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NON-NILPOTENT GRAPH OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with unity. Let $Nil(R)$ be the set of all nilpotent elements of R and $\overline{Nil(R)} = R \setminus Nil(R)$ be the set of all non-nilpotent elements of R. The non-nilpotent graph of R is a simple undirected graph $G_{NN}(R)$ with $\overline{Nil(R)}$ as vertex set and any two distinct vertices x and y are adjacent if and only if $x + y \in Nil(R)$. In this paper, we introduce and discuss the basic properties of the graph $G_{NN}(R)$. We also study the diameter and girth of $G_{NN}(R)$. Further, we determine the domination number and the bondage number of $G_{NN}(R)$. We establish a relation between diameter and domination number of $G_{NN}(R)$. We also establish a relation between girth and bondage number of $G_{NN}(R)$.

1. Introduction

The study of algebraic structures by associating a graph has become an interesting research topic now a days, leading to many fascinating results and questions. This aspect was first introduced by I. Beck [8] in 1988 by defining zero divisor graph of a commutative ring. Following this, Anderson and Naseer [5] modified the Beck's definition. Also this notion was further modified by Anderson and Livingston [4]. After that many researchers have studied the zero divisor graph in the sense of Anderson and Livingston [4]. Since then, the concept of zero divisor graph of ring has been playing a crucial role in its expansion. In 2008,

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Anderson and Badawi [3] introduced the total graph of a commutative ring. Further this notion has been generalised and studied in many different ways (see [1, 2, 6, 11, 12, 23]).

The concepts of dominating sets and domination numbers play a important role in graph theory. Dominating sets are the focus of many books of graph theory (see [15, 14]). But not much research has been done on the domination properties of graphs associated to algebraic structures in terms of algebraic properties. However, some works on domination of graphs associated to rings and modules have appeared recently, for instance see [10, 13, 18, 21, 22, 24].

The study of nilpotent elements of a ring is one of the important aspects of ring theory. Many researchers have characterized nilpotent elements of a ring with the help of graphs, for instance see [7, 16, 20]. It is equally important to study the non-nilpotent elements of a ring as they also play a vital role in ring theory. The set of nilpotent elements of a ring is always an ideal of the ring. The sum of two nilpotent elements of a ring is always nilpotent. However, it is a challenging task to determine whether the sum of two non-nilpotent elements of a ring is nilpotent or not. We try to tackle up this challenge with the help of graph theory by studying the non-nilpotent graph of a commutative ring.

In this paper, we introduce the non-nilpotent graph $G_{NN}(R)$ of a commutative ring R, which is an induced subgraph of the total graph introduced by Anderson and Badawi [3]. We study the basic properties of the graph $G_{NN}(R)$. We also discuss the diameter and girth of $G_{NN}(R)$. Further, we have studied the domination number and bondage number of $G_{NN}(R)$. We establish a relationship between diameter and domination number of $G_{NN}(R)$. Finally, a relationship between girth and bondage number of $G_{NN}(R)$ has been established.

2. Preliminaries

In this section, we recall the basic definitions, concepts and results which are needed in the later sections. Throughout this paper, all rings are commutative with non-zero unity 1_R. Let R be a commutative ring with zero element 0. An element $a \in R$ is called nilpotent if there exists a positive integer n such that $a^n = 0$, and we denote $Nil(R)$ to be the set of all nilpotent elements of R. $Nil(R)$ is always an ideal of R. In fact, $Nil(R)$ is the intersection of prime ideals of R. Clearly, the sum of two nilpotent elements of R is again nilpotent and also if $a \in R$ is a nilpotent element then for all $r \in R$, ra and ar are also nilpotent. Let $\overline{Nil(R)} = R \setminus Nil(R)$ be the set of all non-nilpotent elements of R. The sum of two non-nilpotent elements of R may not be nilpotent. For any undefined terminology in rings and modules we refer to [17, 19].

