Journal of Algebraic Systems
Vol. 12, No. 1, (2024), pp 123-133

ON INVERSE LIMIT OF A PROJECTIVE SYSTEM OF BL-ALGEBRAS

R. TAYEBI KHORAMI

Abstract. In this paper, the inverse limits of a projective system of basic logic algebras (BL-algebras) are introduced, and their basic properties are studied. The set of congruences of a BL-algebra is considered as a poset. Then, a quotient inverse system and a quotient inverse limit on it are constructed. Moreover, by setting filters of a BL-algebra, quotient projective and inverse systems are constructed.

1. Introduction

H'ajek [4] introduced the basic logic (BL) as the logic of continuous triangular norms and their residua. This BL is fuzzy logic; that is, it is complete with respect to linearly ordered models. The algebraic semantics of BL includes a variety of BL-algebras. Cignoli, Esteva, Godo, and Torrens proved that a variety of BL-algebras is solely generated by the continuous triangular norms on the interval $[0,1]$ of reals.

The origin of studying the inverse limits dates back to the 1920s. The classical theory of inverse systems and inverse limits is essential in the extension of homology and cohomology theory. An exhaustive discussion of inverse systems in some classical categories, such as (sets) Set, (topological spaces) Top, (groups) Grp, and (rings) Rng, is given in [1] and presented by Eilenberg and Steenrod [3], which is a milestone in the development of that theory.

DOI: 10.22044/JAS.2022.12222.1643.
MSC(2010): Primary: 03B47; Secondary: 03G25, 06D99.
Keywords: BL-algebra; Inverse Limit; Projective System.
Received: 19 August 2022, Accepted: 23 December 2022.

As this is the case with products, the inverse limits might not generally exist in any category, whereas inverse systems exist in every category. Considering [3], the inverse limits exist in every category where products of objects and the equalizers [1] of pairs of morphisms exist. In other words, the inverse limits exist in every category if the category is complete in the sense of [1].

Additionally, the inverse system has, at most, one limit. If the inverse limit of an inverse system exists in each category C, then this limit is unique up to C-isomorphism. The inverse limits always exist in the categories Set, Top, Grp, and Rng. It should be noted that the inverse limits are generally restricted to diagrams over directed sets.

An inverse limit (also called the projective limit) is a construction that allows one to “glue together” several related objects. The precise gluing process is specified by morphisms between the objects. The inverse limits are a particular case of the concept of limit in the category theory. In this paper, we apply the concept of inverse limit in the sense of category theory to some collections of the BL-algebras. This notion in the category theory has been studied in different kinds of categories. In this paper, the concepts of the inverse limit of an inverse system of BL-algebras are introduced.

2. Preliminaries

Definition 2.1. [4] A BL-algebra is an algebra $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

(BL1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,

(BL2) $(A, \odot, 1)$ is an abelian monoid,

(BL3) $x \odot z \leq y$ if and only if $z \leq x \rightarrow y$,

(BL4) $x \odot (x \rightarrow y) = x \wedge y$,

(BL5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ for all $x, y \in A$.

In this paper, we denote BL-algebra $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ with A.

Definition 2.2. [4] Let A be a BL-algebra. A nonempty subset F of A is called a filter of A if F satisfies the following conditions:

$(F1)$ If $x \in F,\ x \leq y$ and $y \in A$, then $y \in F$,

$(F2)$ $x \odot y \in F$ for every $x, y \in A$; that is, F is a subsemigroup of A.

A mapping $f : X \longrightarrow Y$ is a BL-homomorphism if

$$f(x \odot y) = f(x) \odot f(y)$$

and $f(x \rightarrow y) = f(x) \rightarrow f(y)$ for all $x, y \in A$.

Definition 2.3. [2] Let I be a set and let "$\leq$" be a binary operation on I. We call $I = (I, \leq)$ a directed partially ordered set or directed poset if it satisfies the following conditions:

(i) $i \leq i$, for all $i \in I$,
(ii) $i \leq j$ and $j \leq i$ imply $i = j$, for $i, j \in I$,
(iii) $i \leq j$ and $j \leq k$ imply $i \leq k$, for $i, j, k \in I$,
(iv) if $i, j \in I$, then there exists some $k \in I$ such that $i, j \leq k$.

