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BORDERED GE-ALGEBRAS

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Abstract. The notions of (transitive, commutative, antisymmetric) bordered GE-algebras are introduced, and their properties are investigated. Relations between a bordered GE-algebra and a bounded Hilbert algebra are stated, and the conditions under which every bordered GE-algebra (resp., bounded Hilbert algebra) can be a bounded Hilbert algebra (resp., bordered GE-algebra) are found. The concept of duplex bordered GE-algebras is introduced, and its properties are investigated. Relations between an antisymmetric bordered GE-algebra and a duplex bordered GE-algebra are discussed, and the conditions under which an antisymmetric bordered GE-algebra can be a duplex GE-algebra are established. A characterization of a duplex bordered GE-algebra is provided. A new bordered GE-algebra called cross bordered GE-algebra which is wider than duplex bordered GE-algebra is introduced, and its properties are investigated. Relations between a duplex bordered GE-algebra and a cross bordered GE-algebra are considered.

1. Introduction

In the middle of the last century, Hilbert algebras were introduced by Henkin and Skolem. Since then, several authors have been doing research Hilbert algebra (see [4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]). It is well known that these algebras are the algebraic counterpart of the $\rightarrow$-fragment of the intuitionistic propositional calculus. The study of

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generalization of one known algebraic structure is also an important research task. Bandaru et al. introduced GE-algebra as a generalization of Hilbert algebra, and studied its properties (see [2]). As a follow-up study to [2], Bandaru et al. introduced two kinds of GE-filter called voluntary GE-filter and belligerent GE-filters in GE-algebras, and investigated related properties (see [1, 3]). Rezaei et al. [17] introduced prominent GE-filters in GE-algebras, and studied its properties.

In this article, we introduce the notion of (transitive, commutative, antisymmetric) bordered GE-algebras, and investigate their properties. We consider relations between a commutative bordered GE-algebra and an antisymmetric bordered GE-algebra, and also consider relations between a commutative bordered GE-algebra and a transitive bordered GE-algebra. We discuss relations between a bordered GE-algebra and a bounded Hilbert algebra. We find the conditions under which every bordered GE-algebra can be a bounded Hilbert algebra, also every bounded Hilbert algebra can be a bordered GE-algebra. We introduce the concept of duplex bordered GE-algebras, and investigate its properties. We discuss relations between an antisymmetric bordered GE-algebra and a duplex bordered GE-algebra. We investigate the conditions under which an antisymmetric bordered GE-algebra can be a duplex GE-algebra. We provide a characterization of a duplex bordered GE-algebra. We introduce a new bordered GE-algebra called cross bordered GE-algebra which is wider than duplex bordered GE-algebra, and investigate its properties. We consider relations between a duplex bordered GE-algebra and a cross bordered GE-algebra.

2. Preliminaries

For basic information about (generalized) Hilbert algebras, refer to [5], [4], [6], [10], [11], [14], [15] and [16].

Definition 2.1. A *Hilbert algebra* is defined to be an algebra $(A, *, 1)$ satisfying the following axioms:

(H1) $a * (b * a) = 1$,
(H2) $(a * (b * c)) * ((a * b) * (a * c)) = 1$,
(H3) $a * b = 1 = b * a$ implies $a = b$

for all $a, b, c \in A$.

Definition 2.2. A *generalized Hilbert algebra* (briefly, *g-Hilbert algebra*) is defined to be an algebra $(A, *, 1)$ satisfying the following axioms:

(gH1) $1 * a = a$,
(gH2) $a * a = 1$,
(gH3) $a * (b * c) = b * (a * c)$,

(gH4) $a * (b * c) = (a * b) * (a * c)$
for all $a, b, c \in A$.

A (generalized) Hilbert algebra A is said to be *bounded* or *has zero* if there exists an element 0 in A such that $0 * a = 1$ for all $a \in A$.

Definition 2.3 ([2]). A *GE-algebra* is a non-empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms:
(GE1) $u * u = 1$,
(GE2) $1 * u = u$,
(GE3) $u * (v * w) = u * (v * (u * w))$
for all $u, v, w \in X$.