By a graph G, we mean a simple undirected graph without loops. For a graph G, we denote by $V(G)$ and $E(G)$ the set of all vertices and edges respectively. We recall that a graph is finite if both $V(G)$ and $E(G)$ are finite sets, and we use the symbol $|G|$ to denote the number of vertices in the graph G. We say that G is a null graph if $E(G) = \phi$. Two vertices x and y of a graph G are connected if there is a path in G connecting them. Also, a graph G is connected if there is a path between any two distinct vertices. A graph G is disconnected if it is not connected. A graph G is complete if any two distinct vertices are adjacent. We denote the complete graph on n vertices by K^n. If the vertex set $V(G)$ of the graph G are partitioned into two non-empty disjoint sets X and Y of cardinality $|X| = m$ and $|Y| = n$, and two vertices are adjacent if and only if they are not in the same partite set, then G is called a bipartite graph. A graph G is called a complete bipartite graph if every vertex in X is connected to every vertex in Y. We denote the complete bipartite graph on m and n vertices by $K^{m,n}$. For vertices $x, y \in G$ one defines the distance $d(x, y)$, as the length of the shortest path between x and y, if the vertices $x, y \in G$ are connected and $d(x, y) = \infty$, if they are not. Then, the diameter of the graph G is

$$diam(G) = sup\{d(x, y)|x, y \in G\}.$$

The cycle is a closed path which begins and ends in the same vertex. The cycle of n vertices is denoted by C^n. The girth of the graph G,denoted by $gr(G)$ is the length of the shortest cycle in G and $gr(G) = \infty$ if G has no cycles.

For a subset $S \subseteq V$, $< S >$ denotes the subgraph of G induced by S. For a vertex $v \in V$, $deg(v)$ is the degree of the vertex v,

$$N(v) = \{u \in V \mid u \text{ is adjacent to } v\}$$

and $N[v] = N(v) \cup \{v\}$. A subset S of V is called a dominating set if every vertex in $V - S$ is adjacent to atleast one vertex in S. The domination number $\gamma(G)$ of G is defined to be minimum cardinality of a dominating set in G and such a dominating set is called γ-set of G. If G is a trivial graph, then $\gamma(G) = 0$. The bondage number $b(G)$ is the minimum number of edges whose removal increases the domination number. For basic definitions and results in domination we refer to [9, 15, 14] and for any undefined graph-theoretic terminology we refer to [9].

Now we present some preliminary results on domination number and bondage number of a graph which will be needed for the later sections.

Lemma 2.1. [9, 15]

(1) *If G is a graph of order n, then $1 \leq \gamma(G) \leq n$. A graph G of order n has domination number 1 if and only if G contains a vertex v of degree $n-1$; while $\gamma(G) = n$ if and only if $G \cong \overline{K^n}$.*
(2) *$\gamma(K^n) = 1$ for a complete graph K^n, but the converse is not true, in general and $\gamma(\overline{K^n}) = n$ for a null graph $\overline{K^n}$.*
(3) *Let G be a complete r-partite graph ($r \geq 2$) with partite sets $V_1, V_2, ..., V_r$. If $|V_i| \geq 2$ for $1 \leq i \leq r$, then $\gamma(G) = 2$; because one vertex of V_1 and one vertex of V_2 dominate G. If $|V_i| = 1$ for some i, then $\gamma(G) = 1$.*
(4) *If G is a union of disjoint subgraphs $G_1, G_2, ..., G_k$, then*
$$\gamma(G) = \gamma(G_1) + \gamma(G_2) + ... + \gamma(G_k).$$
(5) *Domination number of a bistar graph is 2; because the set consisting of two centres of the graph is a minimal dominating set.*

Lemma 2.2. [15, 14]

(1) *If G is a simple graph of order n, then $1 \leq b(G) \leq n-1$.*
(2) *$b(K^n) = n-1$ for a complete graph K^n, but the converse is not true, in general and $b(\overline{K^n}) = 0$ for a null graph $\overline{K^n}$.*
(3) *Let G be a complete r-partite graph with partite sets $V_1, V_2, ..., V_r$. Then*
$$b(G) = min\{|V_1|, |V_2|, ..., |V_r|\}.$$
In particular, $b(K^{m,n}) = min\{m, n\}$.
(4) *If G is a union of disjoint subgraphs $G_1, G_2, ..., G_k$, then*
$$b(G) = min\{b(G_1), b(G_2), ..., b(G_k)\}.$$

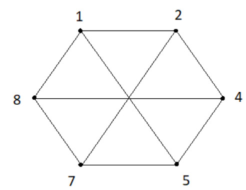
3. Non-nilpotent graph $G_{NN}(R)$ of Commutative rings

In this section we introduce the Non-nilpotent Graph $G_{NN}(R)$ of a commutative ring R and study some of its basic properties. We begin with the following definition.

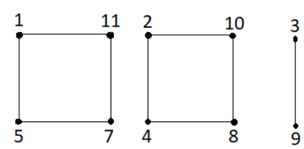
Definition 3.1. Let R be a commutative ring with nonzero identity. The Non-nilpotent graph $G_{NN}(R)$ of R is an undirected simple graph with all elements of $\overline{Nil(R)}$ as the vertices and two distinct vertices $x, y \in \overline{Nil(R)}$ are adjacent if and only if $x + y \in Nil(R)$.

