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DUAL RICKART (BAER) MODULES AND PRERADICALS

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Abstract. In this work, we introduce dual Rickart (Baer) modules via the concept of preradicals. It is shown that W is τ-d-Rickart if and only if $W = \tau(W) \oplus L$ such that $\tau(W)$ is a dual Rickart module. We prove that a module W is τ-d Baer if and only if W is τ-d-Rickart and W satisfies strongly summand sum property for d.s. submodules of W contained in $\tau(W)$. Via $\tau(R_R)$, we characterize right τ-d Baer rings.

1. Introduction

We may say a functor $\tau : Mod-R \to Mod-R$ is a *preradical* if for τ we have the following:

(1) For any right R-module W, $\tau(W)$ is a submodule of W,
(2) If $f : W \to K$ is an R-module homomorphism, then $f(\tau(W)) \subseteq \tau(K)$ and $\tau(f)$ is the restriction of f to $\tau(W)$.

Note that if D is a d.s. submodule (direct summand) of W, then $\tau(D) = \tau(W) \cap D$ for a preradical τ. An interested reader for more information about preradicals, may check [2].

For a module W, a new submodule defined as

$$\overline{Z}(W) = \bigcap\{Kerf \mid f\colon W \to U, U \in \mathcal{U}\}.$$

In this definition, $\mathcal{U}$ stands for the class of all small right R-modules. By the way, the module W is said to be *cosingular* (*noncosingular*) in

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case $\overline{Z}(W) = 0$ ($\overline{Z}(W) = W$)([11]). Consider a ring R such that every simple right R-module is injective. Then R is a right V-ring. Note that as a famous result a ring R is a right V-ring if and only if the radical of every right R-module is zero. Let W be a module and N a submodule of W. Then W is said to be *fully invariant*, denoted by $N \trianglerighteq W$, in case for each $g \in End_R(W)$ we have $g(N) \subseteq N$. Note that $Soc(W)$, $Rad(W)$ and $\overline{Z}(W)$ are some known fully invariant submodules of W.

In the last three decades, some researchers tried to study lifting modules and their various generalizations via some aspects of preradicals. For instance, in [1] and [12] we can see that lifting modules and one of their generalizations were studied via some kind of fully invariant submodules (note that, a fully invariant submodule in fact introduces a preradical). A module M is said to be dual Rickart provided for each $\varphi : M \to M$, the image is a direct summand of M ([6]). Also, in [9] a new generalization of dual Rickart modules (applying a homological approach) were introduced and studied. There are many works about dual Rickart modules and their generalizations ([1, 6, 9],...). Till now, non of works has been done related to dual Rickart modules has a torsionally approach. By the way, in this work we try to make a preradically approach to the concepts dual Rickart modules and dual Baer modules. Via preradicals, we define and study τ-d-Rickart modules and τ-d Baer modules. Somewhere in the manuscript, we apply some known preradicals such as Soc, Rad and $\overline{Z}$. Some general properties of τ-d-Rickart (Baer) modules are also investigated. We tried to achieve some conditions under which a module can be τ-d-Rickart (Baer). Any undefined terminology here, may be found in [8] and [13].

2. τ-d-Rickart modules and τ-d Baer modules

We may start the section with the key definition. Throughout this section τ will denotes a preradical.

Definition 2.1. Suppose W is a module. If for each g in $End_R(W)$, the submodule $g(\tau(W))$ is a d.s. submodule of W, then we say W is τ-d-Rickart.

It can be worth to say that a dual Rickart module W may not be τ-d-Rickart. Consider the $\mathbb{Z}$-module $W = \mathbb{Z}_{p^\infty}$. Then W is dual Rickart while it is not τ-dual-Rickart. Here τ indicates the preradical Soc (see Example 2.2).

Example 2.2. Let W be a module. Suppose that $\tau(W)$ is a nonzero small submodule of W, then for every g in $End_R(W)$, the submodule

$g(\tau(W))$ is a small submodule of W. So that $g(\tau(W))$ may not be a d.s. submodule of W for some endomorphism g of W. Therefore, W is not a τ-d-Rickart module. Now, $\mathbb{Z}$-module $W = \mathbb{Z}_{p^\infty}$ is not τ-d-Rickart if we choose τ to be Soc.