3. Projective Systems

Definition 3.1. An inverse or projective system of BL-algebras over a directed poset I, consists of a collection $\{A_i \mid i \in I\}$ of BL-algebras indexed by I, and a collection of BL-homomorphisms $\{\varphi_{ij} : A_i \to A_j\}$, defined whenever $i \geq j$, such that $\varphi_{ii} = id_{A_i},\ \varphi_{jk}\varphi_{ij} = \varphi_{ik}$ for $i, j, k \in I$ and $i \geq j \geq k$.

We denote an inverse system by $\{A_i, \varphi_{ij}, I\}$ or $\{A_i, \varphi_{ij}\}$.

Definition 3.2. Let B be a BL-algebra, let $\{A_i, \varphi_{ij}, I\}$ be an inverse system of BL-algebras, and let $\psi_i : B \to A_i$ be a BL-homomorphism for each $i \in I$. These mappings ψ_i are called to be compatible if $\varphi_{ij}\psi_i = \psi_j$ whenever $i \geq j$.

Definition 3.3. Let $\{A_i, \varphi_{ij}, I\}$ be an inverse system of BL-algebras. Then a subalgebra A of $\prod_{i\in I} A_i$ together with compatible homomorphisms $\varphi_i : A \to A_i$ is an inverse limit or a projective limit of $\{A_i, \varphi_{ij}, I\}$ if the following universal property is satisfied: whenever B is a BL-algebra and $\{\psi_i : B \to A_i\}$ is a set of compatible homomorphisms, then there is a unique homomorphism $\psi : B \to A$ such that $\varphi_i\psi = \psi_i$ for all $i \in I$.

Moreover, we say ψ induced by the compatible homomorphisms ψ_i. The maps $\varphi_i : A \to A_i$ are called projections. We shall denote the inverse limit of the inverse system $\{A_i, \varphi_{ij}, I\}$ by $\varprojlim_{i\in I} A_i$ or (A, φ_i).

Theorem 3.4. *Let $\{A_i, \varphi_{ij}, I\}$ be an inverse system of BL-algebras. Then (A, φ_i) is an inverse limit of the inverse system $\{A_i, \varphi_{ij}, I\}$, where*

$$A = \Big\{(x_i) \in \prod_{i\in I} A_i \mid \varphi_{ij}(x_i) = x_j \textit{ for all } i, j \in I \textit{ and } i \geq j\Big\}$$

and $\varphi_i : A \to A_i$ is the restriction of the natural projection

$$\pi_i : \prod_{i\in I} A_i \to A_i.$$

Proof. Let $(x_i), (y_i) \in A$. Then $\varphi_{ij}(x_i) = x_j$ and $\varphi_{ij}(y_i) = y_j$, whenever $i \geq j$. We have $(x_i) \odot (y_i) = (x_i \odot y_i)$ and $(x_i) \to (y_i) = (x_i \to y_i)$. Also

$$\varphi_{ij}(x_i \odot y_i) = \varphi_{ij}(x_i) \odot \varphi_{ij}(y_i) = x_j \odot y_j$$

and

$$\varphi_{ij}(x_i \to y_i) = \varphi_{ij}(x_i) \to \varphi_{ij}(y_i) = x_j \to y_j.$$

Therefore, $(x_i) \odot (y_i) \in A$ and $(x_i) \to (y_i) \in A$. Thus A is a subalgebra of $\prod_{i \in I} A_i$.

Let $(x_i) \in A$. Then

$$\varphi_{ij}\varphi_i((x_i)) = \varphi_{ij}(\varphi_i(x_i)) = \varphi_{ij}(x_i) = x_j = \varphi_j((x_i));$$

that is, φ_i is compatible for all $i \in I$.