We now discuss some examples.

Example 3.2. Let us consider the ring $\mathbb{Z}_9$. Clearly, $Nil(\mathbb{Z}_9) = \{0, 3, 6\}$ and $\overline{Nil(\mathbb{Z}_9)} = \{1, 2, 4, 5, 7, 8\}$. The Non-nilpotent graph $G_{NN}(\mathbb{Z}_9)$ is shown in the figure below.



Example 3.3. Let us consider the ring $\mathbb{Z}_{12}$. Clearly, $Nil(\mathbb{Z}_{12}) = \{0, 6\}$ and $\overline{Nil(\mathbb{Z}_{12})} = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11\}$. The Non-nilpotent graph $G_{NN}(\mathbb{Z}_{12})$ is shown in the figure below.



Theorem 3.4. *Let R be a commutative ring such that $|Nil(R)| = \alpha$, $|R/Nil(R)| = \beta$ and $2 = 1_R + 1_R \in Nil(R)$. Then $G_{NN}(R)$ is the union of $\beta - 1$ disjoint K^{α}'s.*

Proof. Let us consider $2 = 1_R + 1_R \in Nil(R)$ and let $x \in R/Nil(R)$. Since $Nil(R)$ is an ideal of R and $2 \in Nil(R)$, therefore, each coset $x + Nil(R)$ is complete subgraph of $G_{NN}(R)$ as

$$(x + z_1) + (x + z_2) = 2x + z_1 + z_2 \in Nil(R)$$

for all $z_1, z_2 \in Nil(R)$. We take distinct cosets $x + Nil(R)$ and $y + Nil(R)$ for some $x, y \in R/Nil(R)$. If $x + x'$ and $y + y'$ are adjacent for some $x', y' \in Nil(R)$ then

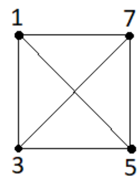
$$x + x' = (x + z_1) + (x' + z_2) - (z_1 + z_2) \in Nil(R)$$

and $x - x' = x + x' - 2x \in Nil(R)$ as $Nil(R)$ is an ideal of R. Thus $x + Nil(R) = x' + Nil(R)$, which is a contradiction. Therefore G_{NN} is the union of $\beta - 1$ disjoint subgraphs $x + Nil(R)$, each of which is a complete graph K^{α}, where $\alpha = |Nil(R)| = |x + Nil(R)|$. $\square$

Example 3.5. Let us consider the ring $\mathbb{Z}_8$. Then we have

$$Nil(\mathbb{Z}_8) = \{0, 2, 4, 6\}$$

and so $\overline{Nil(\mathbb{Z}_8)} = \{1, 3, 5, 7\}$. Clearly, $2 \in Nil(\mathbb{Z}_8)$, $|Nil(\mathbb{Z}_8)| = 4$ and $|\mathbb{Z}_8/Nil(\mathbb{Z}_8)| = 2$. Hence by the above Theorem 3.4, we have $G_{NN}(\mathbb{Z}_8)$ is a complete graph K^4 which can be seen from the following figure.



Theorem 3.6. *Let R be an integral domain such that $|Nil(R)| = \alpha$ and $|R/Nil(R)| = \beta$. Then $G_{NN}(R)$ is the union of $(\beta - 1)/2$ disjoint $K^{\alpha,\alpha}$'s.*

Proof. Let R be an integral domain such that $|Nil(R)| = \alpha$ and $|R/Nil(R)| = \beta$. Then, $Nil(R) = \{0\}$ which gives $\alpha = 1$ and

$$2 = 1_R + 1_R \notin Nil(R).$$

Let $x \in R/Nil(R)$. Then no two distinct elements in $x + Nil(R)$ are adjacent because $(x+z_1)+(x+z_2) \in Nil(R)$ for $z_1, z_2 \in Nil(R)$ which implies that $2x \in Nil(R)$, a contradiction as $2 \notin Nil(R)$.