Following result expresses an important characterization of τ-d-Rickart modules which will be used freely throughout the paper.

Theorem 2.3. *If W is a module, then below statements coincide:*

(1) *W is τ-d-Rickart;*

(2) *W decomposed to a submodule L and $\tau(W)$ such that $\tau(W)$ is dual Rickart.*

Proof. (1) $\Rightarrow$ (2) By assumption, for a module W, $\tau(W)$ is a d.s. submodule of W. Set $W = \tau(W) \oplus L$ for a submodule L of W. Suppose that g is an endomorphism $\tau(W)$. Then $h = j \circ g \circ \pi$ is an endomorphism of W such that j is the inclusion from $\tau(W)$ to W and π is the projection of W on $\tau(W)$. Being W a τ-d-Rickart module implies $h(\tau(W)) = Img$ is a d.s. submodule of W and consequently a d.s. submodule of $\tau(W)$ as $h(\tau(W))$ is contained in $\tau(W)$.

(2) $\Rightarrow$ (1) Let $W = \tau(W) \oplus L$ such that $\tau(W)$ is dual Rickart. Suppose that g is an endomorphism of W. Then $\lambda = \pi \circ g \circ j$ will be an endomorphism of $\tau(W)$ where $j : \tau(W) \to W$ is the inclusion and $\pi : W \to \tau(W)$ is the projection on $\tau(W)$. As $\lambda(\tau(W)) = g(\tau(W))$ and $\tau(W)$ is a dual Rickart module, then $g(\tau(W))$ is a d.s. submodule of $\tau(W)$ and consequently of W, as required. □

Example 2.4. (1) Let F be a field and $R = \prod_{i=1}^{\infty} F_i$ where $F_i = F$ for each $i \in \mathbb{N}$. Then R is a von Neumann regular V-ring. Take $W = R$ and $\tau = Soc$. Then $\tau(W) = Soc(R) = \oplus_{i=1}^{\infty} F$ is an essential submodule of W. It follows by Theorem 2.3, W is not τ-d-Rickart, although $\tau(W)$ itself is dual Rickart.

(2) Let R be a right Noethrian right V-ring (we can consider the ring R in [3, Example]). As every simple right R-module is injective and R is right Noetherian, we conclude that $Soc(R_R)$ is injective and hence a d.s. submodule of R_R. Therefore, R_R is τ-d-Rickart where $\tau = Soc$.

(3) According to [7, C29, Page 255], there is a field F with derivation δ such that the differential polynomial ring $F[x, \delta]$ is a simple non-regular V-domain. Note that $R = F[x, \delta]$ is a right and left Noetherian ring, so that by (2) $Soc(R_R)$ is a d.s. submodule of R_R. Hence, R_R is τ-d-Rickart while R_R is not a dual Rickart module.

Remark 2.5. Let W be an indecomposable module such that $\tau(W) \neq 0$. Then W is τ-d-Rickart if and only if $\tau(W) = W$ is dual Rickart. In other words, if $\tau(W)$ is a nontrivial submodule of W, then W can not be τ-d-Rickart. For instance, a local module W with $\tau(W) \neq 0$ is not a τ-d-Rickart module, where $\tau = Rad$.

Proposition 2.6. *Each d.s. submodule of a τ-d-Rickart module is τ-d-Rickart.*

Proof. Suppose W is a τ-d-Rickart module and N is a d.s. submodule of W. Set $W = N \oplus K$. Consider an arbitrary endomorphism h of N. It follows that $f = j \circ h \circ \pi$ is an endomorphism of W, so that $f(\tau(W)) = h(\tau(N))$ is a d.s. submodule of W as W is a dual τ-Rickart module. Note that $j : N \to W$ is the inclusion and $\pi : W \to N$ is the projection of W on N. It follows that $h(\tau(N))$ is a d.s. submodule of N, which completes the proof. □

Recall from [5], a module M is said to be dual Baer in case for every $N \leq M$, there exists an idempotent e in $S = End_R(M)$ such that $D(N) = \{f \in S | Imf \subseteq N\} = eS$.