Now let B be a BL-algebra and let $\{\psi_i : B \to A_i\}$ be a set of compatible BL-homomorphisms. We define

$$\psi : B \to A$$

by $\psi(y) = (\psi_i(y))$ for each $y \in B$. Then first, we show that $\psi(y) \in A$. Since $\psi_i^{'}s$ are compatible, so

$$\varphi_{ij}(\psi_i(y)) = \psi_j(y) \in A_j.$$

Thus $\psi(y) = (\psi_i(y)) \in A$. Now, we will show that ψ is unique. Let $\varphi : B \to A$ be another BL-homomorphism such that $\varphi_i\varphi = \psi_i$. Also, suppose that there exists $y \in B$ such that $\psi(y) \neq \varphi(y)$. By the definition of A, there exists $i \in I$ such that

$$\varphi_i(\psi(y)) = \psi_i(y) \neq \varphi_i(\varphi(y)),$$

which is a contradiction. Hence, $\psi(y) = \varphi(y)$ for all $y \in B$. So ψ is unique. Therefore (A, φ_i) is an inverse limit of the inverse system $\{A_i, \varphi_{ij}, I\}$. □

Lemma 3.5. *Let $\{A_i, \varphi_{ij}, I\}$ be an inverse system of BL-algebras and let (A, φ_i) be an inverse limit of $\{A_i, \varphi_{ij}, I\}$. Then the homomorphism $id_A : A \to A$ satisfies $\varphi_i id_A = \varphi_i$ for all $i \in I$, and it is the only homomorphism with this property.*

Proof. Let (A, φ_i) be an inverse limit of BL-algebras and let $\{A_i, \varphi_{ij}, I\}$. By the definition of (A, φ_i), the maps $\varphi_i : A \to A_i$ are compatible. Thus, the universal property of the inverse limit shows that there exists a unique BL-homomorphism $\varphi : A \to A$ such that $\varphi_i\varphi = \varphi_i$ for all $i \in I$. On the other hand, since $\varphi_i id_A = \varphi_i$ for all $i \in I$, and φ is unique, $\varphi = id_A$. □

Theorem 3.6. *Let (A, φ_i) and (B, ψ_i) be two inverse limits of the inverse system $\{A_i, \varphi_{ij}, I\}$. Then there exists a unique isomorphism $\varphi : A \to B$ such that $\psi_i\varphi = \varphi_i$ for all $i \in I$.*

Proof. Since the maps $\psi_i : B \to A_i$ are compatible, the universal property of the inverse limit (A, φ_i) shows that there exists a unique BL-homomorphism $\psi : B \to A$ such that $\varphi_i\psi = \psi_i$ for all $i \in I$. Similarly, there exists a unique BL-homomorphism $\varphi : A \to B$ such that $\psi_i\varphi = \varphi_i$ for all $i \in I$. It follows that $\psi_i = \varphi_i\psi = \psi_i\varphi\psi$ for all $i \in I$. So, by Lemma 3.5, $\varphi\psi = id_B$. Similarly, $\psi\varphi = id_A$. Therefore, φ is an isomorphism. □

Proposition 3.7. *Let $\{A_i, \varphi_{ij}, I\}$ be an inverse system of BL-algebras and let $A = (A, \varphi_i)$ be an inverse limit of $\{A_i, \varphi_{ij}, I\}$. Suppose that $i_0 \in I$ such that $i_0 \geq i_1, \ldots, i_t$ and $\varphi_{i_0 i_k}(X_{i_0}) \subseteq A_{i_k}$, where $X_{i_k} \subseteq A_{i_k}$ for all $k = 0, 1, \ldots, t$. Then*

$$A \bigcap \Big[\Big(\prod_{i \neq i_0} A_i\Big) \times X_{i_0}\Big] = A \bigcap \Big[\Big(\prod_{i \neq i_0, i_1, \ldots, i_t} A_i\Big) \times X_{i_0} \times \cdots \times X_{i_t}\Big].$$

Proof. Let

$$(x_i) \in A \cap [(\prod_{i \neq i_0} A_i) \times X_{i_0}].$$

Then $(x_i) \in A$ and $(x_i) \in (\prod_{i \neq i_0} A_i) \times X_{i_0}$. Thus, $\varphi_{ij}(x_i) = x_j$ for all $i \geq j$ and $x_{i_0} \in X_{i_0}$. Hence, $\varphi_{i_0 j}(x_{i_0}) = x_j$ for all $j \leq i_0$.