In a GE-algebra X, a binary relation "$\leq$" is defined by

$$(\forall x, y \in X)\,(x \leq y \;\Leftrightarrow\; x * y = 1)\,. \tag{2.1}$$

Definition 2.4 ([2, 1, 3]). A GE-algebra X is said to be

- *transitive* if it satisfies:
$$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,. \tag{2.2}$$
- *commutative* if it satisfies:
$$(\forall x, y \in X)\,((x * y) * y = (y * x) * x)\,. \tag{2.3}$$
- *left exchangeable* if it satisfies:
$$(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z))\,. \tag{2.4}$$
- *antisymmetric* if the binary relation "$\leq$" is antisymmetric.

Proposition 2.5 ([2]). *Every GE-algebra X satisfies the following items.*

$$(\forall u \in X)\,(u * 1 = 1)\,. \tag{2.5}$$
$$(\forall u, v \in X)\,(u * (u * v) = u * v)\,. \tag{2.6}$$
$$(\forall u, v \in X)\,(u \leq v * u)\,. \tag{2.7}$$
$$(\forall u, v, w \in X)\,(u * (v * w) \leq v * (u * w))\,. \tag{2.8}$$
$$(\forall u \in X)\,(1 \leq u \;\Rightarrow\; u = 1)\,. \tag{2.9}$$
$$(\forall u, v \in X)\,(u \leq (v * u) * u)\,. \tag{2.10}$$
$$(\forall u, v \in X)\,(u \leq (u * v) * v)\,. \tag{2.11}$$
$$(\forall u, v, w \in X)\,(u \leq v * w \;\Leftrightarrow\; v \leq u * w)\,. \tag{2.12}$$

If X is transitive, then

$$(\forall u, v, w \in X)\,(u \leq v \;\Rightarrow\; w * u \leq w * v,\; v * w \leq u * w)\,. \tag{2.13}$$
$$(\forall u, v, w \in X)\,(u * v \leq (v * w) * (u * w))\,. \tag{2.14}$$

Lemma 2.6 ([2])**.** *In a GE-algebra X, the following facts are equivalent each other.*

$$(\forall u, v, w \in X)\,(u * v \leq (w * u) * (w * v))\,. \tag{2.15}$$

$$(\forall u, v, w \in X)\,(u * v \leq (v * w) * (u * w))\,. \tag{2.16}$$

3. Bordered GE-algebras

Definition 3.1. If a GE-algebra X has a special element, say 0, that satisfies $0 \leq x$ for all $x \in X$, we call X the *bordered GE-algebra.*

For every element x of a bordered GE-algebra X, we denote $x * 0$ by x^0, and $(x^0)^0$ is denoted by x^{00}.

Example 3.2. Let $X = \{0, 1, a, b, c\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b	c
0	1	1	1	1	1
1	0	1	a	b	c
a	0	1	1	1	0
b	c	1	1	1	c
c	1	1	1	b	1

Then X is a bordered GE-algebra.

Definition 3.3. If a bordered GE-algebra X satisfies the condition (2.2), we say that X is a *transitive bordered GE-algebra.*

Example 3.4. Let $X = \{0, 1, a, b, c, d\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b	c	d
0	1	1	1	1	1	1
1	0	1	a	b	c	d
a	d	1	1	b	d	d
b	0	1	a	1	c	c
c	a	1	a	1	1	1
d	a	1	a	1	1	1

Then X is a transitive bordered GE-algebra.

Definition 3.5. If a bordered GE-algebra X satisfies the condition (2.3), we say that X is a *commutative bordered GE-algebra.*

Example 3.6. Let $X = \{0, 1\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1
0	1	1
1	0	1

Then X is a commutative bordered GE-algebra.

Definition 3.7. A bordered GE-algebra X is said to be *antisymmetric* if the binary operation "$\leq$" is antisymmetric.

Example 3.8. Let $X = \{0, 1, a, b\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b
0	1	1	1	1
1	0	1	a	b
a	b	1	1	b
b	a	1	a	1

Then X is an antisymmetric bordered GE-algebra.

