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$\mathcal{R}$-CONVEX SUBSETS OF BIMODULES OVER $*$-RINGS

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Abstract. Let $\mathcal{M}$ and $\mathcal{N}$ be bimodules over the unital $*$-rings $\mathcal{R}$ and $\mathcal{B}$, respectively. We investigate the notion of $\mathcal{R}$-convexity and the corresponding notion of $\mathcal{R}$-extreme points. We discuss the effect of an f-homomorphism on $\mathcal{R}$-convex subsets and its $\mathcal{R}$-extreme points. Namely, we declare how an f-homomorphism from $\mathcal{M}$ to $\mathcal{N}$ carries $\mathcal{R}$-convex subsets and its $\mathcal{R}$-extreme points to $\mathcal{B}$-convex subsets and its $\mathcal{B}$-extreme points and vice versa. Moreover, we confirm that the $\mathcal{R}$-convex hull of invariant subsets under f-homomorphisms remains invariant.

1. Introduction

The study of noncommutative convexity or C^*-convexity was initiated by Loebl and Paulsen in [10] as a non-commutative analog of the linear convexity. Then, the notion of C^*-extreme points was studied as a non-commutative analog of linear extreme points. It is evident that every C^*-convex set is convex in the usual sense but the converse does not hold in general. Moreover, it was determined whether C^*-extremeness is distinct from linear extremeness by Hopenwasser, Moore, and Paulsen [7]. Farenick [5] proved the set of C^*-extreme points of compact C^*-convex subsets of the finite dimensional algebra $M_n(\mathbb{C})$ is nonempty and Morenz [14] proved the appropriate variant of the Krein-Milman theorem for C^*-convex subsets in matrix algebras, cf. [4, 6, 13]. Some other results of the linear convexity have been

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generalized to C^*-convexity, for instance, a version of the so-called Hahn-Banach theorem and separation theorem [3, 7]. Later another version of the non-commutative convexity was studied in the context of the quantum information theory in [9].

It makes sense in a C^*-algebra or a $*$-ring and, more generally, for bimodules over C^*-algebras or $*$-rings there is a concept of convexity that incorporates algebra-valued or ring-valued convex coefficients in a natural way, cf. [11, 12, 2, 15].

In this paper, we consider the notion of $\mathcal{R}$-convexity and the corresponding notion of $\mathcal{R}$-extreme points in the bimodules over unital $*$-rings. We prove that an f-homomorphism g, under certain conditions, carries $\mathcal{R}$-convex subsets and $\mathcal{R}$-extreme points of its domain to the $\mathcal{B}$-convex subsets and $\mathcal{B}$-extreme points of its range. We show that the $\mathcal{R}$-convex hull of invariant subsets is invariant under g. For more details on bimodules over rings, we refer the readers to [8].

2. $\mathcal{R}$-convex sets of bimodules over $*$-rings

In this section, we distinguish the properties of f-homomorphisms on $\mathcal{R}$-convex sets of bimodules over $*$-rings and we verify how an f-homomorphism carries $\mathcal{R}$-convex subsets of its domain to $\mathcal{B}$-convex subsets of its range and vice versa. We identify the invariance of the $\mathcal{R}$-convex hull of invariant subsets under f-homomorphisms.

Definition 2.1. Let $\mathcal{M}$ be a bimodule over a unital $*$-ring $\mathcal{R}$. A set $\mathcal{K} \subset \mathcal{M}$ is called $\mathcal{R}$-convex, if $\mathcal{K}$ is closed under the formation of finite sums of the type $\sum_i t_i^* x_i t_i$, where $t_i \in \mathcal{R},\ x_i \in \mathcal{K}$ and $\sum_i t_i^* t_i = 1$.

This formation of finite sums is called an $\mathcal{R}$-convex combination in $\mathcal{K}$ and the coefficients t_i are called $\mathcal{R}$-convex coefficients. If the coefficients t_i are invertible in $\mathcal{R}$, then they are called proper $\mathcal{R}$-convex coefficients and the $\mathcal{R}$-convex combination is called a proper $\mathcal{R}$-convex combination.

By definition it is clear that every subbimodule of $\mathcal{M}$ is $\mathcal{R}$-convex and furthermore, if $\mathcal{K} \subset \mathcal{M}$ is $\mathcal{R}$-convex, $\mathcal{R}_1$ is a $*$-subring of $\mathcal{R}$, and $1 \in \mathcal{R}_1$, then $\mathcal{K}$ is $\mathcal{R}_1$-convex. We remark that any module over a commutative ring is automatically a bimodule. Indeed, if $\mathcal{M}$ is a left module, we can define the multiplication on the right to be the same as the multiplication on the left. So, if the unital $*$-ring $\mathcal{R}$ is commutative, then we have, $amb = (ab)m = m(ab)$ for $m \in \mathcal{M}$ and $a, b \in \mathcal{R}$. Therefore, the $\mathcal{R}$-convex combinations are of the form $\sum_i a_i x_i$, where $a_i \in \mathcal{R}^+$ and $x_i \in \mathcal{K}$. Note that $\mathcal{R}^+$ denotes the cone of positive elements in $\mathcal{R}$. Such $\mathcal{R}$-convex combinations are called linear $\mathcal{R}$-convex combinations.

