\rangle \in Ł^\delta_\lambda$. Let $x, y \in H$ and

$$t \in (0, 0.5] \subseteq (0, 1]$$

be such that $x \leq y$ and $\langle x/t\rangle \in Ł^\delta_\lambda$. The condition (3.3) induces

$$Ł^\delta_\lambda(x \to y) = Ł^\delta_\lambda(1) \geq Ł^\delta_\lambda(x) \geq t,$$

that is, $\langle (x \to y)/t\rangle \in Ł^\delta_\lambda$. Since $\langle (x \odot 1)/t\rangle = \langle x/t\rangle \in Ł^\delta_\lambda$, we have $\langle y/t\rangle = \langle (y \odot 1)/t\rangle \, q \, Ł^\delta_\lambda$ by (3.15), and so $Ł^\delta_\lambda(y) > 1 - t \geq t$ since $t \leq 0.5$. Thus $\langle y/t\rangle \in Ł^\delta_\lambda$. Therefore $Ł^\delta_\lambda$ is a Łukasiewicz fuzzy filter of $H$. Now, assume that $Ł^\delta_\lambda$ satisfies (3.16). Let $t_b, t_c \in (0.5, 1] \subseteq (0, 1]$ and $y, z \in H$ be such that $\langle y/t_b\rangle \in Ł^\delta_\lambda$ and $\langle z/t_c\rangle \in Ł^\delta_\lambda$. Then

$$\langle (1 \to y)/t_b\rangle = \langle y/t_b\rangle \in Ł^\delta_\lambda$$

and $\langle(1\odot z)/t_c\rangle = \langle z/t_c\rangle \in Ł_\lambda^\delta$, which imply that $Ł_\lambda^\delta(1\to y)\geq t_b > 1-t_b$ and $Ł_\lambda^\delta(1\odot z)\geq t_c > 1-t_c$ since $t_b,t_c\in(0.5,1]$, that is, $\langle(1\to y)/t_b\rangle\, q\, Ł_\lambda^\delta$ and $\langle(1\odot z)/t_c\rangle\, q\, Ł_\lambda^\delta$. Using (3.16) leads to $\langle(y\odot z)/\min\{t_b,t_c\}\rangle\in Ł_\lambda^\delta$. Let $x,y\in H$ and $t\in(0.5,1]\subseteq(0,1]$ be such that $x\leq y$ and $\langle x/t\rangle\in Ł_\lambda^\delta$. The condition (3.3) induces

$$Ł_\lambda^\delta(x\to y) = Ł_\lambda^\delta(1)\geq Ł_\lambda^\delta(x)\geq t > 1-t,$$

that is, $\langle(x\to y)/t\rangle\, q\, Ł_\lambda^\delta$. Since $Ł_\lambda^\delta(x\odot 1) = Ł_\lambda^\delta(x)\geq t > 1-t$, we get $\langle(x\odot 1)/t\rangle\, q\, Ł_\lambda^\delta$. Hence $\langle y/t\rangle = \langle(y\odot 1)/t\rangle\in Ł_\lambda^\delta$ by (3.16), and therefore $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy filter of $H$. □

We explore the conditions under which $\in$-set, $q$-set, and $O$-set of the Łukasiewicz fuzzy set can be filters.

**Theorem 3.14.** *Let $Ł_\lambda^\delta$ be a Łukasiewicz fuzzy set in $H$. Then the nonempty $\in$-set $(Ł_\lambda^\delta, t)_\in$ of $Ł_\lambda^\delta$ is a filter of $H$ for all $t\in(0.5,1]$ if and only if $Ł_\lambda^\delta$ satisfies:*

$$(\forall x,y\in H)\left(\min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(y)\}\leq\max\{Ł_\lambda^\delta(x\odot y), 0.5\}\right), \tag{3.17}$$

$$(\forall x,y\in H)\left(x\leq y\ \Rightarrow\ Ł_\lambda^\delta(x)\leq\max\{Ł_\lambda^\delta(y), 0.5\}\right). \tag{3.18}$$

*Proof.* Assume that the nonempty $\in$-set $(Ł_\lambda^\delta, t)_\in$ of $Ł_\lambda^\delta$ is a filter of $H$ for all $t\in(0.5,1]$. If there exist $a,b\in H$ such that

$$\min\{Ł_\lambda^\delta(a), Ł_\lambda^\delta(b)\} > \max\{Ł_\lambda^\delta(a\odot b), 0.5\},$$

then $Ł_\lambda^\delta(a), Ł_\lambda^\delta(b)\in(0.5,1]$ and

$$Ł_\lambda^\delta(a\odot b) < \min\{Ł_\lambda^\delta(a), Ł_\lambda^\delta(b)\}.$$

Since $\langle a/Ł_\lambda^\delta(a)\rangle\in Ł_\lambda^\delta$ and $\langle b/Ł_\lambda^\delta(b)\rangle\in Ł_\lambda^\delta$, we get $a,b\in(Ł_\lambda^\delta, t)_\in$ and so $a\odot b\in(Ł_\lambda^\delta, t)_\in$ where $t:=\min\{Ł_\lambda^\delta(a), Ł_\lambda^\delta(b)\}$. Hence

$$Ł_\lambda^\delta(a\odot b)\geq t = \min\{Ł_\lambda^\delta(a), Ł_\lambda^\delta(b)\},$$

a contradiction. Thus $\min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(y)\}\leq\max\{Ł_\lambda^\delta(x\odot y), 0.5\}$ for all $x,y\in H$. If (3.18) is not valid, then there exist $a,b\in H$ such that $a\leq b$ and $Ł_\lambda^\delta(a) > \max\{Ł_\lambda^\delta(b), 0.5\}$. Then $Ł_\lambda^\delta(a)\in(0.5,1]$ and $Ł_\lambda^\delta(b) < Ł_\lambda^\delta(a)$. Since $\langle a/Ł_\lambda^\delta(a)\rangle\in Ł_\lambda^\delta$, i.e., $a\in(Ł_\lambda^\delta, Ł_\lambda^\delta(a))_\in$, we get $b\in(Ł_\lambda^\delta, Ł_\lambda^\delta(a))_\in$. Thus $Ł_\lambda^\delta(b)\geq Ł_\lambda^\delta(a)$, a contradiction. Therefore $Ł_\lambda^\delta$ satisfies the condition (3.18).