Proposition 3.9. *In a bordered GE-algebra X, the following assertions are valid.*

$$1^0 = 0,\ 0^0 = 1. \tag{3.1}$$

$$(\forall x \in X)\left(x \leq x^{00},\ 0 \leq x^{00}\right). \tag{3.2}$$

$$(\forall x \in X)\left(0 \leq x * x^0 = x^0\right). \tag{3.3}$$

$$(\forall x, y \in X)\left(x * y^0 \leq y * x^0\right). \tag{3.4}$$

$$(\forall x, y \in X)\left(x \leq y^0 \;\Leftrightarrow\; y \leq x^0\right). \tag{3.5}$$

$$(\forall x, y \in X)\left(x * y^0 = x * (y * x^0)\right). \tag{3.6}$$

If X is a transitive bordered GE-algebra, then

$$(\forall x, y \in X)\left(x \leq y \;\Rightarrow y^0 \leq x^0\right). \tag{3.7}$$

$$(\forall x, y \in X)\left(x * y \leq y^0 * x^0\right). \tag{3.8}$$

If X is an antisymmetric bordered GE-algebra, then

$$(\forall x, y \in X)\left(x * y^0 = y * x^0\right). \tag{3.9}$$

If X is a transitive and antisymmetric bordered GE-algebra, then

$$(\forall x \in X)\left(x^{000} = x^0\right). \tag{3.10}$$

Proof. The result (3.1) follows from (GE2) and (GE1), respectively. If we put $u = x$ and $v = 0$ in (2.11), then $x \leq (x * 0) * 0 = x^{00}$. If we put $u = 0$ and $v = x$ in (2.10), then $0 \leq (x * 0) * 0 = x^{00}$. Putting $u = x$ and $v = 0$ in (2.6); and $u = 0$ and $v = x$ in (2.7) induces (3.3). If we put $u = x$, $v = y$ and $w = 0$ in (2.8), then $x * y^0 \leq y * x^0$. (3.5) is induced by taking $u = x$, $v = y$ and $w = 0$ in (2.12). If we take $u = x$, $v = y$ and $w = 0$ in (GE3), then we have (3.6). Assume that X is a transitive bordered GE-algebra. Putting $u = x$, $v = y$ and $w = 0$ in (2.13) and (2.14) induces (3.7) and (3.8). It is clear that if X is an antisymmetric bordered GE-algebra, then $x * y^0 = y * x^0$ for all $x, y \in X$. Suppose that X is a transitive and antisymmetric bordered GE-algebra. By (3.2), $x \leq x^{00}$ for all $x \in X$, which implies from (3.7) that $x^{000} \leq x^0$. Since $x^0 \leq x^{000}$, we have $x^{000} = x^0$ for all $x \in X$ by the antisymmetricity of X. □

We consider relations between a commutative bordered GE-algebra and an antisymmetric bordered GE-algebra.

Lemma 3.10. *Every commutative bordered GE-algebra is an antisymmetric bordered GE-algebra.*

Proof. Let X be a commutative bordered GE-algebra. It is sufficient to show that if $x \leq y$ and $y \leq x$, then $x = y$. Let $x, y \in X$ be such that $x \leq y$ and $y \leq x$. Then $x * y = 1 = y * x$. It follows from the commutativity of X and (GE2) that

$$x = 1 * y = (x * y) * y = (y * x) * x = 1 * y = y.$$

Therefore X is an antisymmetric bordered GE-algebra. □

Corollary 3.11. *Every commutative bordered GE-algebra is a left exchangeable bordered GE-algebra.*

Proof. Straightforward. □

The following example shows that the converse of Lemma 3.10 and Corollary 3.11 is not true in general.

Example 3.12. 1. Let $X = \{0, 1, a, b, c\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b	c
0	1	1	1	1	1
1	0	1	a	b	c
a	b	1	1	b	1
b	c	1	a	1	c
c	b	1	a	b	1

Then X is an antisymmetric bordered GE-algebra. But it is not commutative since $(a * 0) * 0 = b * 0 = c \neq a = 1 * a = (0 * a) * a$.

2. Let $X = \{0, 1, a, b, c\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b	c
0	1	1	1	1	1
1	0	1	a	b	c
a	c	1	1	1	c
b	c	1	a	1	c
c	b	1	1	b	1

Then X is a left exchangeable bordered GE-algebra. But X is not commutative since $(a * b) * b = 1 * b = b \neq 1 = a * a = (b * a) * a$.

We consider relations between a commutative bordered GE-algebra and a transitive bordered GE-algebra.