Since $i_0 \geq i_1, \ldots, i_t$, we have $\varphi_{i_0 i_k}(x_{i_0}) = x_{i_k}$ for all $k = 1, \ldots, t$. It follows from $\varphi_{i_0 i_k}(X_{i_0}) \subseteq X_{i_k}$ for all $k = 1, \ldots, t$ that $x_{i_k} \in X_{i_k}$ for all $k = 1, \ldots, t$. This implies that

$$(x_i) \in \Big(\prod_{i \neq i_0, i_1, \ldots, i_t} A_i\Big) \times X_{i_0} \times \cdots \times X_{i_t}.$$

Since $(x_i) \in A$, we get

$$(x_i) \in A \bigcap \Big[\Big(\prod_{i \neq i_0, i_1, \ldots, i_t} A_i\Big) \times X_{i_0} \times \cdots \times X_{i_t}\Big].$$

The other inclusion is easy to show. □

Corollary 3.8. *Let $\{A_i, \varphi_{ij}, I\}$ be an inverse system of BL-algebras and let $A = (A, \varphi_i)$ be an inverse limit of $\{A_i, \varphi_{ij}, I\}$. Then*

$$A \bigcap \Big[\Big(\prod_{i \neq i_0} A_i\Big) \times \{0_{A_{i_0}}\}\Big] = A \bigcap \Big[\Big(\prod_{i \neq i_0, i_1, \ldots, i_t} A_i\Big) \times \{0_{A_{i_0}}\} \times \cdots \times \{0_{A_{i_t}}\}\Big],$$

where $i_0 \geq i_1, \ldots, i_t$.

Proof. Since $\varphi_{i_0 i_k}$ is a BL-homomorphism for all $i_0 \geq i_1, \ldots, i_t$ and for all $k = 1, \ldots, t$, by Theorem 3.7, $\varphi_{i_0 i_k}(0_{A_{i_0}}) = 0_{A_{i_k}}$ for all $k = 1, \ldots, t$. Thus, $\varphi_{i_0 i_k}(\{0_{A_{i_0}}\}) = 0_{A_{i_k}}$ for all $i_0 \geq i_1, \ldots, i_t$ and for all $k = 1, \ldots, t$. Then by the property of BL-homomorphisms, the result holds. □

Theorem 3.9. *Let $\{A_i, \varphi_{ij}, I\}$ be an inverse system of BL-algebras and let $A = (A, \varphi_i)$ be its corresponding inverse limit, where $\varphi_i : A \to A_i$ is the restriction of the natural projection $\pi_i : \prod_{i \in I} A_i \to A_i$. Then for all $j \in I$, it holds that*

$$A \bigcap \Big[\Big(\prod_{i \neq j} A_i \Big) \times \{0_{A_j}\} \Big] = \ker \varphi_j.$$

Proof. For any $j \in I$, we have

$$\begin{aligned} \ker \varphi_j &= \big\{ (x_i) \in A \mid \varphi_j((x_i)) = \pi_j((x_i)) = x_j = 0_{A_j} \big\} \\ &= \big\{ (x_i) \in A \mid x_j = 0_{A_j} \big\} \\ &= A \bigcap \Big[\Big(\prod_{i \neq j} A_i \Big) \times \{0_{A_j}\} \Big]. \end{aligned}$$

□

4. Projective Systems and Congruence Relations

Proposition 4.1. *Let A be a BL-algebra and let I be the set of congruences of A. Then $(I, \leq)$ is a directed poset, where the binary operation " $\leq$ " on I is defined by $\theta \leq \phi$ whenever $\phi \subseteq \theta$ for all $\theta, \phi \in I$.*

Theorem 4.2. *Let I be the set of congruences of BL-algebra A. Then $\{A/\phi, \varphi_{\phi\theta}, I\}$ is a projective system where $\theta \leq \phi$ whenever $\phi \subseteq \theta$ for all $\phi, \theta \in I$ and $\varphi_{\phi\theta} : A/\phi \to A/\theta$ is the BL-homomorphism defined by $\varphi_{\phi\theta}([x]_\phi) = [x]_\theta$.*

Proof. By Proposition 4.1, $(I, \leq)$ is a directed poset. Now, we have

$$\varphi_{\theta\lambda}\varphi_{\phi\theta}([x]_\phi) = \varphi_{\theta\lambda}(\varphi_{\phi\theta}([x]_\phi)) = \varphi_{\theta\lambda}([x]_\theta) = [x]_\lambda = \varphi_{\phi\lambda}([x]_\phi).$$