Again, the two cosets $x + Nil(R)$ and $-x + Nil(R)$ are disjoint, and each element of $x+Nil(R)$ is adjacent to each element of $-x+Nil(R)$. Hence, $(x + Nil(R)) \cup (-x + Nil(R))$ is a complete bipartite subgraph of $G_{NN}(R)$. Also, if $y+z_1$ is adjacent to $x+z_2$ for some $y \in R/Nil(R)$ and $z_1, z_2 \in Nil(R)$, then $x + y \in Nil(R)$, and hence

$$y + Nil(R) = -x + Nil(R).$$

Thus, $G_{NN}(R)$ is the union of $(\beta - 1)/2$ disjoint subgraphs

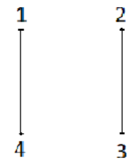
$$(x + Nil(R)) \cup (-x + Nil(R)),$$

each of which is a complete bipartite graph $K^{\alpha,\alpha}$, where

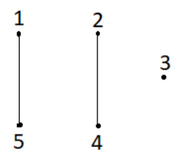
$$\alpha = |Nil(R)| = |x + Nil(R)|.$$

□

Example 3.7. Let us consider the integral domain $\mathbb{Z}_5$. Then we have $Nil(\mathbb{Z}_5) = \{0\}$ and so $\overline{Nil(\mathbb{Z}_5)} = \{1, 2, 3, 4\}$. Clearly, $2 \notin Nil(\mathbb{Z}_5)$, $|Nil(\mathbb{Z}_5)| = 1$ and $|\mathbb{Z}_5/Nil(\mathbb{Z}_5)| = 5$. Hence by the above Theorem 3.6, we have $G_{NN}(\mathbb{Z}_5)$ is the union of 2 complete bipartite graph $K^{1,1}$, which can be seen from the following figure.



Remark 3.8. The above Theorem 3.6, is not true in general if the ring R is not an integral domain. For example, let us consider the reduced ring $\mathbb{Z}_6$. Then we have $Nil(\mathbb{Z}_6) = \{0\}$ and so $\overline{Nil(\mathbb{Z}_6)} = \{1, 2, 3, 4, 5\}$. Clearly, $2 \notin Nil(\mathbb{Z}_6)$, $|Nil(\mathbb{Z}_6)| = 1$ and $|\mathbb{Z}_6/Nil(\mathbb{Z}_6)| = 6$. But the graph $G_{NN}(\mathbb{Z}_6)$ is the union of 2 complete bipartite graph $K^{1,1}$ and an isolated vertex, which can be seen from the following figure.



Theorem 3.9. *Let R be a commutative ring. Then the graph $G_{NN}(R)$ is complete if and only if either $R/Nil(R) \cong \mathbb{Z}_2$ or*
$$R \cong R/Nil(R) \cong \mathbb{Z}_3.$$

Proof. Let us assume that $|Nil(R)| = \alpha$ and $|R/Nil(R)| = \beta$. By Theorems 3.4 and 3.6, we have $G_{NN}(R)$ is complete if and only if $G_{NN}(R)$ is a single K^{α} or $K^{1,1}$.
If $2 \in Nil(R)$, then $\beta - 1 = 1$. So, $\beta = 2$, and hence $R/Nil(R) \cong \mathbb{Z}_2$. Again, if R is an integral domain then $Nil(R) = \{0\}$ which gives $2 \notin Nil(R)$. Therefore, $\alpha = 1$ and $(\beta - 1)/2 = 1$ which implies $\beta = 3$. Hence, $R \cong R/Nil(R) \cong \mathbb{Z}_3$. □

Theorem 3.10. *Let R be a commutative ring such that $|Nil(R)| = \alpha$, $|R/Nil(R)| = \beta$ and $2 = 1_R + 1_R \in Nil(R)$. Then the graph $G_{NN}(R)$ is connected if and only if $R/Nil(R) \cong \mathbb{Z}_2$.*

Proof. By Theorem 3.4, we have $G_{NN}(R)$ is connected if and only if $G_{NN}(R)$ is a single K^{α}. Thus, $\beta - 1 = 1$ which implies $\beta = 2$. Hence, $R/Nil(R) \cong \mathbb{Z}_2$. □

Theorem 3.11. *Let R be an integral domain. Then the graph $G_{NN}(R)$ is connected if and only if $R \cong R/Nil(R) \cong \mathbb{Z}_3$.*

Proof. By Theorem 3.6, we have $G_{NN}(R)$ is connected if and only if $G_{NN}(R)$ is a single $K^{1,1}$. Therefore, $(\beta - 1)/2 = 1$. Also, as R is an integral domain we have $Nil(R) = \{0\}$ which gives $\alpha = 1$. Hence, $\beta = 3$ and so $R \cong R/Nil(R) \cong \mathbb{Z}_3$. □

Remark 3.12. The above Theorem 3.11, is not true in general if the ring R is not an integral domain which can be observed from the following examples.