Conversely, suppose that $Ł_\lambda^\delta$ satisfies two conditions (3.17) and (3.18). If $x,y\in(Ł_\lambda^\delta, t)_\in$ for $t\in(0.5,1]$, then

$$\max\{Ł_\lambda^\delta(x\odot y), 0.5\}\geq\min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(y)\}\geq t > 0.5$$

and thus $Ł_\lambda^\delta(x \odot y) \geq t$, i.e., $x \odot y \in (Ł_\lambda^\delta, t)_\in$. Now, let $x, y \in H$ be such that $x \leq y$ and $x \in (Ł_\lambda^\delta, t)_\in$ for $t \in (0.5, 1]$. Using (3.18) leads to $\max\{Ł_\lambda^\delta(y), 0.5\} \geq Ł_\lambda^\delta(x) \geq t > 0.5$, and so $Ł_\lambda^\delta(y) \geq t$, that is, $y \in (Ł_\lambda^\delta, t)_\in$. Consequently, $(Ł_\lambda^\delta, t)_\in$ is a filter of $H$ for all $t \in (0.5, 1]$. □

**Theorem 3.15.** *Let $Ł_\lambda^\delta$ be a Łukasiewicz fuzzy set in $H$. Then the nonempty $\in$-set $(Ł_\lambda^\delta, t)_\in$ of $Ł_\lambda^\delta$ is a filter of $H$ for all $t \in (0.5, 1]$ if and only if $Ł_\lambda^\delta$ satisfies:*

$$(\forall x \in H) \left( Ł_\lambda^\delta(x) \leq \max\{Ł_\lambda^\delta(1), 0.5\} \right), \tag{3.19}$$

$$(\forall x, y \in H) \left( \min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(x \to y)\} \leq \max\{Ł_\lambda^\delta(y), 0.5\} \right). \tag{3.20}$$

*Proof.* Assume that the nonempty $\in$-set $(Ł_\lambda^\delta, t)_\in$ of $Ł_\lambda^\delta$ is a filter of $H$ for all $t \in (0.5, 1]$. If $Ł_\lambda^\delta(a) > \max\{Ł_\lambda^\delta(1), 0.5\}$ for some $a \in H$, then $Ł_\lambda^\delta(a) \in (0.5, 1]$ and $1 \notin (Ł_\lambda^\delta, Ł_\lambda^\delta(a))_\in$. This is a contradiction, and so $Ł_\lambda^\delta(x) \leq \max\{Ł_\lambda^\delta(1), 0.5\}$ for all $x \in H$. If

$$\min\{Ł_\lambda^\delta(a), Ł_\lambda^\delta(a \to b)\} > \max\{Ł_\lambda^\delta(b), 0.5\}$$

for some $a, b \in H$, then $t_a := Ł_\lambda^\delta(a) \in (0.5, 1]$,

$$t_b := Ł_\lambda^\delta(a \to b) \in (0.5, 1],$$

and $a$, $a \to b \in (Ł_\lambda^\delta, t)_\in$ where $t = \min\{Ł_\lambda^\delta(a), Ł_\lambda^\delta(a \to b)\}$. Hence $b \in (Ł_\lambda^\delta, t)_\in$, and so $Ł_\lambda^\delta(b) \geq t$ which is a contradiction. Thus

$$\min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(x \to y)\} \leq \max\{Ł_\lambda^\delta(y), 0.5\}$$

for all $x, y \in H$.

Conversely, let $t \in (0.5, 1]$ and suppose that $Ł_\lambda^\delta$ satisfies two conditions (3.19) and (3.20). For every $x \in (Ł_\lambda^\delta, t)_\in$, we have

$$\max\{Ł_\lambda^\delta(1), 0.5\} \geq Ł_\lambda^\delta(x) \geq t > 0.5$$

and thus $Ł_\lambda^\delta(1) \geq t$, i.e., $1 \in (Ł_\lambda^\delta, t)_\in$. Let $x, y \in H$ be such that $x \in (Ł_\lambda^\delta, t)_\in$ and $x \to y \in (Ł_\lambda^\delta, t)_\in$. Using (3.20) leads to

$$\max\{Ł_\lambda^\delta(y), 0.5\} \geq \min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(x \to y)\} \geq t > 0.5$$

and so $Ł_\lambda^\delta(y) \geq t$, i.e., $y \in (Ł_\lambda^\delta, t)_\in$. Therefore, $(Ł_\lambda^\delta, t)_\in$ is a filter of $H$ for all $t \in (0.5, 1]$ by Proposition 2.4. □