Then, $\varphi_{\theta\lambda}\varphi_{\phi\theta} = \varphi_{\phi\lambda}$ whenever $\phi \geq \theta \geq \lambda$. Therefore, $\varphi_{\phi\phi} : A/\phi \to A/\phi$ is defined by $\varphi_{\phi\phi}([x]_\phi) = [x]_\phi$. Thus, $\varphi_{\phi\phi} = id_{A/\phi}$, which implies that $\{A/\phi, \varphi_{\phi\theta}, I\}$ is a projective system. □

Theorem 4.3. *Let I be the set of congruences of BL-algebra A. Consider the inverse system $\{A/\phi, \varphi_{\phi\theta}, I\}$. Then*

$$\hat{A} = \Big\{([x]_\phi) \in \prod_{\phi \in I} A/\phi \mid \varphi_{\phi\theta}([x]_\phi) = [x]_\theta \text{ for all } \phi, \theta \in I \ , \ \phi \geq \theta\Big\},$$

called the completion of A, *together with the projections* $\varphi_\phi : A \to A/\phi$, *for all* $\phi \in I$, *is an inverse limit of* $\{A/\phi, \varphi_{\phi\theta}, I\}$.

Proof. The result follows from Theorem 3.4. □

Lemma 4.4. *Let* A *be a BL-algebra and let* $\{A/\phi, \varphi_{\phi\theta}, I\}$ *be the defined inverse system of the completion* $\hat{A}$ *of* A. *Then the canonical epimorphisms* $\varphi_\theta : A \to A/\theta$ *are compatible, where* $\theta \in I$.

Proof. Let $x \in A$. Then

$$\varphi_{\phi\theta}(\varphi_\phi(x)) = \varphi_{\phi\theta}([x]_\phi) = [x]_\theta = \varphi_\theta(x)$$

whenever $\phi \geq \theta$. Also, $\varphi_{\phi\phi}([x]_\phi) = [x]_\phi = id_{A/\phi}$. Therefore, the canonical epimorphisms φ_θ are compatible. □

Theorem 4.5. *Let* A *be a BL-algebra and let* $\{A/\phi, \varphi_{\phi\theta}, I\}$ *be the defined inverse system of the completion* $\hat{A}$ *of* A. *Then the canonical epimorphisms* $\varphi_\theta : A \to A/\theta$ *induce a homomorphism* $\gamma : A \to \hat{A}$ *defined by* $\gamma(x) = (\varphi_\theta(x)) = ([x]_\theta)$.

Proof. By Lemma 4.4, $\{\varphi_\theta : A \to A/\theta\}$ is a set of compatible BL-homomorphisms. Since A is a BL-algebra and $\hat{A} = \varprojlim_{\theta \in I} A/\theta$, by Definition 3.3 and from the proof of Theorem 3.4, there exists a homomorphism $\gamma : A \to \hat{A}$ defined by $\gamma(x) = (\varphi_\theta(x)) = ([x]_\theta)$ for each $x \in A$. □

Theorem 4.6. *Let* A *be a BL-algebra and let* $\{A/\phi, \varphi_{\phi\theta}, I\}$ *be the defined inverse system of the completion* $\hat{A}$ *of* A. *Then*

$$\gamma^{-1}(([x]_\theta)) = \bigcap_{\theta \in I} [x]_\theta,$$

for all $([x]_\theta) \in \hat{A}$.

Proof. Let $y \in \gamma^{-1}(([x]_\theta))$. Then $\gamma(y) = ([x]_\theta)$. By the definition of γ, we have $\gamma(y) = ([y]_\theta)$. Thus, $([y]_\theta) = ([x]_\theta)$. Hence, $[y]_\theta = [x]_\theta$ for all $\theta \in I$. This implies that $y \in \bigcap_{\theta \in I} [x]_\theta$. It follows that

$$\gamma^{-1}(([x]_\theta)) \subseteq \bigcap_{\theta \in I} [x]_\theta.$$

Let $y \in \bigcap_{\theta \in I} [x]_\theta$. Then $y \in [x]_\theta$ for all $\theta \in I$. Thus, $[y]_\theta = [x]_\theta$ for all $\theta \in I$. Hence, $([y]_\theta) = ([x]_\theta)$. It follows from $\gamma(y) = ([y]_\theta)$ that $\gamma(y) = ([x]_\theta)$. Hence, $y \in \gamma^{-1}(([x]_\theta))$, which implies $\bigcap_{\theta \in I} [x]_\theta \subseteq \gamma^{-1}(([x]_\theta))$.