(1) Let us consider the ring $R = \mathbb{Z}_{27}$. Then we have
$$Nil(R) = \{0, 3, 6, 9, 12, 15, 18, 21, 24\}.$$
Clearly, $|Nil(R)| = 9$ and $|R/Nil(R)| = 3$ which shows that $R/Nil(R) \cong \mathbb{Z}_3$. We observe that the graph $G_{NN}(R)$ is connected.

(2) Let us consider the ring $R = \mathbb{Z}_{25}$. Then we have
$$Nil(R) = \{0, 5, 10, 15, 20\}.$$
Clearly, $|Nil(R)| = 5$ and $|R/Nil(R)| = 5$ which shows that $R/Nil(R) \cong \mathbb{Z}_5$. But the graph $G_{NN}(R)$ is connected.

Theorem 3.13. *Let R be a commutative ring.*

(1) *Let H be an induced subgraph of $G_{NN}(R)$, and two distinct vertices x and y of H are connected by a path in H. Then there is a path in H of length at most 2 between x and y. In particular, if $G_{NN}(R)$ is connected, then $diam(G_{NN}(R)) \leq 2$.*

(2) *Let x and y be distinct elements of R that are connected by a path. If x and y are not adjacent, then $x - (-x) - y$ and $x - (-y) - y$ are paths of length 2 between x and y in $G_{NN}(R)$.*

Proof. (1) It is enough to show that if x_1, x_2, x_3 and x_4 are distinct vertices of H and there is a path $x_1 - x_2 - x_3 - x_4$ from x_1 and x_4, then x_1 and x_4 are adjacent.

Now, $x_1 + x_2, x_2 + x_3, x_3 + x_4 \in Nil(R)$. Implies
$$x_1 + x_4 = (x_1 + x_2) - (x_2 + x_3) + (x_3 + x_4) \in Nil(R),$$
as $Nil(R)$ is an ideal of R. Thus x_1 and x_4 are adjacent.

(2) Since x and y are not adjacent, so $x + y \notin Nil(R)$. Then there is a $z \in \overline{Nil(R)}$ such that $x - z - y$ is a path of length 2, by above part (1). Thus, $x + z, z + y \in Nil(R)$, and hence $x - y = (x + z) - (z + y) \in Nil(R)$, as $Nil(R)$ is an ideal of R. Also, $x \neq -x$ and $y \neq -x$ since $x + y \notin Nil(R)$. Hence, $x - (-x) - y$ and $x - (-y) - y$ are paths of length 2 between x and y in $G_{NN}(R)$.

□

Theorem 3.14. *Let R be a commutative ring. The following conditions are equivalent:*

(1) *The graph $G_{NN}(R)$ is connected.*

(2) *Either $x + y \in Nil(R)$ or $x - y \in Nil(R)$ for all $x, y \in \overline{Nil(R)}$.*

(3) *Either* $x+y \in Nil(R)$ *or* $x+2y \in Nil(R)$ *for all* $x, y \in \overline{Nil(R)}$. *In particular, either* $2x \in Nil(R)$ *or* $3x \in Nil(R)$ *(but not both) for all* $x \in \overline{Nil(R)}$.

(4) $R/Nil(R) \cong \mathbb{Z}_2$ *and* $2 \in Nil(R)$.

Proof. (1) $\implies$ (2) Let the graph $G_{NN}(R)$ is connected. Also, let us assume that $x, y \in \overline{Nil(R)}$. If $x = y$, then $x - y \in Nil(R)$. Suppose $x \neq y$. If $x + y \notin Nil(R)$ then $x - (-y) - y$ is a path from x to y, by Theorem 3.13(2). This implies that $x - y \in Nil(R)$.

(2) $\implies$ (3) Let $x, y \in \overline{Nil(R)}$, and suppose that $x + y \notin Nil(R)$. Since $(x + y) - y = x \notin Nil(R)$, thus

$$x + 2y = (x + y) + y \in Nil(R),$$

by the hypothesis. In particular if $x \in \overline{Nil(R)}$, then either $2x \in Nil(R)$ or $3x \in Nil(R)$. If both $2x$ and $3x$ are in $Nil(R)$ then

$$3x - 2x = x \in Nil(R),$$

a contradiction. Thus both $2x$ and $3x$ cannot be in $Nil(R)$.

(3) $\implies$ (1) Let $x, y \in \overline{Nil(R)}$ be such that $x + y \notin Nil(R)$ then $x+2y \in Nil(R)$, by the hypothesis. Since $Nil(R)$ is an ideal of R and $x+2y \in Nil(R)$, this implies that $2y \notin Nil(R)$. Thus $3y \in Nil(R)$, by the hypothesis. Since $x + y \notin Nil(R)$ and $3y \in Nil(R)$, we conclude that $x \neq 2y$, and thus $x - 2y - y$ is a path from x to y in $G_{NN}(R)$. Hence, $G_{NN}(R)$ is connected.