Theorem 3.13. *Every commutative bordered GE-algebra is a transitive bordered GE-algebra.*

Proof. Let X be a commutative bordered GE-algebra. Using Corollary 3.11, (2.5) and the commutativity of X, we have

$$\begin{aligned}(x * y) * ((z * x) * (z * y)) &= (z * x) * ((x * y) * (z * y)) \\ &= (z * x) * (z * ((x * y) * y)) \\ &= (z * x) * (z * ((y * x) * x)) \\ &= (z * x) * ((y * x) * (z * x)) \\ &= (y * x) * ((z * x) * (z * x)) \\ &= (y * x) * 1 = 1,\end{aligned}$$

and so $(x * y) \leq (z * x) * (z * y)$ for all $x, y, z \in X$. Therefore X is a transitive bordered GE-algebra. □

The converse of Theorem 3.13 is not true in general as seen in the following example.

Example 3.14. Let $X = \{0, 1, a, b, c\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b	c
0	1	1	1	1	1
1	0	1	a	b	c
a	c	1	1	1	c
b	0	1	a	1	c
c	a	1	a	b	1

Then X is a transitive bordered GE-algebra. But X is not commutative since

$$(a*b)*b = 1*b = b \neq 1 = a*a = (b*a)*a.$$

We discuss relations between bordered GE-algebra and bounded Hilbert algebra. It is clear that every bounded Hilbert algebra is a bordered GE-algebra.

The following example shows that any bordered GE-algebra is not a bounded Hilbert algebra in general.

Example 3.15. Let $X = \{0, 1, a, b, c\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b	c
0	1	1	1	1	1
1	0	1	a	b	c
a	c	1	1	1	c
b	a	1	a	1	1
c	b	1	1	b	1

Then X is a bordered GE-algebra. But X is not a bounded Hilbert algebra since

$$(a*(b*c))*((a*b)*(a*c)) = (a*1)*(1*c) = 1*c = c \neq 1.$$

We find the conditions under which every bordered GE-algebra can be a bounded Hilbert algebra, also every bounded Hilbert algebra can be a bordered GE-algebra.

Theorem 3.16. *Every commutative bordered GE-algebra is a bounded Hilbert algebra.*

Proof. Let X be a commutative bordered GE-algebra. Then $0*x = 1$, i.e., $0 \leq x$, and

$$x*(y*x) = x*(y*(x*x)) = x*(y*1) = x*1 = 1$$

for all $x, y \in X$. Let $x, y \in X$ be such that $x*y = 1 = y*x$. Then

$$x = 1*x = (y*x)*x = (x*y)*y = 1*y = y.$$

We know that

$$\begin{aligned} x*(y*z) &= y*(x*z) \\ &\leq (x*y)*(x*(x*z)) \\ &= (x*y)*(x*z), \end{aligned}$$

and so $(x*(y*z))*((x*y)*(x*z)) = 1$ for all $x, y, z \in X$. Thus X is a bounded Hilbert algebra. $\square$

Theorem 3.17. *Every bounded g-Hilbert algebra is a bordered GE-algebra.*

Proof. Let X be a bounded g-Hilbert algebra. Then X satisfies (GE1), (GE2) and $0 * x = 1$, i.e., $0 \leq x$ for all $x \in X$. For every $x, y, z \in X$, we have

$$\begin{aligned} x * (y * z) &= (x * y) * (x * z) \\ &= (x * y) * (x * (x * z)) \\ &= x * (y * (x * z)). \end{aligned}$$

Thus X is a bordered GE-algebra. □

The converse of Theorems 3.16 and 3.17 is not true in general as seen in the following example.

Example 3.18. 1. Let $X = \{0, 1, a, b, c\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b	c
0	1	1	1	1	1
1	0	1	a	b	c
a	b	1	1	b	1
b	a	1	a	1	c
c	0	1	a	b	1

Then X is a bounded Hilbert algebra. But X is not a commutative bordered GE-algebra since $(a*c)*c = 1*c = c \neq 1 = a*a = (c*a)*a$.

2. Let $X = \{0, 1, a, b, c\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b	c
0	1	1	1	1	1
1	0	1	a	b	c
a	c	1	1	1	c
b	1	1	a	1	1
c	b	1	a	b	1

Then X is a bordered GE-algebra. But X is not a bounded g-Hilbert algebra since $a * (b * c) = a * 1 = 1 \neq c = 1 * c = (a * b) * (a * c)$.