**Theorem 3.16.** *If a Łukasiewicz fuzzy set $Ł_\lambda^\delta$ in $H$ satisfies:*

$$(\forall x \in H)(\forall t \in (0.5, 1]) \left( \langle x/t \rangle \, q \, Ł_\lambda^\delta \;\Rightarrow\; \langle 1/t \rangle \in Ł_\lambda^\delta \right), \tag{3.21}$$

$$(\forall x, y \in H)(\forall t_a, t_b \in (0.5, 1]) \left( \begin{array}{c} \langle x/t_a \rangle \, q \, Ł_\lambda^\delta, \; \langle (x \to y)/t_b \rangle \, q \, Ł_\lambda^\delta \\ \Rightarrow \; \langle y/\min\{t_a, t_b\} \rangle \in Ł_\lambda^\delta \end{array} \right), \tag{3.22}$$

*then the nonempty* $\in$*-set* $(Ł_\lambda^\delta, t)_\in$*, where* $t := \min\{t_a, t_b\}$*, of* $Ł_\lambda^\delta$ *is a filter of* $H$ *for all* $t_a, t_b \in (0.5, 1]$.

*Proof.* Let $t_a, t_b \in (0.5, 1]$ and $t := \min\{t_a, t_b\}$, and suppose that $(Ł_\lambda^\delta, t)_\in$ is nonempty. Then there exists $x \in (Ł_\lambda^\delta, t)_\in$, and so

$$Ł_\lambda^\delta(x) \geq t > 1 - t,$$

that is, $\langle x/t\rangle \, q \, Ł_\lambda^\delta$. Thus $\langle 1/t\rangle \in Ł_\lambda^\delta$ by (3.21) which shows $1 \in (Ł_\lambda^\delta, t)_\in$. Let $x, y \in H$ be such that $x \in (Ł_\lambda^\delta, t)_\in$ and $x \to y \in (Ł_\lambda^\delta, t)_\in$. Since $t := \min\{t_a, t_b\} > 0.5$, it follows that $Ł_\lambda^\delta(x) \geq t > 1 - t$ and

$$Ł_\lambda^\delta(x \to y) \geq t > 1 - t,$$

that is, $\langle x/t\rangle \, q \, Ł_\lambda^\delta$ and $\langle (x \to y)/t\rangle \, q \, Ł_\lambda^\delta$. Using (3.22) leads to $\langle y/t\rangle \in Ł_\lambda^\delta$. Hence $y \in (Ł_\lambda^\delta, t)_\in$, and therefore $(Ł_\lambda^\delta, t)_\in$ is a filter of $H$ where $t := \min\{t_a, t_b\}$ by Proposition 2.4. □

**Corollary 3.17.** *If a Łukasiewicz fuzzy set* $Ł_\lambda^\delta$ *in* $H$ *satisfies:*

$$(\forall x \in H)(\forall t \in (0.5, 1]) \left( \langle x/t\rangle \, q \, Ł_\lambda^\delta \;\Rightarrow\; \langle 1/t\rangle \in Ł_\lambda^\delta \right), \tag{3.23}$$

$$(\forall x, y \in H)(\forall t_a, t_b \in (0.5, 1]) \left( \begin{array}{c} \langle x/t_a\rangle \, q \, Ł_\lambda^\delta, \, \langle (x \to y)/t_b\rangle \, q \, Ł_\lambda^\delta \\ \Rightarrow \; \langle y/\max\{t_a, t_b\}\rangle \in Ł_\lambda^\delta \end{array} \right), \tag{3.24}$$

*then the nonempty* $\in$*-set* $(Ł_\lambda^\delta, t)_\in$*, where* $t := \max\{t_a, t_b\}$*, of* $Ł_\lambda^\delta$ *is a filter of* $H$ *for all* $t_a, t_b \in (0.5, 1]$.

**Theorem 3.18.** *If* $Ł_\lambda^\delta$ *is a Łukasiewicz fuzzy filter of* $H$*, then its q-set* $(Ł_\lambda^\delta, t)_q$ *is a filter of* $H$ *for all* $t \in (0, 1]$.

*Proof.* Let $Ł_\lambda^\delta$ be a Łukasiewicz fuzzy filter of $H$ and let $t \in (0, 1]$. If $1 \notin (Ł_\lambda^\delta, t)_q$, then $\langle 1/t\rangle \, \overline{q} \, Ł_\lambda^\delta$, i.e., $Ł_\lambda^\delta(1) + t \leq 1$. Since $\langle x/Ł_\lambda^\delta(x)\rangle \in Ł_\lambda^\delta$ for all $x \in H$, we get $\langle 1/Ł_\lambda^\delta(x)\rangle \in Ł_\lambda^\delta$ for all $x \in H$ by (3.5). Hence $Ł_\lambda^\delta(1) \geq Ł_\lambda^\delta(x)$ for $x \in (Ł_\lambda^\delta, t)_q$, and so $Ł_\lambda^\delta(x) \leq Ł_\lambda^\delta(1) \leq 1 - t$. This shows that $\langle x/t\rangle \, \overline{q} \, Ł_\lambda^\delta$, that is, $x \notin (Ł_\lambda^\delta, t)_q$, a contradiction. Thus $1 \in (Ł_\lambda^\delta, t)_q$. Let $x, y \in H$ be such that $x \in (Ł_\lambda^\delta, t)_q$ and $x \to y \in (Ł_\lambda^\delta, t)_q$. Then $\langle x/t\rangle \, q \, Ł_\lambda^\delta$ and $\langle (x \to y)/t\rangle \, q \, Ł_\lambda^\delta$, that is, $Ł_\lambda^\delta(x) > 1 - t$ and $Ł_\lambda^\delta(x \to y) > 1 - t$. It follows from (3.6) that

$$Ł_\lambda^\delta(y) \geq \min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(x \to y)\} > 1 - t.$$

Hence $\langle y/t\rangle \, q \, Ł_\lambda^\delta$, and so $y \in (Ł_\lambda^\delta, t)_q$. Therefore $(Ł_\lambda^\delta, t)_q$ is a filter of $H$ by Proposition 2.4. □

**Corollary 3.19.** *If* $\lambda$ *is a fuzzy filter of* $H$*, then the q-set* $(Ł_\lambda^\delta, t)_q$ *of* $Ł_\lambda^\delta$ *is a filter of* $H$ *for all* $t \in (0, 1]$.