Definition 2.2. Let $\mathcal{M}$ and $\mathcal{N}$ be bimodules over unital $*$-rings $\mathcal{R}$ and $\mathcal{B}$, respectively and $f : \mathcal{R} \to \mathcal{B}$ a $*$-homomorphism. We say the mapping $g : \mathcal{M} \to \mathcal{N}$ is an f-homomorphism whenever

i) $g(m_1 + m_2) = g(m_1) + g(m_2)$, for all $m_1, m_2 \in \mathcal{M}$,
ii) $g(amb) = f(a)g(m)f(b)$, for all $a, b \in \mathcal{R}$ and $m \in \mathcal{M}$.

It is clear that if f is the identity mapping and $\mathcal{R} = \mathcal{B}$, then g is clearly an $\mathcal{R}$-bimodule homomorphism from $\mathcal{M}$ into $\mathcal{N}$, i.e., g is an additive mapping such that $g(amb) = ag(m)b$ for all $a, b \in \mathcal{R}$ and $m \in \mathcal{M}$.

One may consider $\mathcal{R}$ and $\mathcal{B}$ as bimodules over themselves. Let $f : \mathcal{R} \to \mathcal{B}$ be a $*$-homomorphism. Then, f is an f-homomorphism, $2f$ is an f-homomorphism, and $-f$ is an f-homomorphism.

An injective f-homomorphism is called an f-monomorphism and a surjective f-homomorphism is called an f-epimorphism.

Definition 2.3. [1] A $*$-ring is said to satisfy the positive square-root axiom (briefly, the (PSR)-axiom) in case, for every $x > 0$, there exists $y \in \{x\}''$ with $y > 0$ and $x = y^2$.

Definition 2.4. Let $\mathcal{M}$ be a bimodule over a unital $*$-ring $\mathcal{R}$. For $x, y \in \mathcal{M}$, the $\mathcal{R}$-segment connecting x and y is defined by

$$[x, y]_{\mathcal{R}} := \Big\{ \sum_i t_i^* x t_i + \sum_j v_j^* y v_j : \sum_i t_i^* t_i + \sum_j v_j^* v_j = 1, t_i, v_j \in \mathcal{R} \Big\}.$$

Note that in this article the formation of all sums are finite sums.

Proposition 2.5. *Let $\mathcal{M}$ be a bimodule over a unital $*$-ring $\mathcal{R}$. For $x, y \in \mathcal{M}$, the $\mathcal{R}$-segment $[x, y]_{\mathcal{R}}$ is an $\mathcal{R}$-convex set that contains both of x and y.*

Proof. Let $x_1, \ldots, x_n \in [x, y]_{\mathcal{R}}$. We prove $\sum_k t_k^* x_k t_k \in [x, y]_{\mathcal{R}}$, where $\sum_k t_k^* t_k = 1$. Since $x_k \in [x, y]_{\mathcal{R}}$, there exist $a_{ik}, b_{jk} \in \mathcal{R}$ such that

$$x_k = \sum_i a_{ik}^* x a_{ik} + \sum_j b_{jk}^* y b_{jk}, \quad \sum_i a_{ik}^* a_{ik} + \sum_j b_{jk}^* b_{jk} = 1.$$

We have

$$\begin{aligned} \sum_k t_k^* x_k t_k &= \sum_k t_k^* \Big(\sum_i a_{ik}^* x a_{ik} + \sum_j b_{jk}^* y b_{jk} \Big) t_k \\ &= \sum_k \sum_i t_k^* a_{ik}^* x a_{ik} t_k + \sum_k \sum_j t_k^* b_{jk}^* y b_{jk} t_k, \end{aligned}$$

where

$$\sum_k \sum_i t_k^* a_{ik}^* a_{ik} t_k + \sum_k \sum_j t_k^* b_{jk}^* b_{jk} t_k = \sum_k t_k^* \Big(\sum_i a_{ik}^* a_{ik} + \sum_j b_{jk}^* b_{jk} \Big) t_k$$
$$= \sum_k t_k^* (1) t_k = 1.$$

So, the $\mathcal{R}$-segment $[x, y]_{\mathcal{R}}$ is closed under the formation of finite sums of the desired type. One may write $x = 1x1 + 0_{\mathcal{R}} y 0_{\mathcal{R}}$, $y = 0_{\mathcal{R}} x 0_{\mathcal{R}} + 1y1$, and hence x and y belong to $[x, y]_{\mathcal{R}}$. $\square$

W remark that Proposition 2.5 is a generalization of Proposition 2.6 (i) of [2]. Let $\mathcal{R}$ be a unital $*$-ring. Considering $\mathcal{M} = \mathcal{R}$ in Proposition 2.5, the unital $*$-ring $\mathcal{R}$ is a bimodule over itself and so we get Proposition 2.6 (i) of [2]. Let $\mathcal{B}(\mathcal{H})$ denote the $*$-ring of bounded linear operators on a (separable) Hilbert space $\mathcal{H}$. Considering $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\mathcal{R} = \mathcal{B}(\mathcal{H})$ in Proposition 2.5, $\mathcal{B}(\mathcal{H})$ is a bimodule over itself and so we reach Lemma 12 of [10].