□

Let A be a BL-algebra. Then there is a one to one correspondence between the set of congruences of A and the set of filters of A; see [6]. For every congruence θ, there is a corresponding $\equiv_F$, where F is a filter, such that $\theta = \equiv_F$. With the above discussion, we can write every congruence θ as a filter F in A.

Theorem 4.7. *Let A be a BL-algebra, and let $\{F_i : i \in I\}$ be the family of filters of A such that $F_i \subseteq F_j$ whenever $i \geq j$. Then* $\ker \gamma = \bigcap_{i \in I} F_i$.

Proof. Let A be a BL-algebra. Then

$$\begin{aligned}
\ker \gamma &= \{x \in A : \gamma(x) = ([0_A]_{F_i}), \text{ for all } i \in I\} \\
&= \{x \in A : ([x]_{F_i}) = ([0_A]_{F_i}), \text{ for all } i \in I\} \\
&= \{x \in A : x \in [0_A]_{F_i} = F_i, \text{ for all } i \in I\} \\
&= \{x \in A : (x \to 0) \odot (0 \to x) \in F_i, \text{ for all } i \in I\} \\
&= \{x \in A : x \in F_i, \text{ for all } i \in I\} \\
&= \bigcap_{i \in I} F_i.
\end{aligned}$$

□

Theorem 4.8. *Let A be a BL-algebra and let $\mathcal{I}$ be a family of filters of A such that $F \cap G \in \mathcal{I}$ for every $F, G \in \mathcal{I}$. Then $\{A/F, \varphi_{FG}, \mathcal{I}\}$ is an inverse system of BL-algebras, where $\varphi_{FG} : A/F \to A/G$ is the epimorphism defined by $\varphi_{FG}([x]_F) = [x]_G$, whenever $F \subseteq G$.*

Proof. We prove that $\{A/F, \varphi_{FG}, \mathcal{I}\}$ is an inverse system. Note that $\varphi_{FF} : A/F \to A/F$ is defined by $\varphi_{FF}([x]_F) = [x]_F$. So, $\varphi_{FF} = id_{A/F}$. Now we show that, $\varphi_{GH}\varphi_{FG} = \varphi_{FH}$, whenever $F \geq G \geq H$. Then

$$\begin{aligned}
\varphi_{GH}\varphi_{FG}([x]_F) &= \varphi_{GH}(\varphi_{FG}([x]_F)) \\
&= \varphi_{GH}([x]_G) \\
&= [x]_H \\
&= \varphi_{FH}([x]_F).
\end{aligned}$$

Thus, $\varphi_{GH}\varphi_{FG} = \varphi_{FH}$, whenever $F \geq G \geq H$. Therefore, $\{A/F, \varphi_{FG}, \mathcal{I}\}$ is an inverse system. □

If $\mathcal{I}$ is the family of all filters of A such that A/F is finite for all $F \in \mathcal{I}$, then the inverse limit of the inverse system $\{A/F, \varphi_{FG}, \mathcal{I}\}$ is called the completion of A.

Theorem 4.9. *Let $\mathcal{I}$ be a set of filters of BL-algebra A. Consider the inverse system $\{A/F, \varphi_{FG}, \mathcal{I}\}$. Then*

$$\hat{A} = \left\{([x]_F) \in \prod_{F \in \mathcal{I}} A/F \mid \varphi_{FG}([x]_F) = [x]_G \text{ for all } F, G \in \mathcal{I},\ F \geq G\right\}$$

together with the projections $\varphi_F : A \to A/F$ for all $F \in \mathcal{I}$ is an inverse limit of $\{A/F, \varphi_{FG}, \mathcal{I}\}$.