**Proposition 3.20.** *If the $q$-set of a Łukasiewicz fuzzy set $Ł_\lambda^\delta$ in $H$ is a filter of $H$, then the arguments below are valid.*

$$(\forall t \in (0, 0.5])(1 \in (Ł_\lambda^\delta, t)_\in), \tag{3.25}$$

$$(\forall x, y \in H)(\forall t_a, t_b \in (0, 0.5]) \left( \begin{array}{l} \langle x/t_a\rangle \, q \, Ł_\lambda^\delta, \ \langle (x \to y)/t_b\rangle \, q \, Ł_\lambda^\delta \\ \Rightarrow \ \langle y/\max\{t_a, t_b\}\rangle \in Ł_\lambda^\delta \end{array} \right), \tag{3.26}$$

$$(\forall x, y \in H)(\forall t \in (0, 0.5]) \left( x \leq y, \ \langle x/t\rangle \, q \, Ł_\lambda^\delta \ \Rightarrow \ \langle y/t\rangle \in Ł_\lambda^\delta \right). \tag{3.27}$$

*Proof.* Assume that the $q$-set $(Ł_\lambda^\delta, t)_q$ of $Ł_\lambda^\delta$ is a filter of $H$. If $1 \notin (Ł_\lambda^\delta, t)_\in$ for some $t \in (0, 0.5]$, then $\langle 1/t\rangle \,\overline{\in}\, Ł_\lambda^\delta$. Hence $Ł_\lambda^\delta(1) < t \leq 1 - t$ since $t \in (0, 0.5]$, and so $\langle 1/t\rangle \,\overline{q}\, Ł_\lambda^\delta$, i.e., $1 \notin (Ł_\lambda^\delta, t)_q$. This is a conradiction, and thus $1 \in (Ł_\lambda^\delta, t)_\in$. Let $x, y \in H$ and $t_a, t_b \in (0, 0.5]$ be such that $\langle x/t_a\rangle \, q \, Ł_\lambda^\delta$ and $\langle (x \to y)/t_b\rangle \, q \, Ł_\lambda^\delta$. Then

$$x \in (Ł_\lambda^\delta, t_a)_q \subseteq (Ł_\lambda^\delta, \max\{t_a, t_b\})_q$$

and

$$(x \to y) \in (Ł_\lambda^\delta, t_b)_q \subseteq (Ł_\lambda^\delta, \max\{t_a, t_b\})_q,$$

from which $y \in (Ł_\lambda^\delta, \max\{t_a, t_b\})_q$ is derived. Hence

$$Ł_\lambda^\delta(y) > 1 - \max\{t_a, t_b\} \geq \max\{t_a, t_b\},$$

i.e., $\langle y/\max\{t_a, t_b\}\rangle \in Ł_\lambda^\delta$. Hence $y \in (Ł_\lambda^\delta, \max\{t_a, t_b\})_\in$. Let $x, y \in H$ and $t \in (0, 0.5]$ be such that $x \leq y$ and $\langle x/t\rangle \, q \, Ł_\lambda^\delta$. Then $x \in (Ł_\lambda^\delta, t)_q$ and $x \to y = 1 \in (Ł_\lambda^\delta, t)_q$, which imply that $y \in (Ł_\lambda^\delta, t)_q$. Hence

$$Ł_\lambda^\delta(y) > 1 - t \geq t,$$

and so $\langle y/t\rangle \in Ł_\lambda^\delta$. □

**Theorem 3.21.** *If a Łukasiewicz fuzzy set $Ł_\lambda^\delta$ in $H$ satisfies:*

$$(\forall x \in H)(\forall t \in (0, 0.5]) \left( \langle x/t\rangle \in Ł_\lambda^\delta \ \Rightarrow \ \langle 1/t\rangle \, q \, Ł_\lambda^\delta \right), \tag{3.28}$$

$$(\forall x, y \in H)(\forall t_a, t_b \in (0, 0.5]) \left( \begin{array}{l} \langle x/t_a\rangle \in Ł_\lambda^\delta, \ \langle (x \to y)/t_b\rangle \in Ł_\lambda^\delta \\ \Rightarrow \ \langle y/\min\{t_a, t_b\}\rangle \, q \, Ł_\lambda^\delta \end{array} \right), \tag{3.29}$$

*then the nonempty $q$-set $(Ł_\lambda^\delta, t)_q$, where $t := \min\{t_a, t_b\}$, of $Ł_\lambda^\delta$ is a filter of $H$ for all $t_a, t_b \in (0, 0.5]$.*