Theorem 2.6. *Suppose that $\mathcal{M}$ is a bimodule over $\mathcal{R}$, $\mathcal{K} \subset \mathcal{M}$, and the unital $*$-ring $\mathcal{R}$ satisfies the (PSR)-axiom. Then the set $\mathcal{K}$ is $\mathcal{R}$-convex if and only if the $\mathcal{R}$-segment $[x, y]_{\mathcal{R}}$ is contained in $\mathcal{K}$ for every $x, y \in \mathcal{K}$.*

Proof. If the set $\mathcal{K}$ is $\mathcal{R}$-convex, then clearly the $\mathcal{R}$-segment $[x, y]_{\mathcal{R}}$ is contained in $\mathcal{K}$ for every $x, y \in \mathcal{K}$. Conversely, suppose the $\mathcal{R}$-segment $[x, y]_{\mathcal{R}}$ is contained in $\mathcal{K}$ for every $x, y \in \mathcal{K}$. We show that $\mathcal{K}$ is closed under the $\mathcal{R}$-convex combination of the form $\sum_{i=1}^m t_i^* x_i t_i$, where $x_i \in \mathcal{K}$, $t_i \in \mathcal{R}$, and $\sum_{i=1}^m t_i^* t_i = 1$. Let $x := \sum_{i=1}^m t_i^* x_i t_i$. We prove that $x \in \mathcal{K}$. The proof is given by induction in m. The case of $m = 1$ is evident (since the only 1-term $\mathcal{R}$-convex combinations are of the form is $1^* x_1 1 = x_1 \in \mathcal{K}$). Assume that we already know that any $\mathcal{R}$-convex combination of $m - 1$ vectors, $m \geq 2$, from $\mathcal{K}$ is again a vector from $\mathcal{K}$, and let us prove that this statement remains valid also for all $\mathcal{R}$-convex combinations of m vectors from $\mathcal{K}$. Let the representation of x be such an $\mathcal{R}$-convex combination. We can assume that $t_m^* t_m < 1$, since otherwise there is nothing to prove (indeed, if $t_m^* t_m = 1$, then the remaining t_i's should be zero, since all $t_i^* t_i$'s are nonnegative with the unit sum, and we have $x = t_m^* x_m t_m \in \mathcal{K}$). Assuming $t_m^* t_m < 1$ and noting that $\mathcal{R}$ satisfies the

(PSR)-axiom, we can write

$$x = (1 - t_m^* t_m)^{\frac{1}{2}} \Big[\sum_{i=1}^{m-1} (1 - t_m^* t_m)^{-\frac{1}{2}} t_i^* x_i t_i (1 - t_m^* t_m)^{-\frac{1}{2}} \Big] (1 - t_m^* t_m)^{\frac{1}{2}} + t_m^* x_m t_m,$$

whence

$$\sum_{i=1}^{m-1} (1 - t_m^* t_m)^{-\frac{1}{2}} t_i^* t_i (1 - t_m^* t_m)^{-\frac{1}{2}} = (1 - t_m^* t_m)^{-\frac{1}{2}} (1 - t_m^* t_m)(1 - t_m^* t_m)^{-\frac{1}{2}} = 1.$$

So what is in the brackets, clearly is an $\mathcal{R}$-convex combination of $m - 1$ points from $\mathcal{K}$ and therefore, by the inductive hypothesis, this is a point, let it be called z, from $\mathcal{K}$; we have

$$x = (1 - t_m^* t_m)^{\frac{1}{2}} z (1 - t_m^* t_m)^{\frac{1}{2}} + t_m^* x_m t_m$$

with z and $x_m \in \mathcal{K}$, and so $x \in [z, x_m]_{\mathcal{R}} \subseteq \mathcal{K}$ by the assumption. □

W may remark that Theorem 2.6 is a generalization of [10, Theorem 15]. Let M_n denote the $*$-ring of complex $n \times n$ matrices. Considering $\mathcal{M} = M_n$ and $\mathcal{R} = M_n$ in Theorem 2.6, the unital $*$-ring M_n satisfies the (PSR)-axiom and it is a bimodule over itself and so we get [10, Theorem 15].

Definition 2.7. Let $\mathcal{M}$ be a bimodule over a unital $*$-ring $\mathcal{R}$ and $\mathcal{S} \subset \mathcal{M}$. The convex hull of $\mathcal{S}$ in $\mathcal{M}$ over $\mathcal{R}$ is the smallest $\mathcal{R}$-convex set containing $\mathcal{S}$. We denote it by $\mathcal{R}-\mathsf{conv}\mathcal{S}$.

It is clear that

$$\mathcal{R}-\mathsf{conv}\mathcal{S} = \left\{ \sum_i t_i^* x_i t_i : t_i \in \mathcal{R}, x_i \in \mathcal{S}, \sum_i t_i^* t_i = 1 \right\}.$$

If the unital $*$-ring $\mathcal{R}$ is commutative, then $\mathcal{R}-\mathsf{conv}\{m\} = \{m\}$ for $m \in \mathcal{M}$.