Definition 3.19. By a duplex bordered element in a bordered GE-algebra X, we mean an element x of X which satisfies $x^{00} = x$.

The set of all duplex bordered elements of a bordered GE-algebra X is denoted by $0^2(X)$, and is called the duplex bordered set of X. It is clear that $0, 1 \in 0^2(X)$.

Definition 3.20. A bordered GE-algebra X is said to be *duplex* if every element of X is a duplex bordered element, that is, $X = 0^2(X)$.

Example 3.21. 1. Let $X = \{0, 1, a, b, c, d\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b	c	d
0	1	1	1	1	1	1
1	0	1	a	b	c	d
a	b	1	1	b	1	b
b	a	1	a	1	1	1
c	d	1	1	d	1	d
d	c	1	1	1	c	1

Then X is a duplex bordered GE-algebra.

2. Let $X = \{0, 1, a, b, c, d\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b	c	d
0	1	1	1	1	1	1
1	0	1	a	b	c	d
a	b	1	1	b	c	1
b	a	1	a	1	c	a
c	d	1	d	1	1	d
d	b	1	1	b	c	1

Then X is not a duplex bordered GE-algebra since

$$0^2(X) = \{0, 1, a, b\} \neq X.$$

In general, any antisymmetric bordered GE-algebra may not be duplex as seen in the following example.

Example 3.22. Let $X = \{0, 1, a, b, c, d\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b	c	d
0	1	1	1	1	1	1
1	0	1	a	b	c	d
a	b	1	1	b	1	1
b	c	1	a	1	c	1
c	d	1	a	1	1	d
d	b	1	a	b	1	1

Then X is an antisymmetric bordered GE-algebra. But X is not a duplex bordered GE-algebra since $0^2(X) = \{0, 1\} \neq X$.

We investigate the conditions under which antisymmetric bordered GE-algebra can be duplex.

Proposition 3.23. *Let X be a bordered GE-algebra. If X is antisymmetric and duplex, then X satisfies:*

$$(\forall x, y \in X)\left(x * y = y^0 * x^0\right). \tag{3.11}$$

Proof. Let $x, y \in X$. Since $y = y^{00}$, it follows from (3.9) that

$$x * y = x * y^{00} = y^0 * x^0$$

which proves (3.11). □

Proposition 3.24. *If an antisymmetric bordered GE-algebra X satisfies (3.11), then*

$$(\forall x, y \in X)\left(x^0 * y = y^0 * x\right). \tag{3.12}$$

Proof. Let X be an antisymmetric bordered GE-algebra X satisfying (3.11). Then $x^0 * y = y^0 * x^{00}$ and $y^0 * x = x^0 * y^{00}$ by (3.11). Since $x^0 * y^{00} = y^0 * x^{00}$ by (3.9), it follows that $x^0 * y = y^0 * x$ for all $x, y \in X$. □

Corollary 3.25. *Every antisymmetric duplex bordered GE-algebra X satisfies (3.12).*

Proposition 3.26. *If a bordered GE-algebra X satisfies (3.12), then*

$$(\forall x, y \in X)\left(x^0 \leq y \;\Rightarrow\; y^0 \leq x\right). \tag{3.13}$$

Proof. Straightforward. □

Proposition 3.27. *If an antisymmetric bordered GE-algebra X satisfies (3.13), then X is duplex.*

Proof. By (3.2), $x \leq x^{00}$ for all $x \in X$. Since $x^0 \leq x^0$ for all $x \in X$, it follows from (3.13) that $x^{00} \leq x$. Hence $x = x^{00}$ for all $x \in X$, and therefore X is duplex. □

Combining Propositions 3.23, 3.24, 3.26 and 3.27 induces the characterization of duplex bordered GE-algebras.