(4) $\iff$ (1) This follows directly from Theorem 3.10. □

Now we discuss the diameter and girth of the graph $G_{NN}(R)$.

Theorem 3.15. *Let* R *be a commutative ring. Then,*

(1) $diam(G_{NN}(R)) = 0$ *if and only if* $R \cong \mathbb{Z}_2$.

(2) $diam(G_{NN}(R)) = 1$ *if and only if either* $R/Nil(R) \cong \mathbb{Z}_2$ *and* $R \ncong \mathbb{Z}_2$ *(i.e.,* $|Nil(R)| \geq 2$*) or* $R \cong R/Nil(R) \cong \mathbb{Z}_3$.

(3) $diam(G_{NN}(R)) = \infty$ *if* R *is an integral domain such that* $R \ncong \mathbb{Z}_2, \mathbb{Z}_3$.

Proof.

(1) The $diam(G_{NN}(R)) = 0$ if and only if $G_{NN}(R)$ is a null graph if and only if $G_{NN}(R)$ is the complete graph K^1 if and only if $R \cong \mathbb{Z}_2$.

(2) The $diam(G_{NN}(R)) = 1$ if and only if $G_{NN}(R)$ is a single K^2 or $K^{1,1}$. So by Theorems 3.10 and 3.11, we get $R/Nil(R) \cong \mathbb{Z}_2$ or $R \cong R/Nil(R) \cong \mathbb{Z}_3$.

(3) If R is an integral domain such that $R \ncong \mathbb{Z}_2, \mathbb{Z}_3$, then by Theorem 3.6 we have $G_{NN}(R)$ is a disjoint union of more than

one $K^{1,1}$ which gives $G_{NN}(R)$ is a disconnected graph. Hence, $diam(G_{NN}(R)) = \infty$.

□

Theorem 3.16. *Let R be a commutative ring. Then,*

(1) $gr(G_{NN}(R)) = 3$ *if and only if* $2 \in Nil(R)$ *and* $|Nil(R)| \geq 3$.
(2) $gr(G_{NN}(R)) = \infty$ *if R is an integral domain.*

Proof. (1) Suppose $2 \in Nil(R)$ and $|Nil(R)| \geq 3$. Then $G_{NN}(R)$ is the disjoint union of complete graph K^α, by Theorem 3.4, and so it must contain a shortest cycle of length 3. Hence, $gr(G_{NN}(R)) = 3$.
Conversely, let $gr(G_{NN}(R)) = 3$. If $G_{NN}(R)$ contains a cycle, then $G_{NN}(R)$ is the disjoint union of either complete graph K^α or the complete bipartite graph $K^{\alpha,\alpha}$, by Theorem 3.4 and 3.6. Since, $gr(G_{NN}(R)) = 3$, so the graph $G_{NN}(R)$ cannot be $K^{\alpha,\alpha}$ as $K^{\alpha,\alpha}$ cannot have an odd cycle. Thus, $G_{NN}(R)$ is the disjoint union of complete graph $K^\alpha, \alpha \geq 3$. This implies that $2 \in Nil(R)$, by Theorem 3.4 and $\alpha = |Nil(R)| \geq 3$.
(2) If R is an integral domain, then by Theorem 3.6 we have $G_{NN}(R)$ is a disjoint union of $K^{1,1}$ which gives $G_{NN}(R)$ contains no cycle. Hence, $gr(G_{NN}(R)) = \infty$.

□

Remark 3.17. (1) The condition for which $gr(G_{NN}(R)) = \infty$ as mentioned in the statement (2) of Theorem 3.16 is only necessary but not a sufficient condition.
From Remark 3.8, we can observe that the graph $G_{NN}(\mathbb{Z}_6)$ of the reduced ring $\mathbb{Z}_6$ is a disjoint union of two $K^{1,1}$ and an isolated vertex which gives $gr(G_{NN}(\mathbb{Z}_6)) = \infty$ but $\mathbb{Z}_6$ is not an integral domain.
(2) From Example 3.2, we can observe that the girth of the graph $G_{NN}(\mathbb{Z}_9)$ is $gr(G_{NN}(\mathbb{Z}_9)) = 4$. Also, there are certain rings whose Non-nilpotent graphs have girth 4.