Proposition 2.8. *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are bimodules over the unital $*$-rings $\mathcal{R}$ and $\mathcal{B}$, respectively and $f : \mathcal{R} \to \mathcal{B}$ is a unital $*$-homomorphism. If $g : \mathcal{M} \to \mathcal{N}$ is an f-homomorphism and $\mathcal{S} \subset \mathcal{M}$, then*

$$g(\mathcal{R}-\mathsf{conv}\mathcal{S}) \subseteq f(\mathcal{R})-\mathsf{conv}g(\mathcal{S}).$$

The equality holds, when $f^{-1}(1) = \{1\}$.

Proof. We have

$$\begin{aligned}
f(\mathcal{R})-\mathsf{conv}\ g(\mathcal{S})\\
&= \left\{\sum_i f(t_i)^* g(x_i) f(t_i) : t_i \in \mathcal{R}, x_i \in \mathcal{S}, \sum_i f(t_i)^* f(t_i) = 1\right\}\\
&= \left\{g(\sum_i t_i^* x_i t_i) : t_i \in \mathcal{R}, x_i \in \mathcal{S}, f(\sum_i t_i^* t_i) = 1\right\}\\
&\subseteq \left\{g(\sum_i t_i^* x_i t_i) : t_i \in \mathcal{R}, x_i \in \mathcal{S}, \sum_i t_i^* t_i = 1\right\}\\
&= g(\mathcal{R}-\mathsf{conv}\mathcal{S}).
\end{aligned}$$

□

If the unital $*$-ring $\mathcal{B}$ is commutative, then $g(\mathcal{R}-\mathsf{conv}\{m\}) = \{g(m)\}$ for $m \in \mathcal{M}$.

Corollary 2.9. *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are bimodules over the unital $*$-rings $\mathcal{R}$ and $\mathcal{B}$, respectively and*

$$f : \mathcal{R} \to \mathcal{B}$$

is a unital $*$-homomorphism. If $g : \mathcal{M} \to \mathcal{N}$ is an f-homomorphism, then

$$g([x, y]_{\mathcal{R}}) \subseteq [g(x), g(y)]_{f(\mathcal{R})}$$

for every $x, y \in \mathcal{M}$. Moreover, the equality holds, when $f^{-1}(1) = \{1\}$.

Proof. In view of Proposition 2.8, one can see that

$$\begin{aligned}
g([x, y]_{\mathcal{R}}) &= g(\mathcal{R}-\mathsf{conv}\{x, y\})\\
&\subseteq f(\mathcal{R})-\mathsf{conv}\{g(x), g(y)\}\\
&= [g(x), g(y)]_{f(\mathcal{R})}.
\end{aligned}$$

□

Proposition 2.10. *Let $\mathcal{M}$ and $\mathcal{N}$ be bimodules over the unital $*$-rings $\mathcal{R}$ and $\mathcal{B}$, respectively and $f : \mathcal{R} \to \mathcal{B}$ a $*$-homomorphism such that $f^{-1}(1) = \{1\}$. If $g : \mathcal{M} \to \mathcal{N}$ is an f-homomorphism, $\mathcal{K} \subset \mathcal{M}$ is $\mathcal{R}$-convex, and the unital $*$-ring $\mathcal{B}$ satisfies the (PSR)-axiom, then $g(\mathcal{K})$ is $f(\mathcal{R})$-convex.*

Proof. Note that $f(\mathcal{R})$ is a $*$-subring of $\mathcal{B}$ and $g(\mathcal{M})$ is a subbimodule of $\mathcal{N}$ over the unital $*$-subring $f(\mathcal{R})$. According to Theorem 2.6, we show that $[g(x), g(y)]_{f(\mathcal{R})} \subseteq g(\mathcal{K})$ for every $x, y \in \mathcal{K}$. Since $\mathcal{K}$ is

$\mathcal{R}$-convex, $[x,y]_{\mathcal{R}} \subseteq \mathcal{K}$ and so $g([x,y]_{\mathcal{R}}) \subseteq g(\mathcal{K})$. By using Corollary 2.9, we reach

$$[g(x), g(y)]_{f(\mathcal{R})} = g([x,y]_{\mathcal{R}}) \subseteq g(\mathcal{K}).$$

□

Corollary 2.11. *Under the hypotheses of Proposition 2.10, if f is an epimorphism, then $\mathsf{Im}g$ is a $\mathcal{B}$-convex subset of $\mathcal{N}$.*

Corollary 2.12. *Let $\mathcal{M}$ and $\mathcal{N}$ be bimodules over the unital $*$-ring $\mathcal{R}$. If $g : \mathcal{M} \to \mathcal{N}$ is an $\mathcal{R}$-bimodule homomorphism, $\mathcal{K} \subset \mathcal{M}$ is $\mathcal{R}$-convex, and the unital $*$-ring $\mathcal{R}$ satisfies the (PSR)-axiom, then $g(\mathcal{K})$ is $\mathcal{R}$-convex.*

Proof. Suppose that $f : \mathcal{R} \to \mathcal{R}$ is the identity mapping and apply Proposition 2.10. □

We state the converse of Proposition 2.10 as follows:

Proposition 2.13. *Let $\mathcal{M}$ and $\mathcal{N}$ be bimodules over the unital $*$-rings $\mathcal{R}$ and $\mathcal{B}$, respectively and $f : \mathcal{R} \to \mathcal{B}$ is a unital $*$-homomorphism. If $g : \mathcal{M} \to \mathcal{N}$ is an f-monomorphism, $\mathcal{K} \subset \mathcal{M}$, $g(\mathcal{K})$ is $f(\mathcal{R})$-convex, and the unital $*$-ring $\mathcal{R}$ satisfies the (PSR)-axiom, then $\mathcal{K}$ is $\mathcal{R}$-convex.*

Proof. According to Theorem 2.6, we show that $[x,y]_{\mathcal{R}} \subseteq \mathcal{K}$ for every $x, y \in \mathcal{K}$. Since $g(\mathcal{K})$ is $f(\mathcal{R})$-convex, $[g(x), g(y)]_{f(\mathcal{R})} \subseteq g(\mathcal{K})$ for every $x, y \in \mathcal{K}$ and so $g([x,y]_{\mathcal{R}}) \subseteq g(\mathcal{K})$, by Corollary 2.9. Since g is an f-monomorphism, the desired result follows. □

Corollary 2.14. *Let $\mathcal{M}$ and $\mathcal{N}$ be bimodules over the unital $*$-ring $\mathcal{R}$. If $g : \mathcal{M} \to \mathcal{N}$ is an $\mathcal{R}$-bimodule monomorphism, $\mathcal{K} \subset \mathcal{M}$, $g(\mathcal{K})$ is $\mathcal{R}$-convex, and the unital $*$-ring $\mathcal{R}$ satisfies the (PSR)-axiom, then $\mathcal{K}$ is $\mathcal{R}$-convex.*

Proof. Suppose that $f : \mathcal{R} \to \mathcal{R}$ is the identity mapping and apply Proposition 2.13. □

Corollary 2.15. *Let $\mathcal{M}$ and $\mathcal{N}$ be bimodules over the unital $*$-rings $\mathcal{R}$ and $\mathcal{B}$, respectively and $f : \mathcal{R} \to \mathcal{B}$ is a $*$-epimorphism such that $f^{-1}(1) = \{1\}$. If $g : \mathcal{M} \to \mathcal{N}$ is an f-monomorphism and the unital $*$-rings $\mathcal{R}$ and $\mathcal{B}$ satisfy the (PSR)-axiom, then $\mathcal{K} \subset \mathcal{M}$ is $\mathcal{R}$-convex if and only if $g(\mathcal{K}) \subset \mathcal{N}$ is $\mathcal{B}$-convex.*

Proposition 2.16. *Let $\mathcal{M}$ and $\mathcal{N}$ are bimodules over the unital $*$-rings $\mathcal{R}$ and $\mathcal{B}$, respectively and $f : \mathcal{R} \to \mathcal{B}$ is a unital $*$-homomorphism. If $g : \mathcal{M} \to \mathcal{N}$ is an f-homomorphism, $S \subset \mathcal{N}$ is $\mathcal{B}$-convex, and the unital $*$-ring $\mathcal{R}$ satisfies the (PSR)-axiom, then $g^{-1}(S)$ is an $\mathcal{R}$-convex subset of $\mathcal{M}$.*

Proof. By applying Theorem 2.6, we prove that $[x,y]_{\mathcal{R}} \subseteq g^{-1}(S)$ for every $x, y \in g^{-1}(S)$. Since S is $\mathcal{B}$-convex and $g(x), g(y) \in S$,

$$[g(x), g(y)]_{\mathcal{B}} \subseteq S.$$

So, it follows from Corollary 2.9 that

$$g([x,y]_{\mathcal{R}}) \subseteq [g(x), g(y)]_{f(\mathcal{R})} \subseteq [g(x), g(y)]_{\mathcal{B}} \subseteq S.$$

Hence, $[x,y]_{\mathcal{R}} \subseteq g^{-1}(S)$. □

Corollary 2.17. *Under the hypothesis of Proposition 2.16,* $\mathsf{Ker}g$ *is an* $\mathcal{R}$*-convex subset of* $\mathcal{M}$.

Proof. It follows from the fact that, $\mathsf{Ker}g = g^{-1}(\{0\})$ and $\{0\}$ is a $\mathcal{B}$-convex subset of $\mathcal{N}$. □

Corollary 2.18. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be bimodules over the unital* $*$*-ring* $\mathcal{R}$*. If* $g : \mathcal{M} \to \mathcal{N}$ *is an* $\mathcal{R}$*-bimodule homomorphism,* $S \subset \mathcal{N}$ *is* $\mathcal{R}$*-convex, and the unital* $*$*-ring* $\mathcal{R}$ *satisfies the (PSR)-axiom, then* $g^{-1}(S)$ *is* $\mathcal{R}$*-convex.*

Proof. Suppose that $f : \mathcal{R} \to \mathcal{R}$ is the identity mapping and apply Proposition 2.16. □

Definition 2.19. Let $\mathcal{M}$ be a bimodule over a unital $*$-ring $\mathcal{R}$ and $\mathcal{S} \subseteq \mathcal{M}$. The subset $\mathcal{S}$ is invariant under an f-homomorphism $g : \mathcal{M} \to \mathcal{M}$ whenever $g(\mathcal{S}) \subseteq \mathcal{S}$.