It is natural to define an analogue for d. Baer modules in τ-case.

Definition 2.7. Let W be a module. We say that W is τ-d Baer provided for every right ideal I of $End_R(W)$ the submodule

$$I\tau(W) = \sum_{g \in I} g(\tau(W))$$

is a d.s. submodule of W.

Theorem 2.8. *For a module W, the below listed statements coincide:*

(1) *W is τ-d Baer;*

(2) *$\tau(W)$ is a d. Baer d.s. submodule of W;*

(3) *W is τ-d-Rickart and W satisfies strong summand sum property for d.s. submodules of W included in $\tau(W)$;*

(4) *The submodule $\sum_{g \in B} g(\tau(W))$ is a d.s. submodule of W, where B is an arbitrary subset of $End_R(W)$,*

Proof. (1) ⇒ (2) Consider $S = End_R(W)$ as an ideal of itself. Then by (1), $S\tau(W) = \sum_{g \in S} g(\tau(W)) = \tau(W)$ is a d.s. submodule of W. Now, let I be a right ideal of $End_R(\tau(W))$ and consider the inclusion $j : \tau(W) \to W$ and the projection $\pi_{\tau(W)} : W \to \tau(W)$. Consider the subset $I_0 = \{j \circ \lambda \circ \pi_{\tau(W)} \mid \lambda \in I\}$ of S. Then $J = I_0 S$ is a right ideal of S. As

$$I\tau(W) = \sum_{g \in I} g(\tau(W)) = \sum_{g \in J} g(\tau(W)) = J\tau(W)$$

and W is a τ-d Baer module, we conclude that $I\tau(W) = J\tau(W)$ is a d.s. submodule of W and consequently is a d.s. submodule of $\tau(W)$, as well. It follows from [5, Theorem 2.1], $\tau(W)$ is a d. Baer module.

(2) ⇒ (1) Let I be a right ideal of S and $B = \{\pi_{\tau(W)} \circ g \,|_{\tau(W)} | \, g \in I\}$. Note that $J = BEnd_R(\tau(W))$ is a right ideal of $End_R(\tau(W))$. Since

$J\tau(W) = I\tau(W)$ and $\tau(W)$ is a d. Baer module, we conclude that $J\tau(W)$ is a d.s. submodule of $\tau(W)$ and hence a d.s. submodule of W.

(1) $\Rightarrow$ (3) Let $g \in S$. As W is τ-d Baer and $< g > \tau(W) = g(\tau(W))$, then $g(\tau(W))$ is a d.s. submodule of W. Let $\{e_\gamma \mid \gamma \in \Gamma\}$ be a set of idempotents of S such that $Ime_\gamma \subseteq \tau(W)$ for each $\gamma \in \Gamma$. Suppose $I =< \sum_{\gamma\in\Gamma} e_\gamma >$ that is an ideal of S. Now,

$$I\tau(W) = \sum_{g\in I} g(\tau(W)) \subseteq \sum_{\gamma\in\Gamma} e_\gamma(W).$$

As $e_\gamma(W)$ is contained in $\sum_{g\in I} g(\tau(W))$, it follows that

$$\sum_{\gamma\in\Gamma} e_\gamma(W) = \sum_{g\in I} g(\tau(W)) = I\tau(W)$$

is a d.s. submodule of W (note that W is τ-d Baer).

(3) $\Rightarrow$ (4) It follows from the fact that $\tau(W)$ is fully invariant in W.

(4) $\Rightarrow$ (1) It is obvious. □

By Theorem 2.8, every τ-d Baer module is τ-d-Rickart.

Proposition 2.9. *Let W be a regular module. If W satisfies strong summand sum property on d.s. submodules of W contained in $\tau(W)$, then W is τ-d Baer.*

Proof. Let g be an arbitrary endomorphism of W. Note that

$$g(\tau(W)) = \sum_{x\in g(\tau(W))} xR.$$

Being W regular, we conclude $g(\tau(W))$ is a d.s. submodule of W. □

In the light of Theorem 2.8, we have the following remark.