4. Domination Number and Bondage Number of $G_{NN}(R)$

In this section we determine the domination number and the bondage number of the Non-nilpotent graph $G_{NN}(R)$. We establish a relationship between diameter and domination number of $G_{NN}(R)$. We also establish a relationship between girth and bondage number of $G_{NN}(R)$. We begin with the following Theorem.

Theorem 4.1. *Let R be a commutative ring such that $|Nil(R)| = \alpha$, $|R/Nil(R)| = \beta$ and $2 = 1_R + 1_R \in Nil(R)$. Then $\gamma(G_{NN}(R)) = \beta - 1$.*

Proof. Let R be a commutative ring such that $|Nil(R)| = \alpha$, $|R/Nil(R)| = \beta$ and $2 \in Nil(R)$. Then by Theorem 3.4, we have the graph $G_{NN}(R)$ is the union of $\beta - 1$ disjoint K^{α}'s. But $\gamma(K^{\alpha}) = 1$ which gives $\gamma(G_{NN}(R)) = \beta - 1$. □

Example 4.2. From Example 3.5, for the ring $\mathbb{Z}_8$, we have $|Nil(\mathbb{Z}_8)| = 4$, $|\mathbb{Z}_8/Nil(\mathbb{Z}_8)| = 2$ and $2 \in Nil(\mathbb{Z}_8)$. We observe that $G_{NN}(\mathbb{Z}_8)$ is a complete graph K^4. Thus, $\gamma(G_{NN}(\mathbb{Z}_8)) = 1$.

Theorem 4.3. *Let R be an integral domain such that $|Nil(R)| = \alpha$ and $|R/Nil(R)| = \beta$. Then $\gamma(G_{NN}(R)) = \beta - 1$.*

Proof. Let R be an integral domain such that $|Nil(R)| = \alpha$ and $|R/Nil(R)| = \beta$. Then, $Nil(R) = \{0\}$ which gives $\alpha = 1$ and $2 = 1_R + 1_R \notin Nil(R)$. Now, by Theorem 3.6, we have the graph $G_{NN}(R)$ is the union of $(\beta - 1)/2$ disjoint $K^{1,1}$'s. But $\gamma(K^{\alpha,\alpha}) = 2$ which implies $\gamma(G_{NN}(R)) = (\beta - 1)/2 \times 2 = \beta - 1$. □

Example 4.4. From Example 3.7, for the integral domain $\mathbb{Z}_5$, we have $|Nil(\mathbb{Z}_5)| = 1$ and $|\mathbb{Z}_5/Nil(\mathbb{Z}_5)| = 5$. We observe that $G_{NN}(\mathbb{Z}_5)$ is a disjoint union of two complete bipartite graph $K^{1,1}$. Thus,

$$\gamma(G_{NN}(\mathbb{Z}_5)) = 2 + 2 = 4.$$

Theorem 4.5. *Let R be a commutative ring such that*

$$2 = 1_R + 1_R \in Nil(R).$$

Then $\gamma(G_{NN}(R)) = 1$ if and only if $R/Nil(R) \cong \mathbb{Z}_2$.

Proof. Let us suppose that $\gamma(G_{NN}(R)) = 1$. Then $G_{NN}(R)$ is connected. Since $2 \in Nil(R)$, so by Theorem 3.10 we have

$$R/Nil(R) \cong \mathbb{Z}_2.$$

Conversely, let $R/Nil(R) \cong \mathbb{Z}_2$. Then by Theorem 3.9, $G_{NN}(R)$ is complete and hence $\gamma(G_{NN}(R)) = 1$. □

Theorem 4.6. *Let R be an integral domain. Then $\gamma(G_{NN}(R)) = 1$ if and only if $R \cong R/Nil(R) \cong \mathbb{Z}_3$.*

Proof. Let us assume that $\gamma(G_{NN}(R)) = 1$. Then $G_{NN}(R)$ is connected and so by Theorem 3.11 we have $R \cong R/Nil(R) \cong \mathbb{Z}_3$.
Conversely, let $R \cong R/Nil(R) \cong \mathbb{Z}_3$. Then by Theorem 3.9 we have $G_{NN}(R)$ is complete and hence $\gamma(G_{NN}(R)) = 1$. □

In the following corollary a relationship between diameter and domination number of $G_{NN}(R)$ has been established.

Corollary 4.7. *Let R be a commutative ring. Then,*

$$diam(G_{NN}(R)) = 1$$

if and only if $\gamma(G_{NN}(R)) = 1$.