Proposition 2.20. *Suppose that* $\mathcal{M}$ *is a bimodule over the unital* $*$*-ring* $\mathcal{R}$ *and* $f : \mathcal{R} \to \mathcal{R}$ *is a unital* $*$*-homomorphism. If* $g : \mathcal{M} \to \mathcal{M}$ *is an* f*-homomorphism and* $S \subset \mathcal{M}$ *is invariant under* g*, then* $\mathcal{R}$*-convex hull of* $\mathcal{S}$ *is invariant under* g.

Proof. Taking into account that $g(\mathcal{S}) \subseteq \mathcal{S}$ we have

$$\mathcal{R}\mathsf{-conv}g(\mathcal{S}) \subseteq \mathcal{R}\mathsf{-conv}\mathcal{S}.$$

Consequently, by Corollary 2.9, we deduce

$$\begin{aligned} g(\mathcal{R}\mathsf{-conv}\mathcal{S}) &\subseteq f(\mathcal{R})\mathsf{-conv}g(\mathcal{S}) \\ &\subseteq \mathcal{R}\mathsf{-conv}g(\mathcal{S}) \\ &\subseteq \mathcal{R}\mathsf{-conv}\mathcal{S}. \end{aligned}$$

□

Corollary 2.21. *Suppose that* $\mathcal{M}$ *is a bimodule over the unital* $*$*-ring* $\mathcal{R}$*. If* $g : \mathcal{M} \to \mathcal{M}$ *is an* $\mathcal{R}$*-bimodule homomorphism and* $S \subset \mathcal{M}$ *is invariant under* g*, then* $\mathcal{R}$*-convex hull of* $\mathcal{S}$ *is invariant under* g.

Proof. According to Corollary 2.9 and Proposition 2.20 and for the identity mapping $f : \mathcal{R} \to \mathcal{R}$, one has

$$g(\mathcal{R}-\mathsf{conv}\mathcal{S}) \subseteq \mathcal{R}-\mathsf{conv}g(\mathcal{S}) \subseteq \mathcal{R}-\mathsf{conv}\mathcal{S}.$$

□

3. $\mathcal{R}$-extreme points of $\mathcal{R}$-convex sets

In this section, we discuss the effect of an f-homomorphism on $\mathcal{R}$-extreme points of $\mathcal{R}$-convex subsets of bimodules over unital $*$-rings.

Definition 3.1. Let $\mathcal{M}$ be a bimodule over a unital $*$-ring $\mathcal{R}$ and $\mathcal{K} \subset \mathcal{M}$ an $\mathcal{R}$-convex subset of $\mathcal{M}$. The point $x \in \mathcal{K}$ is called $\mathcal{R}$-extreme point in $\mathcal{K}$, if $x = \sum_i t_i^* x_i t_i$ is a proper $\mathcal{R}$-convex combination of elements $x_i \in \mathcal{K}$, then each x_i comes from the unitary orbit of x, i.e., there exist unitary elements $u_i \in \mathcal{R}$ such that for all i, $x = u_i^* x_i u_i$.

If $x \in \mathcal{K}$ is an $\mathcal{R}$-extreme point of $\mathcal{K}$ and $x = \sum_i a_i x_i$ is a proper linear convex combination, then we have $x = x_i$ for each i.

Proposition 3.2. *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are bimodules over unital $*$-rings $\mathcal{R}$ and $\mathcal{B}$, respectively. Let $f : \mathcal{R} \to \mathcal{B}$ be a unital $*$-epimorphism, $g : \mathcal{M} \to \mathcal{N}$ an f-homomorphism and $\mathcal{K} \subset \mathcal{M}$ an $\mathcal{R}$-convex set. If y is a $\mathcal{B}$-extreme point of $g(\mathcal{K})$ and x is an $\mathcal{R}$-extreme point of $g^{-1}(y) \cap \mathcal{K}$, then x is an $\mathcal{R}$-extreme point of $\mathcal{K}$.*

Proof. Let $x = \sum_i t_i^* x_i t_i$ be a proper $\mathcal{R}$-convex combination of a finite number of elements $x_i \in \mathcal{K}$. Then,

$$y = g(x) = g(\sum_i t_i^* x_i t_i) = \sum_i g(t_i^* x_i t_i) = \sum_i f(t_i)^* g(x_i) f(t_i)$$

and

$$\sum_i f(t_i)^* f(t_i) = \sum_i f(t_i^* t_i) = f(\sum_i t_i^* t_i) = f(1) = 1.$$

Since the coefficients $f(t_i)$ are invertible in $\mathcal{B}$ and y is a $\mathcal{B}$-extreme point of $g(\mathcal{K})$, there exist unitary elements $v_i \in \mathcal{R}$ such that

$$y = f(v_i)^* g(x_i) f(v_i).$$

It follows that $y = g(v_i^* x_i v_i)$ and so $v_i^* x_i v_i \in g^{-1}(y)$. We can rewrite

$$x = \sum_i t_i^* x_i t_i = \sum_i t_i^* (v_i v_i^*) x_i (v_i v_i^*) t_i = \sum_i (v_i^* t_i)^* (v_i^* x_i v_i)(v_i^* t_i).$$

This means that x is a proper $\mathcal{R}$-convex combination of elements in $g^{-1}(y) \cap \mathcal{K}$. Since x is an $\mathcal{R}$-extreme point of $g^{-1}(y) \cap \mathcal{K}$, there exist unitary elements $w_i \in \mathcal{R}$ such that $x = w_i^*(v_i^* x_i v_i) w_i$. Let $u_i = v_i w_i$.