Remark 2.10. Let W be an indecomposable module such that $\tau(W) \neq 0$. Then W is τ-d Baer if and only if $\tau(W) = W$ is d. Baer.

Theorem 2.11. *Let W be a module. Then W is τ-d Baer if and only if every d.s. submodule N of W is τ-d Baer.*

Proof. Let W be τ-d Baer and $W = N \oplus N'$ for a submodule N' of W. Then $\tau(W) = \tau(N) \oplus \tau(N')$. Suppose that A is a subset of $End_R(N)$. Then $B = \{j \circ g \circ \pi_N \mid g \in A\}$ in which $\pi_N : W \to N$ is the projection of W on N and j is the inclusion from N to W, is a subset of $End_R(W)$. It is straightforward to check that

$$A\tau(N) = \sum_{g\in A} g(\tau(N)) = \sum_{g\in B} g(\tau(W)).$$

Being W, a τ-d Baer module implies that $A\tau(N)$ is a d.s. submodule of W and hence a d.s. submodule of N. The result follows from Theorem 2.8.

The converse is straightforward. □

Corollary 2.12. *Let W be a module, P a projective module and $f : W \to P$ be an epimorphism such that $Ker f$ is contained in $\tau(W)$. Then, if W is τ-d Baer, then P is τ-d Baer.*

Remark 2.13. Let W be a module. Then

(1) If $Rad(W) \ll W$ and W is Rad-d. Baer module, then $Rad(W) = 0$.

(2) If $Soc(W) \leq_e W$ and W is Soc-d. Baer module, then W is semisimple.

Proof. (1) Since W is finitely generated, $Rad(W)$ is small in W. By Theorem 2.8, $Rad(W)$ is a d.s. submodule of W. Hence $Rad(W) = 0$.

(2) Since W is finitely cogenerated, $Soc(W)$ is essential in W and, by Theorem 2.8, $Soc(W)$ is a d.s. submodule of W. Hence $Soc(W) = W$ and so W is semisimple. □

3. Relatively τ-d-Rickart modules

In this section we shall define relative τ-d-Rickart modules and we will apply this concept to study finite direct sums of τ-d-Rickart modules.

Definition 3.1. Let W and U be R-modules. Then W is said to be *U-τ-d-Rickart* in case the image of $\tau(W)$ under φ for each $\varphi \in End_R(W)$ is a d.s. submodule of U.

Next, we introduce a condition for relatively τ-d-Rickart modules.

Theorem 3.2. *Suppose that W and U are R-modules. Then below listed coincide:*

(1) *The module W is U-τ-d-Rickart;*

(2) *For each d.s. submodule P of W and every submodule C of U, P is C-τ-d-Rickart.*

Proof. (1) ⇒ (2) Let W be U-τ-d-Rickart. Suppose that $P = eW$ for some $e^2 = e \in End_R(W)$ and let C be a submodule of U. Assume that $\psi \in Hom(P, C)$. Since $\psi eW = \psi P \subseteq C \subseteq U$ and W is U-τ-d-Rickart, $\psi e(\tau(W))$ is a d.s. submodule of U. As $\psi e(\tau(W))$ is contained in C, we conclude that $\psi e(\tau(W))$ is a d.s. submodule of C. We shall prove that $\psi(\tau(L))$ is a d.s. submodule of C. Suppose that $W = P \oplus P'$. Next, we have $\tau(W) = \tau(P) \oplus \tau(P')$. Then $e(\tau(W)) = e(\tau(P)) = \tau(P)$. Now $\psi e(\tau(W)) = \psi(\tau(P))$ combining with W is τ-d-Rickart relative to U, we come to a conclusion that $\psi(\tau(P))$ is a d.s. submodule of C.

(2) ⇒ (1) Obvious. □

Proposition 3.3. *Let W be a τ-d-Rickart module. Then*

(1) *The sum of two d.s. submodules of* W *one of them contained in* $\tau(W)$, *is a d.s. submodule of* W.

(2) *The sum of each pair of d.s. submodules of* W *included in* $\tau(W)$, *is a d.s. submodule of* W.