*Proof.* Let $(Ł_\lambda^\delta, t)_q$ be the nonempty $q$-set of $Ł_\lambda^\delta$ where $t := \min\{t_a, t_b\}$ and $t_a, t_b \in (0, 0.5]$. Then there exists $x \in (Ł_\lambda^\delta, t)_q$, and thus $Ł_\lambda^\delta(x) > 1 - t \geq t$, that is, $\langle x/t\rangle \in Ł_\lambda^\delta$. Hence $\langle 1/t\rangle \, q \, Ł_\lambda^\delta$ by (3.28), that is, $1 \in (Ł_\lambda^\delta, t)_q$. Let $x, y \in H$ be such that $x \in (Ł_\lambda^\delta, t)_q$ and $x \to y \in (Ł_\lambda^\delta, t)_q$. Then $\langle x/t\rangle \in Ł_\lambda^\delta$ and $\langle (x \to y)/t\rangle \in Ł_\lambda^\delta$. Thus $\langle y/t\rangle \, q \, Ł_\lambda^\delta$ by (3.29), and so $Ł^{\delta}_{\lambda}(y) > 1 - t \geq t$, that is, $y \in (Ł^{\delta}_{\lambda}, t)_q$. Consequently, $(Ł^{\delta}_{\lambda}, t)_q$ is a filter of $H$. ☐

**Theorem 3.22.** *If a Łukasiewicz fuzzy set* $Ł^{\delta}_{\lambda}$ *in* $H$ *satisfies:*

$$(\forall x, y \in H)(\forall t_a, t_b \in (0, 0.5]) \left( \begin{array}{l} \langle x/t_a \rangle \in Ł^{\delta}_{\lambda}, \ \langle y/t_b \rangle \in Ł^{\delta}_{\lambda} \\ \Rightarrow \ \langle (x \odot y)/\min\{t_a, t_b\} \rangle \, q \, Ł^{\delta}_{\lambda} \end{array} \right) \tag{3.30}$$

$$(\forall x, y \in H)(\forall t \in (0, 0.5]) \left( x \leq y, \ \langle x/t \rangle \in Ł^{\delta}_{\lambda} \ \Rightarrow \ \langle y/t \rangle \, q \, Ł^{\delta}_{\lambda} \right), \tag{3.31}$$

*then the nonempty q-set* $(Ł^{\delta}_{\lambda}, t)_q$*, where* $t := \min\{t_a, t_b\}$*, of* $Ł^{\delta}_{\lambda}$ *is a filter of* $H$ *for all* $t_a, t_b \in (0, 0.5]$.

*Proof.* Let $x, y \in (Ł^{\delta}_{\lambda}, t)_q$ where $t := \min\{t_a, t_b\}$ and $t_a, t_b \in (0, 0.5]$. Then $Ł^{\delta}_{\lambda}(x) > 1 - t \geq t$ and $Ł^{\delta}_{\lambda}(y) > 1 - t \geq t$, i.e., $\langle x/t \rangle \in Ł^{\delta}_{\lambda}$ and $\langle y/t \rangle \in Ł^{\delta}_{\lambda}$. Hence $\langle (x \odot y)/t \rangle \, q \, Ł^{\delta}_{\lambda}$ by (3.30), and thus $x \odot y \in (Ł^{\delta}_{\lambda}, t)_q$. Let $x, y \in H$ be such that $x \leq y$ and $x \in (Ł^{\delta}_{\lambda}, t)_q$. Then $Ł^{\delta}_{\lambda}(x) > 1 - t \geq t$, and so $\langle x/t \rangle \in Ł^{\delta}_{\lambda}$. It follows from (3.31) that $\langle y/t \rangle \, q \, Ł^{\delta}_{\lambda}$, i.e., $y \in (Ł^{\delta}_{\lambda}, t)_q$. Therefore, $(Ł^{\delta}_{\lambda}, t)_q$ is a filter of $H$. ☐

**Theorem 3.23.** *If a Łukasiewicz fuzzy set* $Ł^{\delta}_{\lambda}$ *in* $H$ *satisfies:*

$$(\forall t \in (0.5, 1])(1 \in (Ł^{\delta}_{\lambda}, t)_{\in}), \tag{3.32}$$

$$(\forall x, y \in H)(\forall t \in (0.5, 1]) \left( \langle x/t \rangle \, q \, Ł^{\delta}_{\lambda}, \ \langle (x \to y)/t \rangle \, q \, Ł^{\delta}_{\lambda} \ \Rightarrow \ \langle y/t \rangle \in Ł^{\delta}_{\lambda} \right), \tag{3.33}$$

*then the q-set* $(Ł^{\delta}_{\lambda}, t)_q$ *of* $Ł^{\delta}_{\lambda}$ *is a filter of* $H$ *for all* $t \in (0.5, 1]$.

*Proof.* The condition (3.32) induces $Ł^{\delta}_{\lambda}(1) + t \geq 2t > 1$ and so $\langle 1/t \rangle \, q \, Ł^{\delta}_{\lambda}$. Hence $1 \in (Ł^{\delta}_{\lambda}, t)_q$. Let $x \in (Ł^{\delta}_{\lambda}, t)_q$ and $x \to y \in (Ł^{\delta}_{\lambda}, t)_q$ for every $x, y \in H$. Then $\langle x/t \rangle \, q \, Ł^{\delta}_{\lambda}$ and $\langle (x \to y)/t \rangle \, q \, Ł^{\delta}_{\lambda}$. Using (3.33) leads to $\langle y/t \rangle \in Ł^{\delta}_{\lambda}$, and so $Ł^{\delta}_{\lambda}(y) \geq t > 1 - t$. Hence $y \in (Ł^{\delta}_{\lambda}, t)_q$, and therefore, $(Ł^{\delta}_{\lambda}, t)_q$ is a filter of $H$ by Proposition 2.4. ☐

**Theorem 3.24.** *If* $\lambda$ *is a fuzzy filter of* $H$*, then the non-empty O-set* $O(Ł^{\delta}_{\lambda})$ *of* $Ł^{\delta}_{\lambda}$ *is a filter of* $H$.