Then, u_i is unitary and $x = u_i^* x_i u_i$, i.e., each x_i comes from the unitary orbit of x. □

Theorem 3.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be bimodules over unital $*$-rings $\mathcal{R}$ and $\mathcal{B}$, respectively and let $g : \mathcal{M} \to \mathcal{N}$ be an f-monomorphism and $\mathcal{K} \subset \mathcal{M}$ an $\mathcal{R}$-convex subset.*

(a) *If $f : \mathcal{R} \to \mathcal{B}$ is a $*$-epimorphism such that $f^{-1}(1) = \{1\}$ and x is an $\mathcal{R}$-extreme point of $\mathcal{K}$, then $g(x)$ is a $\mathcal{B}$-extreme point of $g(\mathcal{K})$.*

(b) *If $f : \mathcal{R} \to \mathcal{B}$ is a unital $*$-epimorphism and $g(x)$ is a $\mathcal{B}$-extreme point of $g(\mathcal{K})$, then x is an $\mathcal{R}$-extreme point of $\mathcal{K}$.*

Proof. (a) Let $g(x) = \sum_i b_i^* y_i b_i$ be a proper $\mathcal{B}$-convex combination of a finite number of elements $y_i \in g(\mathcal{K})$. There exist $t_i \in \mathcal{R}$ and $x_i \in \mathcal{K}$ such that $f(t_i) = b_i$ and $g(x_i) = y_i$. We observe that

$$g(x) = \sum_i b_i^* y_i b_i = \sum_i f(t_i)^* g(x_i) f(t_i) = g(\sum_i t_i^* x_i t_i),$$

$$1 = \sum_i b_i^* b_i = \sum_i f(t_i)^* f(t_i) = f(\sum_i t_i^* t_i)$$

and so that $x = \sum_i t_i^* x_i t_i$ with $\sum_i t_i^* t_i = 1$. This $\mathcal{R}$-convex combination is proper. Since x is an $\mathcal{R}$-extreme point of $\mathcal{K}$, there exist unitary elements $u_i \in \mathcal{R}$ such that $x = u_i^* x_i u_i$. Thus,

$$\begin{aligned} g(x) &= g(u_i^* x_i u_i) \\ &= f(u_i^*) g(x_i) f(u_i) \\ &= f(u_i)^* g(x_i) f(u_i) \\ &= f(u_i)^* y_i f(u_i). \end{aligned}$$

This indicates that each y_i comes from the unitary orbit of $g(x)$, i.e., $g(x)$ is a $\mathcal{B}$-extreme point of $g(\mathcal{K})$.

(b) Since g is f-monomorphism, $g^{-1}(y) = \{x\}$, where $y = g(x)$. Hence, $g^{-1}(y) \cap \mathcal{K}$ is a singleton subset containing x and so x is $\mathcal{R}$-extreme. The result now follows from Proposition 3.2 and the fact that y is a $\mathcal{B}$-extreme point of $g(\mathcal{K})$, □

The following corollary provides the necessary and sufficient condition.

Corollary 3.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be bimodules over unital $*$-rings $\mathcal{R}$ and $\mathcal{B}$, respectively and $f : \mathcal{R} \to \mathcal{B}$ a $*$-epimorphism such that $f^{-1}(1) = \{1\}$. If $g : \mathcal{M} \to \mathcal{N}$ is an f-monomorphism and $\mathcal{K} \subset \mathcal{M}$ is $\mathcal{R}$-convex, then x is an $\mathcal{R}$-extreme point of $\mathcal{K}$ if and only if $g(x)$ is a $\mathcal{B}$-extreme point of $g(\mathcal{K})$.*

4. EXAMPLES

Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the collection of bounded linear operators on $\mathcal{H}$ and by $\mathcal{B}_0(\mathcal{H})$ the compact operators. The unit ball of $\mathcal{B}(\mathcal{H})$; that is,

$$B = \{T \in \mathcal{B}(\mathcal{H}) : ||T|| \leq 1\}$$

is $\mathcal{B}(\mathcal{H})$-convex [10].

Example 4.1. Consider a triangle with the wedges $(0, 0)$, $(1, 0)$, and $(1, 1)$ in the plane $\mathbb{R}^2$. This triangle is $\mathbb{R}$-convex in $\mathbb{R}^2$ and its $\mathbb{R}$-extreme points are its wedges.

Example 4.2. Consider the unit disk in the complex plane. This unit disk is $\mathbb{C}$-convex in $\mathbb{C}$ and its $\mathbb{C}$-extreme points are the set

$$\{z \in \mathbb{C} : |z| = 1\}.$$

Example 4.3. [7, Corollary 1.2] The $\mathcal{B}(\mathcal{H})$-extreme points of the unit ball B are precisely the isometries and co-isometries.