Proof. (1) Let $K = eW$ and $L = fW$ for some $e^2 = e \in End_R(W)$ and $f^2 = f \in End_R(W)$. Since $W = fW \oplus (1-f)W$, $L = fW \subseteq \tau(W)$, we have $\tau(W) = fW \oplus \tau((1-f)W)$. Then $((1-e)f)(\tau(W)) = (1-e)fW$. As W is a τ-d-Rickart module, $((1-e)f)(\tau(W)) = (1-e)fW$ is a d.s. submodule of W. Since

$$(1-e)fW = (fW + eW) \cap (1-e)W,$$

$W = ((fW + eW) \cap (1-e)W) \oplus T$ for some $T \leq W$. Hence

$$(1-e)W = ((fW + eW) \cap (1-e)W) \oplus (T \cap (1-e)W).$$

So

$$\begin{aligned} W &= eW \oplus (1-e)W \\ &= eW + ((fW + eW) \cap (1-e)W) \oplus (T \cap (1-e)W) \\ &= (fW + eW) + (T \cap (1-e)W). \end{aligned}$$

Since $(fW+eW)\cap(T\cap(1-e)W) = 0$, $W = (eW+fW)\oplus(T\cap(1-e)W)$. Hence $K + L$ is a d.s. submodule of W.

(2) It is clear by (1). □

Theorem 3.4. *Let* W *be a module. Then* W *is* τ*-d-Rickart if and only if for each f.g right ideal* J *of* $End_R(W)$, *the submodule* $\sum_{\phi \in J} \phi(\tau(W))$ *is a d.s. submodule of* W.

Proof. Assume that J is a f.g right ideal of $End_R(W)$ generated by $\phi_1, \ldots, \phi_n$. As W is τ-d-Rickart, then each $\phi_i(\tau(W))$ is a d.s. submodule of W. By Proposition 3.3, W satisfies summand sum property for d.s. submodules included in $\tau(W)$. Note that $\tau(W)$ is fully invariant, so $\sum_{\phi \in J} \phi(\tau(W)) = \phi_1(\tau(W)) + \cdots + \phi_n(\tau(W))$ is a d.s. submodule of W. The converse is obvious. □

4. Ring version of τ-d Baer

Throughout this section, we shall preradicals τ such that for any ring R, $\tau(R_R)$ is a two-sided ideal of R.

Definition 4.1. A ring R is said to be (left) right τ-d Baer, provided $({}_RR)$ R_R is τ-d Baer.

The following includes a ring R such that R_R is τ-d Baer while ${}_RR$ is not.

Example 4.2. ([10, Example 3.3]) Let D be a commutative local integral domain with field of fractions Q (for example, we might take D the localization of the integers $\mathbb{Z}$ by a prime number p, i.e., D is the subring of the field of rational numbers consisting of fractions a/b such that b is not divisible by p). Let $R = \begin{pmatrix} D & Q \\ 0 & Q \end{pmatrix}$. The operations are given by the ordinary matrix operations. Since D is local, it has a unique maximal ideal, say W and the Jacobson radical of R is $J(R) = \begin{pmatrix} m & Q \\ 0 & 0 \end{pmatrix}$. Then $R/J(R) \cong (D/m) \times Q$. Thus R is semilocal. On the other hand, if we suppose that D has zero socle, then R has zero left socle and so $\overline{Z}(R_R) = Soc(_RR) = 0$. Hence R_R is $\overline{Z}$-d. Baer. But R has non-zero right socle, namely, $\overline{Z}(_RR) = Soc(R_R) = \begin{pmatrix} 0 & Q \\ 0 & Q \end{pmatrix}$. It is known that, $\overline{Z}(_RR) = Soc(R_R)$ is essential in $_RR$ (see [4]). It follows that $_RR$ can not be $\overline{Z}$-d. Baer.

Theorem 4.3. *Let R be a ring. Then the following are equivalent:*

(1) *R is right τ-d Baer;*

(2) *R_R decomposed to $\tau(R_R)$ and a right ideal K where $\tau(R_R)$ is d. Baer;*

(3) *R_R decomposed to $\tau(R_R)$ and a right ideal K such that $\tau(R_R)$ is semisimple.*

Proof. (1) $\Leftrightarrow$ (2) By Theorem 2.8.