Proof. (1). It is clear by Theorems 3.15(2), 4.5 and 4.6. □

Theorem 4.8. *Let* R *be a commutative ring such that* $|Nil(R)| = \alpha$, $|R/Nil(R)| = \beta$ *and* $2 = 1_R + 1_R \in Nil(R)$. *Then* $b(G_{NN}(R)) = \alpha - 1$

Proof. By Theorem 3.4, we see that the graph $G_{NN}(R)$ is the union of $\beta - 1$ disjoint K^{α}'s. Since $b(K^{\alpha}) = \alpha - 1$.
Thus,

$$\begin{aligned} b(G_{NN}(R)) &= min\{b(K^{\alpha}), b(K^{\alpha}), b(K^{\alpha}), ..., b(K^{\alpha})\}_{(\beta-1)copies} \\ &= min\{\alpha - 1, \alpha - 1, \alpha - 1, ..., \alpha - 1\}_{(\beta-1)copies} \\ &= \alpha - 1. \end{aligned}$$

□

Example 4.9. From Example 3.5, for the ring $\mathbb{Z}_8$, we have $|Nil(\mathbb{Z}_8)| = 4$, $|\mathbb{Z}_8/Nil(\mathbb{Z}_8)| = 2$ and $2 \in Nil(\mathbb{Z}_8)$. We observe that $G_{NN}(\mathbb{Z}_8)$ is a complete graph K^4. Thus, $b(G_{NN}(\mathbb{Z}_8)) = 4 - 1 = 3$.

Theorem 4.10. *Let* R *be an integral domain such that* $|Nil(R)| = \alpha$ *and* $|R/Nil(R)| = \beta$. *Then* $b(G_{NN}(R)) = 1$

Proof. Since R is an integral domain, so $Nil(R) = \{0\}$ which gives $\alpha = 1$. Then by Theorem 3.6, we have $G_{NN}(R)$ is the union of $(\beta-1)/2$ disjoint $K^{1,1}$'s and we know that $b(K^{1,1}) = 1$. Hence,

$$G_{NN}(R) = min\{b(K^{1,1}), b(K^{1,1}), ..., b(K^{1,1})\}_{(\beta-1)/2copies} = 1.$$

□

In the following theorem a relationship between girth and bondage number of $G_{NN}(R)$ has been established.

Theorem 4.11. *Let* R *be a commutative ring such that* $|Nil(R)| = \alpha$, $|R/Nil(R)| = \beta$. *Then,*

$$gr(G_{NN}(R)) = 3$$

if and only if $b(G_{NN}(R)) = \alpha - 1$, $\alpha \geq 3$ *and* $2 = 1_R + 1_R \in Nil(R)$.

Proof. If $gr(G_{NN}(R)) = 3$, then $2 \in Nil(R)$ and $\alpha = |Nil(R)| \geq 3$, by Theorem 3.16(1). Since $2 \in Nil(R)$, so by Theorem 3.4, the graph $G_{NN}(R)$ is the union of $\beta - 1$ disjoint K^{α}'s. Therefore,

$$b(G_{NN}(R)) = \alpha - 1.$$

The converse part is clear by Theorem 3.16(1). □

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NON–NILPOTENT GRAPH OF COMMUTATIVE RINGS

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گراف غیرپوچ‌توان حلقه‌های جابه‌جایی

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فرض کنید R حلقه‌ای جابه‌جایی و یکدار باشد. همچنین، فرض کنید $Nil(R)$ مجموعه‌ی همه‌ی عناصر پوچ‌توان R و $\overline{Nil(R)} = R \setminus Nil(R)$ مجموعه‌ی همه‌ی عناصر غیرپوچ‌توان R باشد. گراف غیرپوچ‌توان R گراف ساده و غیرجهتی $G_{NN}(R)$ با مجموعه‌ی رئوس $\overline{Nil(R)}$ است که در آن دو رأس متمایز x و y مجاورند اگر و تنها اگر $x + y \in Nil(R)$. در این مقاله، مفهوم و خواص اولیه گراف $G_{NN}(R)$ معرفی و بررسی شده‌اند. به علاوه، قطر و کمر $G_{NN}(R)$ مطالعه شده‌اند. همچنین، عدد احاطه‌گری و عدد اسارت گراف $G_{NN}(R)$ تعیین شده‌اند. رابطه‌ای بین قطر و عدد احاطه‌گری $G_{NN}(R)$ بیان شده است. به علاوه، رابطه‌ای بین کمر و عدد اسارت $G_{NN}(R)$ بدست آمده است.

کلمات کلیدی: حلقه‌های جابه‌جایی، گراف غیرپوچ توان، عناصر غیرپوچ‌توان.