The numerical range of an operator $T \in \mathcal{B}(\mathcal{H})$, denoted by $W(T)$, is the collection of complex numbers $\langle Th, h\rangle$, where h runs through all vectors in $\mathcal{H}$ of norm 1. The numerical radius of T, $w(T)$, is defined by

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

Let $M_n(\mathbb{C})$ be the set of $n \times n$ matrices with entries in $\mathbb{C}$. We denote by W_1 the set of all matrices $T \in M_n$ such that $w(T) \leq 1$. It is a standard fact that W_1 is linearly convex ($\mathbb{R}$-convex) and in [10] it is shown that W_1 is M_n-convex. We denote by W_1^1 the collection of matrices $T \in M_n$ for which $w(T) = 1$ and $1 \in W(T)$.

Example 4.4. [7, Theorem 2.9] $T \in M_n$ is $\mathbb{C}$-extreme point of W_1 if and only if $W(T)$ is the entire unit disk. Recall that $W(T)$ is an elliptical disk.

Example 4.5. [7, Theorem 2.10] The identity matrix and all nilpotent matrices in W_1^1 are M_n-extreme point in W_1^1.

Example 4.6. [7, Theorem 3.1] Assume that H is infinite dimensional and let $S = \{T \in \mathcal{B}(\mathcal{H}) : 0 \leq T \leq I\}$ be the unit operator interval and $P \in \mathcal{B}(\mathcal{H})$ a projection (unequal to 0 or I).

1. If P has infinite rank and co-rank, then P is $\mathcal{B}(\mathcal{H})$-extreme of S.
2. If P has finite rank, then P is $\mathcal{B}(\mathcal{H})$-extreme of $S \cap \mathcal{B}_0(\mathcal{H})$.
3. If P has finite co-rank, then P is $\mathcal{B}(\mathcal{H})$-extreme of the set

$$\{T \in S : I - T \in \mathcal{B}_0(\mathcal{H})\}.$$

Our goal was to extend the notion of convexity to bimodules over $*$-rings which is available for $*$-rings or $*$-algebras. All of the examples that exist are for $*$-rings or $*$-algebras. So, to clarify the distinction between our new concept of convex sets of bimodules over $*$-rings and the notion of C*-convexity for $*$-rings or $*$-algebras, we include an example which is not a bimodule over itself. In other words, the module is distinct from its ring.

Example 4.7. Let G be a commutative group and consider

$$End(G) := \{f : G \to G : f \text{ is a homomorphism}\}.$$

Then $End(G)$ is a unital non-commutative $*$-ring with

$$\begin{aligned} (f+g)(a) &= f(a) + g(a), \\ (fg)(a) &= f(g(a)), \\ f^*(a) &= f(a) \end{aligned}$$

for every $a \in G$ and $f, g \in End(G)$. In this situation, the group G is an $End(G)$-bimodule by

$$a.f = f.a = f(a)$$

for every $a \in G$ and $f \in End(G)$. A set $\mathcal{K} \subset G$ is $End(G)$-convex, if $\mathcal{K}$ is closed under the formation of finite sums of the type $\sum_i f_i^*.x_i.f_i$, where $f_i \in End(G)$, $x_i \in \mathcal{K}$ and $\sum_i f_i^* f_i = I$, I is the identity homomorphism on G. We note that

$$\sum_i f_i^*.x_i.f_i = \sum_i f_i.x_i.f_i = \sum_i f_i.f_i(x_i) = \sum_i f_i^2(x_i).$$

So, $\mathcal{K}$ is $End(G)$-convex, when $\sum_i f_i^2(x_i) \in \mathcal{K}$. We now provide an $End(G)$-convex set in G. Let $g \in End(G)$ be a fixed homomorphism such that $gf = fg$ for every $f \in End(G)$ and consider

$$S_g = \{x \in G : g(x) = 0\}.$$

Then S_g is $End(G)$-convex in G.

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$\mathcal{R}$-CONVEX SUBSETS OF BIMODULES OVER $*$-RINGS

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زیرمجموعه‌های $\mathcal{R}$-محدب دومدول‌ها روی $*$-حلقه‌ها

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فرض کنیم $\mathcal{M}$ و $\mathcal{N}$ دومدول‌هایی به ترتیب روی $*$-حلقه‌های $\mathcal{R}$ و $\mathcal{B}$ باشند. مفهوم $\mathcal{R}$-تحدب و مفهوم متناظر نقاط $\mathcal{R}$-انتهایی را بررسی می‌کنیم. در مورد اثر یک f-همریختی روی زیرمجموعه‌های $\mathcal{R}$-محدب و نقاط $\mathcal{R}$-انتهایی آن بحث می‌کنیم. به عبارت دیگر، نشان می‌دهیم یک f-همریختی از $\mathcal{M}$ به $\mathcal{N}$ چگونه زیرمجموعه‌های $\mathcal{R}$-محدب و نقاط $\mathcal{R}$-انتهایی را به زیرمجموعه‌های $\mathcal{B}$-محدب و نقاط $\mathcal{B}$-انتهایی منتقل می‌کند و برعکس. به‌علاوه، نشان می‌دهیم پوش $\mathcal{R}$-محدب زیرمجموعه‌های پایا تحت f-همریختی‌ها، پایا باقی می‌ماند.

کلمات کلیدی: مجموعه‌های $\mathcal{R}$-محدب، نقاط $\mathcal{R}$-انتهایی، f-همریختی.