(1) $\Rightarrow$ (3) As R is τ-d Baer, then R_R can be written as a direct sum of $\tau(R_R)$ and a right ideal K. R. We may prove that every submodule of $\tau(R_R)$ is a d.s. submodule. Note that $B = \sum_{b\in B} bR$ and R is τ-d Baer. Hence $\sum_{b\in B} bI$ is a d.s. submodule of R. Therefore, $B\tau(R_R)$ is a d.s. submodule of R. Notice that $B \subseteq \tau(R_R)$, implies $B = BI$ is a d.s. submodule of $\tau(R_R)$. We are done.

(3) $\Rightarrow$ (1) Suppose that (3) holds. Being any semisimple module, d. Baer combining with Theorem 2.8 imply R is τ-d Baer. □

Theorem 4.4. *Below listed statements coincide for a ring R:*

(1) *R_R is τ-d Baer;*

(2) *Every cyclic projective right R-module W is τ-d Baer.*

Proof. (1) $\Rightarrow$ (2) Suppose that W is a cyclic projective right R-module. Then, $W = wR \cong R/r_R(w)$ for some $w \in W$. Therefore, $r_R(w)$ is a d.s. submodule of R. Hence, $R = r_R(w) \oplus J$. As R_R is τ-d Baer, by Theorem 2.11 J is τ-d Baer. Hence W is τ-d Baer.

(2) $\Rightarrow$ (1) It is obvious. □

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5. Direct sum of τ-d-Rickart modules and direct sum of τ-d Baer modules

In this section, we study direct sums of τ-d-Rickart modules and direct sums of τ-d Baer modules.

We provide some conditions which under a direct sum of τ-d-Rickart modules is also τ-dual Rickart.

Proposition 5.1. *Let $W = \oplus_{i=1}^{n} W_i$ and U be modules such that U satisfies summand sum property for d.s. submodules of U included in $\tau(U)$. Then W is U-τ-d-Rickart if and only if each W_i is U-τ-d-Rickart.*

Proof. If W is U-τ-d-Rickart, then each W_i is U-τ-d-Rickart by Theorem 3.2. Conversely, let $\phi : W \to U$ be a homomorphism. Then $\phi = (\phi_i)_{i=1}^{n}$ where each $\phi_i : W_i \to U$ is a homomorphism. By hypothesis, $\phi_i(\tau(W_i))$ is a d.s. submodule of U. As U satisfies summand sum property for d.s. submodules included in $\tau(U)$, we have

$$\begin{aligned}\phi(\tau(W)) &= \phi(\oplus_{i=1}^{n}\tau(W_i))\\ &= \phi_1(\tau(W_1)) + \phi_2(\tau(W_2)) + \cdots + \phi_n(\tau(W_n))\\ &\leq^{\oplus} U.\end{aligned}$$

Therefore W is U-τ-d-Rickart. □

Corollary 5.2. *Let $W = \oplus_{i=1}^{n} W_i$. Then W is W_j-τ-d-Rickart if and only if W_i is τ-d-Rickart relative to W_j for each $1 \leq i \leq n$.*

Theorem 5.3. *Let $\{W_i\}_{i=1}^{n}$ and U be modules. Assume that for each $i \geq j$ with $1 \leq i, j \leq n$, W_i is projective relative to W_j. Then U is $(\oplus_{i=1}^{n} W_i)$-τ-d-Rickart if and only if U is W_j-τ-d-Rickart for all $1 \leq j \leq n$.*