Proof. The result follows from Theorem 3.4. □

Example 4.10. Let $A = \{0, a, b, c, d, 1\}$, with $0 < a < b < 1$ and $0 < c < d < 1$, but a, c and respective b, d are incomparable. We define $\odot$ and $\to$ in Tables 1 and 2, respectively.

TABLE 1. Cayley Table of $\odot$

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

TABLE 2. Cayley Table of $\to$

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Then $(A, \vee, \wedge, \odot, \to, 0, 1)$ is a non-linearly ordered BL-algebra. There is a bijection between the congruence relations and filters of A. Thus, the set $\mathcal{I}$ of congruence relations of A is completely determined by the set of all filters of A. The filters of A are $F_1 = \{1\}$, $F_2 = \{b, 1\}$, $F_3 = \{c, d, 1\}$, and $F_4 = A$. Hence, $\equiv^{F_i}$ is a congruence relation on A for $i = 1, 2, 3, 4$. Thus, $I = \{\equiv^{F_i} : i = 1, 2, 3, 4\}$. We will just denote $\equiv^{F_i}$ by F_i for all $i = 1, 2, 3, 4$. Note that $\equiv^{F_i} \geq \equiv^{F_j}$ if $F_i \subseteq F_j$. Now,

$$\mathcal{I}(A) = \{F_1 = \{1\},\ F_2 = \{b, 1\},\ F_3 = \{c, d, 1\},\ F_4 = A\}$$

is a set of filters of A. Then

$A/F_1 = \{[0]_{F_1}, [a]_{F_1}, [b]_{F_1}, [c]_{F_1}, [d]_{F_1}, [1]_{F_1}\}$, where $[0]_{F_1} = \{0\}$, $[a]_{F_1} = \{a\},\ [b]_{F_1} = \{b\},\ [c]_{F_1} = \{c\},\ [d]_{F_1} = \{d\},\ [1]_{F_1} = \{1\}$;

$A/F_2 = \{[0]_{F_2}, [a]_{F_2}, [b]_{F_2}\}$, where $[0]_{F_2} = \{0, c\},\ [a]_{F_2} = \{a, d\}$, $[b]_{F_2} = \{b, 1\}$;

$A/F_3 = \{[0]_{F_3}, [c]_{F_3}\}$, where $[0]_{F_3} = \{0, a, b\},\ [c]_{F_3} = F_3$; and

$A/F_4 = \{[0]_{F_4}\}$, where $[0]_{F_4} = F_4$.

Therefore, $\{A/F_i, \varphi_{F_iF_j}, \mathcal{I}\}$ is an inverse system of BL-algebras, where $\varphi_{F_iF_j} : A/F_i \to A/F_j$ is the canonical epimorphism whenever $F_i \geq F_j$; that is, $F_i \subseteq F_j$.

Example 4.11. Consider the inverse system $\{A/F_i, \varphi_{F_iF_j}, \mathcal{I}\}$ defined in Example 4.10. Then

$$\varprojlim A/F_i = \Big\{([x]_{F_i}) \in \prod_{i=1}^{4} A/F_i \,|\, \varphi_{F_iF_j}([x]_{F_i}) = [x]_{F_j}, \text{ for all } F_i, F_j \in I, F_i \geq F_j\Big\}.$$

Now, the elements

$$\alpha_1 = ([0]_{F_1}, [0]_{F_2}, [0]_{F_3}, [0]_{F_4}),$$
$$\alpha_2 = ([a]_{F_1}, [a]_{F_2}, [0]_{F_3}, [0]_{F_4}),$$
$$\alpha_3 = ([b]_{F_1}, [1]_{F_2}, [a]_{F_3}, [0]_{F_4}),$$
$$\alpha_4 = ([c]_{F_1}, [0]_{F_2}, [d]_{F_3}, [0]_{F_4}),$$
$$\alpha_5 = ([d]_{F_1}, [a]_{F_2}, [c]_{F_3}, [0]_{F_4}),$$
$$\alpha_6 = ([1]_{F_1}, [b]_{F_2}, [c]_{F_3}, [0]_{F_4})$$

of $\varprojlim A/F_i$ are the only elements that satisfy the condition of $\varprojlim A/F_i$. Thus, $\varprojlim A/F_i = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$.