*Proof.* If $\lambda$ is a fuzzy filter of $H$, then $Ł^{\delta}_{\lambda}$ is a Łukasiewicz fuzzy filter of $H$ (see Theorem 3.6). It is clear that $1 \in O(Ł^{\delta}_{\lambda})$. Let $x, y \in H$ be such that $x \in O(Ł^{\delta}_{\lambda})$ and $x \to y \in O(Ł^{\delta}_{\lambda})$. Then $Ł^{\delta}_{\lambda}(x) > 0$ and $Ł^{\delta}_{\lambda}(x \to y) > 0$. Since $\langle x/Ł^{\delta}_{\lambda}(x) \rangle \in Ł^{\delta}_{\lambda}$ and $\langle (x \to y)/Ł^{\delta}_{\lambda}(x \to y) \rangle \in Ł^{\delta}_{\lambda}$, we have

$$\langle y/\min\left\{Ł^{\delta}_{\lambda}(x), Ł^{\delta}_{\lambda}(x \to y)\right\} \rangle \in Ł^{\delta}_{\lambda}$$

by Theorem 3.3. It follows that

$$Ł^{\delta}_{\lambda}(y) \geq \min\left\{Ł^{\delta}_{\lambda}(x), Ł^{\delta}_{\lambda}(x \to y)\right\} > 0.$$

Hence $y \in O(Ł_\lambda^\delta)$, and therefore $O(Ł_\lambda^\delta)$ is a filter of $H$ by Proposition 2.4. □

**Theorem 3.25.** *If a Łukasiewicz fuzzy set* $Ł_\lambda^\delta$ *in* $H$ *satisfies* (3.5) *and*

$$(\forall x, y \in H)(\forall t_a, t_b \in (0,1]) \left( \begin{array}{l} \langle x/t_a\rangle \in Ł_\lambda^\delta,\ \langle (x \to y)/t_b\rangle \in Ł_\lambda^\delta \\ \Rightarrow\ \langle y/\max\{t_a, t_b\}\rangle \, q \, Ł_\lambda^\delta \end{array} \right). \tag{3.34}$$

*then the nonempty O-set* $O(Ł_\lambda^\delta)$ *of* $Ł_\lambda^\delta$ *is a filter of* $H$.

*Proof.* Let $x \in O(Ł_\lambda^\delta)$. Then $t := Ł_\lambda^\delta(x) > 0$ and so $\langle x/t\rangle \in Ł_\lambda^\delta$ for $t > 0$. Hence $\langle 1/t\rangle \in Ł_\lambda^\delta$ by (3.5), and so $Ł_\lambda^\delta(1) \geq t > 0$. Thus $1 \in O(Ł_\lambda^\delta)$. Let $x \in O(Ł_\lambda^\delta)$ and $x \to y \in O(Ł_\lambda^\delta)$ for every $x, y \in H$. Then $\lambda(x) + \delta > 1$ and $\lambda(x \to y) + \delta > 1$. Since $\langle x/Ł_\lambda^\delta(x)\rangle \in Ł_\lambda^\delta$ and

$$\langle (x \to y)/Ł_\lambda^\delta(x \to y)\rangle \in Ł_\lambda^\delta,$$

it follows from (3.34) that

$$\langle y/\max\{Ł_\lambda^\delta(x),\, Ł_\lambda^\delta(x \to y)\}\rangle \, q \, Ł_\lambda^\delta.$$

If $y \notin O(Ł_\lambda^\delta)$, then $Ł_\lambda^\delta(y) = 0$, and so

$$\begin{aligned} Ł_\lambda^\delta(y) + \max\left\{Ł_\lambda^\delta(x),\, Ł_\lambda^\delta(x \to y)\right\} &= \max\left\{Ł_\lambda^\delta(x),\, Ł_\lambda^\delta(x \to y)\right\} \\ &= \max\big\{ \max\{0, \lambda(x) + \delta - 1\}, \\ &\qquad\quad \max\{0, \lambda(x \to y) + \delta - 1\}\big\} \\ &= \max\left\{\lambda(x) + \delta - 1,\, \lambda(x \to y) + \delta - 1\right\} \\ &= \max\left\{\lambda(x),\, \lambda(x \to y)\right\} + \delta - 1 \\ &\leq 1 + \delta - 1 \\ &\leq 1. \end{aligned}$$

Hence $\langle y/\max\{Ł_\lambda^\delta(x),\, Ł_\lambda^\delta(x \to y)\}\rangle \, \overline{q} \, Ł_\lambda^\delta$, a contradiction. Thus $y \in O(Ł_\lambda^\delta)$, and therefore $OŁ(_\lambda^\delta)$ is a filter of $H$ by Proposition 2.4. □

**Theorem 3.26.** *If a Łukasiewicz fuzzy set* $Ł_\lambda^\delta$ *in* $H$ *satisfies*

$$(\forall x, y \in H)(\forall t_a, t_b \in (0,1]) \left( \begin{array}{l} \langle x/t_a\rangle \in \lambda,\ \langle y/t_b\rangle \in \lambda \\ \Rightarrow\ \langle (x \odot y)/\max\{t_a, t_b\}\rangle \, q \, Ł_\lambda^\delta \end{array} \right) \tag{3.35}$$

$$(\forall x, y \in H)(\forall t \in (0,1]) \left(x \leq y,\ \langle x/t\rangle \in \lambda \ \Rightarrow\ \langle y/t\rangle \, q \, Ł_\lambda^\delta\right), \tag{3.36}$$

*then the nonempty O-set* $O(Ł_\lambda^\delta)$ *of* $Ł_\lambda^\delta$ *is a filter of* $H$.