Proof. First implication holds by Theorem 3.2. For the other side, suppose that U is W_j-τ-d-Rickart for all $1 \leq j \leq n$. We prove by induction on n. Assume that $n = 2$ and U is τ-d-Rickart relative to W_1 and W_2. Let $\phi : U \to W_1 \oplus W_2$ be a homomorphism. Then $\phi = \pi_1\phi + \pi_2\phi$, where π_i is the natural projection from $W_1 \oplus W_2$ to W_i $(i = 1, 2)$. As U is W_2-τ-dual Rickart, $\pi_2\phi(\tau(U))$ is a d.s. submodule of W_2. Let $W_2 = \pi_2\phi(\tau(U)) \oplus W_2'$ for some $W_2' \leq W_2$. Hence

$$W_1 \oplus W_2 = W_1 \oplus \pi_2\phi(\tau(U)) \oplus W_2'.$$

As W_2 is W_1-projective, $\pi_2\phi(\tau(U))$ is W_1-projective. Since

$$W_1 + \phi(\tau(U)) = W_1 \oplus \pi_2\phi(\tau(U))$$

is a d.s. submodule of $W_1 \oplus W_2$, there exists $T \subseteq \phi(\tau(U))$ such that $W_1 + \phi(\tau(U)) = W_1 \oplus T$, by [8, Lemma 4.47]. Thus

$$\phi(\tau(U)) = (\phi(\tau(U)) \cap W_1) \oplus T.$$

Since U is W_1-τ-d-Rickart,

$$\pi_1\phi(\tau(U)) = W_1 \cap (W_2 + \phi(\tau(U))) = W_1 \cap \phi(\tau(U))$$

is a d.s. submodule of W_1. Therefore $\phi(\tau(U))$ is a d.s. submodule of $W_1 \oplus T$. Since $W_1 \oplus T = W_1 \oplus \phi(\tau(U)) \leq^{\oplus} W_1 \oplus W_2$, $\phi(\tau(U))$ is a d.s. submodule of $W_1 \oplus W_2$. Thus U is τ-d-Rickart relative to $W_1 \oplus W_2$. Now, assume that U is τ-dual Rickart relative to $\oplus_{i=1}^{n} W_i$. We show that U is τ-d-Rickart relative to $W_{n+1} \oplus (\oplus_{i=1}^{n} W_i)$. Since W_{n+1} is W_j-projective for each $1 \leq j \leq n$, W_{n+1} is $(\oplus_{i=1}^{n} W_i)$-projective. As U is W_{n+1}-τ-d-Rickart, U is $\oplus_{i=1}^{n+1} W_i$-τ-d-Rickart by a similar argument for the case $n = 2$. □

We mention that in the above theorem we use ideas of the proof of [6, Theorem 5.5].

Corollary 5.4. *Let $\{W_i\}_{i=1}^{n}$ be modules. Assume that for each $i \geq j$ with $1 \leq i, j \leq n$, W_i is W_j-projective. Then $\oplus_{i=1}^{n} W_i$ is τ-d-Rickart if and only if W_i is W_j-τ-d-Rickart for all $1 \leq i, j \leq n$.*

Proof. The first implication follows from Theorem 3.2. Conversely, assume that W_i is W_j-τ-d-Rickart for all $1 \leq j \leq n$. Now $\oplus_{i=1}^{n} W_i$ is W_j-τ-d-Rickart for all $1 \leq j \leq n$ by Corollary 5.2. Therefore, by Theorem 5.3, $\oplus_{i=1}^{n} W_i$ is τ-d-Rickart. □

We may interested in investigating direct sums of τ-d Baer modules.

Theorem 5.5. *Suppose that W_i, for $i = 1, \ldots, n$ are modules, $W = \oplus_{i=1}^{n} W_i$ and $W_i \trianglelefteq W$ for all $i \in \{1, \ldots, n\}$. Then W is a τ-d Baer module if and only if W_i is τ-d Baer for all $i \in \{1, \ldots, n\}$.*

Proof. One way holds by Theorem 2.11. For the other side, let W_i be a τ-d Baer module for all $i \in \{1, \ldots, n\}$ and I be a subset of $End_R(W)$. Then $\tau(W) = \oplus_{i=1}^{n}(\tau(W_i))$. Let

$$\phi = (\phi_{ij})_{i,j \in \{1,\ldots,n\}} \in End_R(W)$$

be arbitrary, where $\phi_{ij} \in Hom(W_j, W_i)$. Since $W_i \trianglelefteq W$ for all $i \in \{1, \ldots, n\}$ and $\tau(W) = \oplus_{i=1}^{n}(\tau(W_i))$, we have