Lemma 4.12. *Let $\mathcal{I}$ and $\mathcal{I}'$ be two directed posets. Then $\mathcal{I} \times \mathcal{I}'$ is a directed poset.*

Theorem 4.13. *Let A and B be two BL-algebras, and let $\mathcal{I}$ and $\mathcal{I}'$ be the sets of all filters of A and B, respectively. Suppose that $\{A/F, \varphi_{FG}, \mathcal{I}\}$ and $\{B/F', \varphi'_{F'G'}, \mathcal{I}'\}$ are the inverse systems of them, respectively. Then $\{A/F \times B/F', (\varphi_{FG}, \varphi'_{F'G'}), \mathcal{I} \times \mathcal{I}'\}$ is an inverse system of $A/F \times B/F'$, whenever*

$$(\varphi_{FG}, \varphi'_{F'G'}) : A/F \times B/F' \to A/G \times B/G'$$

is the epimorphism defined by

$$(\varphi_{FG}, \varphi'_{F'G'})([x]_F, [y]_{F'}) = ([x]_G, [y]_{G'}).$$

Proof. The result follows from Theorem 3.4. □

5. Conclusion

The BL-algebras are the algebraic structures for BL arising from the continuous triangular norms, which are familiar in the frameworks of fuzzy set theory. An inverse limit (also called the projective limit) is a construction that allows one to "glue together" several related objects. The precise gluing process is specified by morphisms between the objects. The inverse limits are a particular case of limit in category theory. In this note, we introduced the inverse limit of an inverse system of the BL-algebras. In addition, we considered the set of congruences of the BL-algebra as the poset. Then, we constructed a quotient inverse system and quotient inverse limit on it. Moreover, we proved that two inverse limits of an inverse system are isomorphism.

Acknowledgments

I would like to thank the reviewers for their thoughtful comments and efforts, which greatly helped improve this manuscript.

References

1. J. Ad'amek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories*, (John Wiley & Sons, Inc., 1990). Volume 17, Springer-Verlag, 1990.
2. J. Albaracin and J. Vilela, Inverse Limit of an Inverse System of BE-algebras, *Eur. J. Pure Appl. Math.*, **14**(2) (2021), 423–430.
3. S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton, New Jersey, Princeton University Press, 1952.
4. P. H'ajek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, 1998.
5. N. C. Phillips, Inverse Limits of C*-Algebras, *J. Operator Theory*, **19**(1) (1998), 159–195.
6. R. Tayebi Khorami and A. Borumand Saeid, Nodal filters of BL-algebras, *J. Intel. Fuzzy Syst.*, **28** (2015), 1159–1167.
7. B. Zhao, The Inverse Limit in the Category of Topological Molecular Lattices, *Fuzzy Sets and Systems*, **118**(3) (2001), 547–554.

Reza Tayebi Khorami
Department of Mathematics, Ahvaz Branch, Islamic Azad University, Ahvaz, Iran.
Email: r.t.khorami@gmail.com

Journal of Algebraic Systems

ON INVERSE LIMIT OF A PROJECTIVE SYSTEM OF BL-ALGEBRAS

R. TAYEBI KHORAMI

حد معکوس سیستم‌های تصویری در BL-جبرها

رضا طیبی خرمی

گروه ریاضی، واحد اهواز، دانشگاه آزاد اسلامی، اهواز، ایران

در این مقاله، حدهای معکوس یک سیستم تصویری از جبرهای منطقی پایه (BL-جبرها) معرفی شده و خواص اساسی آن‌ها مورد بررسی قرار می‌گیرد. مجموعه همنهشتی‌های یک BL-جبر به عنوان یک مجموعه جزئاً مرتب در نظر گرفته می‌شود، سپس یک سیستم معکوس خارج قسمتی و یک حد معکوس خارج قسمتی بر روی آن ساخته می‌شود. علاوه بر این، با در نظر گرفتن مجموعه فیلترهای یک BL-جبر، سیستم‌های تصویری خارج قسمتی و سیستم‌های معکوس ساخته می‌شوند.

کلمات کلیدی: BL-جبر، حد معکوس، سیستم تصویری.