Theorem 3.28. *Given an antisymmetric bordered GE-algebra X, the following are equivalent.*

(i) *X is duplex.*
(ii) *X satisfies (3.11).*
(iii) *X satisfies (3.12).*
(iv) *X satisfies (3.13).*

Proposition 3.29. *In an antisymmetric bordered GE-algebra X, we have*

$$(\forall x, y \in X)\left(x, y \in 0^2(X) \;\Rightarrow\; x^0 * y = y^0 * x\right). \tag{3.14}$$

Proof. Let $x, y \in 0^2(X)$. then $x^{00} = x$ and $y^{00} = y$. Hence

$$x^0 * y = x^0 * y^{00} = y^0 * x^{00} = y^0 * x$$

by (3.9). □

Proposition 3.30. *If X is an antisymmetric bordered GE-algebra, then $x * y = y^0 * x^0$ for all $x \in X$ and $y \in 0^2(X)$.*

Proof. Using (3.9) induces $x * y = x * y^{00} = y^0 * x^0$ for all $x \in X$ and $y \in 0^2(X)$. □

Corollary 3.31. *If X is an antisymmetric duplex bordered GE-algebra, then $x * y = y^0 * x^0$ for all $x, y \in X$.*

We present conditions under which the duplex bordered set $0^2(X)$ can be a GE-subalgebra of X.

Theorem 3.32. *The duplex bordered set $0^2(X)$ of a transitive and antisymmetric bordered GE-algebra X is closed under the binary operation $*$ in X, that is, it is a GE-subalgebra of X and is also bordered.*

Proof. Let $x, y \in 0^2(X)$. Then $x * y \leq (x * y)^{00}$ by (3.2). Using (2.8), (2.11), (3.9), (3.10) and Proposition 3.30, we have

$$\begin{aligned}
1 &= x * ((x * y) * y) \\
&= x * (y^0 * (x * y)^0) \\
&\leq y^0 * (x * (x * y)^0) \\
&= y^0 * (x * (x * y)^{000}) \\
&= y^0 * ((x * y)^{00} * x^0) \\
&\leq (x * y)^{00} * (y^0 * x^0) \\
&= (x * y)^{00} * (x * y)
\end{aligned}$$

and so $(x * y)^{00} * (x * y) = 1$, that is $(x * y)^{00} \leq (x * y)$. Since X is antisymmetric, it follows that $(x * y)^{00} = (x * y)$, that is, $x * y \in 0^2(X)$. This completes the proof. □

The following example shows that if X is a bordered GE-algebra which is neither transitive nor antisymmetric, then the duplex bordered set $0^2(X)$ can not be a GE-subalgebra of X.

Example 3.33. Let $X = \{0, 1, a, b, c, d, e\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b	c	d	e
0	1	1	1	1	1	1	1
1	0	1	a	b	c	d	e
a	b	1	1	b	1	b	1
b	a	1	a	1	1	e	e
c	d	1	e	b	1	d	e
d	c	1	1	1	c	1	1
e	b	1	1	b	1	b	1

Then X is a bordered GE-algebra which is neither transitive nor antisymmetric and $0^2(X) = \{0, 1, a, b, c, d\}$. But $0^2(X)$ is not a GE-subalgebra of X since $b, d \in 0^2(X)$ and $b * d = e \notin 0^2(X)$.

In the following example, we know that the duplex bordered set $0^2(X)$ is not a GE-subalgebra of a transitive bordered GE-algebra X.

Example 3.34. Let $X = \{0, 1, a, b, c, d, e\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b	c	d	e
0	1	1	1	1	1	1	1
1	0	1	a	b	c	d	e
a	b	1	1	b	e	1	e
b	a	1	a	1	1	d	1
c	d	1	d	1	1	d	1
d	c	1	1	e	c	1	e
e	d	1	d	1	1	d	1

Then X is a transitive bordered GE-algebra and $0^2(X) = \{0, 1, a, b, c, d\}$. But $0^2(X)$ is not a GE-subalgebra of X since $b, d \in 0^2(X)$ and $d * b = e \notin 0^2(X)$.

In the following example, we know that the duplex bordered set $0^2(X)$ is not a GE-subalgebra of an antisymmetric bordered GE-algebra X.