*Proof.* Let $x, y \in O(Ł_\lambda^\delta)$. Then $\lambda(x) > 1 - \delta$ and $\lambda(y) > 1 - \delta$, that is, $\langle x/(1-\delta)\rangle \in \lambda$ and $\langle y/(1-\delta)\rangle \in \lambda$. Thus $\langle (x \odot y)/(1-\delta)\rangle \, q \, Ł_\lambda^\delta$ by (3.35), and so $Ł_\lambda^\delta(x \odot y) + 1 - \delta > 1$. Hence $Ł_\lambda^\delta(x \odot y) > \delta > 0$, that is, $x \odot y \in O(Ł_\lambda^\delta)$. Let $x, y \in H$ be such that $x \leq y$ and $x \in O(Ł_\lambda^\delta)$.

Then $\lambda(x) > 1-\delta$, i.e., $\langle x/(1-\delta)\rangle \in \lambda$. It follows from (3.36) that $\langle y/(1-\delta)\rangle \, q \, Ł^{\delta}_{\lambda}$. Thus $Ł^{\delta}_{\lambda}(y) + 1 - \delta > 1$, and so $Ł^{\delta}_{\lambda}(y) > \delta > 0$ which shows $y \in O(Ł^{\delta}_{\lambda})$. Consequently, $O(Ł^{\delta}_{\lambda})$ is a filter of $H$. □

**Theorem 3.27.** *If a Łukasiewicz fuzzy set* $Ł^{\delta}_{\lambda}$ *in* $H$ *satisfies*

$$(\forall x \in H)(\forall t \in (0,1]) \left(\langle x/t\rangle \in \lambda \;\Rightarrow\; \langle 1/t\rangle \, q \, Ł^{\delta}_{\lambda}\right), \tag{3.37}$$

$$(\forall x, y \in H)(\forall t_a, t_b \in (0,1]) \left( \begin{array}{l} \langle x/t_a\rangle \in \lambda,\, \langle (x \to y)/t_b\rangle \in \lambda \\ \Rightarrow\; \langle y/\max\{t_a, t_b\}\rangle \, q \, Ł^{\delta}_{\lambda} \end{array} \right), \tag{3.38}$$

*then the nonempty* $O$*-set* $O(Ł^{\delta}_{\lambda})$ *of* $Ł^{\delta}_{\lambda}$ *is a filter of* $H$*.*

*Proof.* Let $x \in O(Ł^{\delta}_{\lambda})$. Then $\lambda(x) > 1-\delta$, i.e., $\langle x/(1-\delta)\rangle \in \lambda$ which implies from (3.37) that $\langle 1/(1-\delta)\rangle \, q \, Ł^{\delta}_{\lambda}$. Hence $Ł^{\delta}_{\lambda}(1) + 1 - \delta > 1$, and so $Ł^{\delta}_{\lambda}(1) > \delta > 0$. So $1 \in O(Ł^{\delta}_{\lambda})$. Let $x \in O(Ł^{\delta}_{\lambda})$ and $x \to y \in O(Ł^{\delta}_{\lambda})$ for every $x, y \in H$. Then $\lambda(x) > 1-\delta$, i.e., $\langle x/(1-\delta)\rangle \in \lambda$, and $\lambda(x \to y) > 1-\delta$, i.e., $\langle (x \to y)/(1-\delta)\rangle \in \lambda$. It follows from (3.38) that $\langle y/(1-\delta)\rangle \, q \, Ł^{\delta}_{\lambda}$. Thus $Ł^{\delta}_{\lambda}(y) + 1 - \delta > 1$, and so $Ł^{\in}_{\lambda}(y) > \delta > 0$. This shows $y \in O(Ł^{\delta}_{\lambda})$. Therefore, $O(Ł^{\delta}_{\lambda})$ is a filter of $H$ by Proposition 2.4. □

**Theorem 3.28.** *If a Łukasiewicz fuzzy set* $Ł^{\delta}_{\lambda}$ *in* $H$ *satisfies:*

$$(\forall x \in H)(\langle x/\delta\rangle \, q \, \lambda \;\Rightarrow\; \langle 1/\delta\rangle \in Ł^{\delta}_{\lambda}), \tag{3.39}$$

$$(\forall x, y \in H) \left(\langle x/\delta\rangle \, q \, \lambda,\, \langle (x \to y)/\delta\rangle \, q \, \lambda \;\Rightarrow\; \langle y/\delta\rangle \in Ł^{\delta}_{\lambda}\right), \tag{3.40}$$

*then the nonempty* $O$*-set* $O(Ł^{\delta}_{\lambda})$ *of* $Ł^{\delta}_{\lambda}$ *is a filter of* $H$*.*