Example 3.35. Let $X = \{0, 1, a, b, c, d, e, f, g, h, x, y, u, v\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b	c	d	e	f	g	h	x	y	u	v
0	1	1	1	1	1	1	1	1	1	1	1	1	1	v
1	0	1	a	b	c	d	e	f	g	h	x	y	u	v
a	d	1	1	c	c	d	e	e	1	d	e	1	c	e
b	e	1	1	1	1	e	e	e	1	e	e	1	1	e
c	f	1	a	a	1	e	e	f	1	x	x	y	y	x
d	a	1	a	a	1	1	1	a	1	a	a	1	1	a
e	b	1	a	b	c	c	1	a	1	b	a	1	c	a
f	g	1	1	1	1	1	1	1	g	1	1	1	g	g
g	h	1	a	b	c	d	e	x	1	h	x	1	c	x
h	y	1	1	1	1	1	1	y	1	1	1	y	y	v
x	u	1	1	c	c	c	1	y	g	c	1	y	u	g
y	v	1	a	a	1	e	e	x	g	x	x	1	g	v
u	x	1	a	a	1	e	e	x	1	x	x	1	1	x
v	c	1	1	c	c	c	1	1	1	c	1	1	c	v

Then X is an antisymmetric bordered GE-algebra and

$$0^2(X) = \{0, 1, a, b, d, e, x, u\}.$$

But $0^2(X)$ is not a GE-subalgebra of X since $a, b \in 0^2(X)$ and $a * b = c \notin 0^2(X)$.

We introduce a new bordered GE-algebra which is wider than duplex bordered GE-algebra.

Definition 3.36. A bordered GE-algebra X is said to be *cross* if it satisfies:

$$(\forall x \in X)\left((x^{00} * x)^0 = 0\right). \tag{3.15}$$

Example 3.37. Let $X = \{0, 1, a, b, c, d\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	a	b	c	d
0	1	1	1	1	1	1
1	0	1	a	b	c	d
a	0	1	1	1	d	d
b	1	1	1	1	c	c
c	0	1	a	1	1	1
d	1	1	1	b	1	1

Then X is a cross bordered GE-algebra.

We consider relations between duplex bordered GE-algebra and cross bordered GE-algebra.

Theorem 3.38. *Every duplex bordered GE-algebra is a cross bordered GE-algebra.*

Proof. Let X be a duplex bordered GE-algebra. Then $x = x^{00}$ for all $x \in X$. It follows from (GE1) and (3.1) that $(x^{00} * x)^0 = 1^0 = 0$ for all $x \in X$. Therefore X is a cross bordered GE-algebra. □

The converse of Theorem 3.38 is not true in general as seen in the following example.

Example 3.39. The cross bordered GE-algebra X in Example 3.37 is not duplex since $0^2(X) = \{0, 1\} \neq X$.

Before ending this article, we pose an open question: Under what conditions will cross bordered GE-algebra be duplex bordered GE-algebra?

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BORDERED GE-ALGEBRAS

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GE-جبرهای محدود شده

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در این مقاله، مفاهیم GE-جبرهای محدود شده‌ی (متعدی، جابه‌جایی و پادمتقارن) را معرفی می‌کنیم و ویژگی‌های آن‌ها را مورد بررسی قرار می‌دهیم. رابطه‌ی بین یک GE-جبر محدود شده با جبر هیلبرت کراندار بررسی شده و شرایطی را بیان می‌کنیم که تحت آن‌ها، هر GE-جبر محدود شده می‌تواند یک جبر هیلبرت کراندار باشد. همچنین، مفهوم GE-جبر مضاعف کراندار را بیان می‌کنیم و سپس ویژگی‌های آن را مورد بررسی قرار می‌دهیم. رابطه‌ی بین یک GE-جبر محدود شده‌ی پادمتقارن با یک GE-جبر محدود شده‌ی مضاعف مورد بررسی قرار گرفته و شرایطی که یک GE-جبر محدود شده‌ی پادمتقارن می‌تواند یک GE-جبر مضاعف باشد ذکر شده است. یک مشخصه‌سازی از GE-جبر محدود شده‌ی مضاعف بیان می‌کنیم و همچنین GE-جبرهای محدود شده‌ی صلیبی که بسیار وسیع‌تر GE-جبرهای محدود شده‌ی مضاعف هستند را معرفی و سپس ویژگی‌های آن را مورد بررسی قرار می‌دهیم. در آخر، رابطه‌ی بین GE-جبر محدود شده‌ی مضاعف و GE-جبر محدود شده‌ی صلیبی مورد بررسی قرار گرفته است.

کلمات کلیدی: GE-جبر محدود شده (جابه‌جایی، متعدی و پادمتقارن)، GE-جبر محدود شده‌ی مضاعف، GE-جبر محدود شده‌ی صلیبی.