*Proof.* Let $x \in O(Ł^{\delta}_{\lambda})$. Then $\lambda(x) + \delta > 1$, i.e., $\langle x/\delta\rangle \, q \, \lambda$, and so $\langle 1/\delta\rangle \in Ł^{\delta}_{\lambda}$ by (3.39). Thus $Ł^{\delta}_{\lambda}(1) \geq \delta > 0$, that is, $1 \in O(Ł^{\delta}_{\lambda})$. Let $x, y \in H$ be such that $x \in O(Ł^{\delta}_{\lambda})$ and $x \to y \in O(Ł^{\delta}_{\lambda})$. Then $\lambda(x) + \delta > 1$ and $\lambda(x \to y) + \delta > 1$, that is, $\langle x/\delta\rangle \, q \, \lambda$ and $\langle (x \to y)/\delta\rangle \, q \, \lambda$. It follows from (3.40) that $\langle y/\delta\rangle \in Ł^{\delta}_{\lambda}$. Thus $Ł^{\delta}_{\lambda}(y) \geq \delta > 0$, and hence $y \in O(Ł^{\delta}_{\lambda})$. So $O(Ł^{\delta}_{\lambda})$ is a filter of $H$ by Proposition 2.4. □

**Theorem 3.29.** *If a Łukasiewicz fuzzy set* $Ł^{\delta}_{\lambda}$ *in* $H$ *satisfies:*

$$(\forall x, y \in H)(\forall t_a, t_b \in [\delta, 1]) \left(\langle x/t_a\rangle \, q \, \lambda,\, \langle y/t_b\rangle \, q \, \lambda \Rightarrow \langle (x \odot y)/\delta\rangle \in Ł^{\delta}_{\lambda}\right), \tag{3.41}$$

$$(\forall x, y \in H)(\forall t \in [\delta, 1]) \left(x \leq y,\, \langle x/t\rangle \, q \, \lambda \;\Rightarrow\; y \in (Ł^{\delta}_{\lambda}, \delta)_{\in}\right), \tag{3.42}$$

*then the nonempty* $O$*-set* $O(Ł^{\delta}_{\lambda})$ *of* ${}^{\delta}_{\lambda}$ *is a filter of* $H$*.*

*Proof.* Let $t_a, t_b \in [\delta, 1]$ and $x, y \in O(Ł^{\delta}_{\lambda})$ for every $x, y \in H$. Then $\lambda(x) + t_a \geq \lambda(x) + \delta > 1$ and $\lambda(y) + t_b \geq \lambda(x) + \delta > 1$, that is, $\langle x/t_a\rangle \, q \, \lambda$ and $\langle y/t_b\rangle \, q \, \lambda$. Applying (3.41) leads to $\langle (x \odot y)/\delta\rangle \in Ł^{\delta}_{\lambda}$.

Thus $Ł_\lambda^\delta(x \odot y) \geq \delta > 0$, and so $x \odot y \in O(Ł_\lambda^\delta)$. Let $t \in [\delta, 1]$ and $x, y \in H$ be such that $x \leq y$ and $x \in O(Ł_\lambda^\delta)$. Then

$$\lambda(x) + t \geq \lambda(x) + \delta > 1$$

and thus $\langle x/t \rangle \, q \, \lambda$. Using (3.42) leads to $y \in (Ł_\lambda^\delta, \delta)_\in$. So $Ł_\lambda^\delta(y) \geq \delta > 0$, i.e., $y \in O(Ł_\lambda^\delta)$. Consequently, $O(Ł_\lambda^\delta)$ is a filter of $H$. □

**Corollary 3.30.** *If a Łukasiewicz fuzzy set $Ł_\lambda^\delta$ in $H$ satisfies:*

$$(\forall x, y \in H) \left( \langle x/\delta \rangle \, q \, \lambda, \; \langle y/\delta \rangle \, q \, \lambda \; \Rightarrow \; \langle (x \odot y)/\delta \rangle \in Ł_\lambda^\delta \right), \tag{3.43}$$

$$(\forall x, y \in H) \left( x \leq y, \; \langle x/\delta \rangle \, q \, \lambda \; \Rightarrow \; y \in (Ł_\lambda^\delta, \delta)_\in \right), \tag{3.44}$$

*then the nonempty $O$-set $O(Ł_\lambda^\delta)$ of $Ł_\lambda^\delta$ is a filter of $H$.*

## Conclusion

In this paper, by using the concept of the Łukasiewicz fuzzy set to the filter in hoops, the Łukasiewicz fuzzy filter is introduced, and its properties are investigated. The relationship between fuzzy filter and Łukasiewicz fuzzy filter is discussed. Conditions for the Łukasiewicz fuzzy set to be a Łukasiewicz fuzzy filter are provided, and characterizations of Łukasiewicz fuzzy filter are displayed. The conditions under which the three subsets, ∈-set, q-set, and $O$-set, will be filter are explored.

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# ŁUKASIEWICZ FUZZY FILTERS IN HOOPS

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فیلترهای فازی لوکاسیویچ در هوپ‌ها

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با استفاده از مفهوم مجموعه‌های فازی لوکاسیویچ روی فیلترهای هوپ، فیلترهای فازی لوکاسیویچ معرفی شده و خواص آن بررسی می‌شود. رابطه بین فیلتر فازی و فیلتر فازی لوکاسیویچ مورد بحث قرار گرفته است. شرایط برای مجموعه فازی لوکاسیویچ به عنوان یک فیلتر فازی لوکاسیویچ فراهم شده و مشخصه معادل فیلتر فازی لوکاسیویچ نمایش داده می‌شود. همچنین شرایطی بررسی می‌شود که تحت آن سه زیرمجموعه، $\in$ -مجموعه، $q$ -مجموعه و $O$ -مجموعه فیلتر می‌شوند.

کلمات کلیدی: مجموعه‌های فازی لوکاسیویچ، فیلترهای فازی لوکاسیویچ، $\in$ -مجموعه، $q$ -مجموعه، $O$ -مجموعه.