$$\phi(\tau(W)) = \oplus_{i=1}^{n} \phi_{ii}(\tau(W_i)).$$

Hence

$$\textstyle\sum_{\phi \in I} \phi(\tau(W)) = \sum_{\phi \in I_i} \oplus_{i=1}^{n} \phi_{ii}(\tau(W_i)) = \oplus_{i=1}^{n} \sum_{\phi \in I_i} \phi_{ii}(\tau(W_i))$$

where $I_i = \{\phi|_{W_i} : \phi \in I\} \subseteq End_R(W_i)$. As W_i is τ-d Baer for all $i \in \{1, \ldots, n\}$, $\sum_{\phi \in I_i} \phi_{ii}(\tau(W_i))$ is a d.s. submodule of W_i and so $\sum_{\phi \in I} \phi(\tau(W))$ is a d.s. submodule of W. Therefore W is a τ-d Baer module. □

We can prove the following proposition similar to the proof of Theorem 5.5.

Proposition 5.6. *Let $\{W_i\}_{i \in \mathcal{I}}$ be a class of R-modules for an index set $\mathcal{I}$. If for every $i \in \mathcal{I}$, we have $W_i \trianglelefteq \bigoplus_{i \in \mathcal{I}} W_i$, then $\bigoplus_{i \in \mathcal{I}} W_i$ is τ-d Baer if and only if each W_i is τ-d Baer.*

Similar to τ-d-Rickart version, we may define the following.

Definition 5.7. Let W and U be R-modules. Then, W is called U-*τ-d Baer* if for every subset I of $Hom_R(W, U)$, $\sum_{\phi \in I} \phi(\tau(W))$ is a d.s. submodule of U.

Theorem 5.8. *Let $W = W_1 \oplus W_2$ and U be R-modules. If W is U-τ-d Baer, then for any d.s. submodule K of U, each W_i is K-τ-d Baer.*

Proof. Note that $\tau(W) \trianglelefteq W$, so $\tau(W) = \tau(W_1) \oplus \tau(W_2)$. Suppose that A is a subset of $Hom_R(W_1, K)$. Then $B = \{j \circ g \circ \pi_{W_1} \mid g \in A\}$ in which $\pi_{W_1} : M \to W_1$ is the projection of W on W_1 and j is the inclusion from K to U, is a subset of $Hom_R(W, U)$. It is easy to check that $A\tau(W_1) = \sum_{g \in A} g(\tau(W_1)) = \sum_{g \in B} g(\tau(W))$. As W is a U-τ-d Baer module, $A\tau(W_1)$ is a d.s. submodule of U and hence a d.s. submodule of K. □

Proposition 5.9. *Suppose that $\mathcal{J}$ is an index set, $\{W_i\}_{i \in \mathcal{J}}$ a class of R-modules and U is an R-module. Below listed statements hold:*

(1) *If U satisfies summand sum property for d.s. submodules included in $\tau(U)$ and $\mathcal{J}$ is finite, then $\bigoplus_{i \in \mathcal{J}} W_i$ is U-τ-d Baer if and only if each W_i ($i \in \mathcal{J}$) is U-τ-d Baer.*

(2) *If U satisfies strong summand sum property for d.s. submodules included in $\tau(U)$, and $\mathcal{J}$ is arbitrary, then $\bigoplus_{i \in \mathcal{J}} W_i$ is U-τ-d Baer if and only if each W_i is U-τ-d Baer.*

Proof. (1) One way holds from Theorem 5.8. For the necessity, suppose that A is a subset of $Hom_R(\bigoplus_{i \in \mathcal{J}} W_i, U)$. Then

$$B_i = \{\phi j_i \mid \phi \in A\}$$

in which j_i is the inclusion from W_i to $\bigoplus_{i \in \mathcal{J}} W_i$, is a subset of $Hom_R(W_i, U)$.

Assume that ϕ is a homomorphism from $\bigoplus_{i \in \mathcal{J}} W_i$ to U. Then $\phi = (\phi_i)_{i \in \mathcal{J}}$ where $\phi_i = \phi j_i$ is a homomorphism from W_i to U for each $i \in \mathcal{J}$. By hypothesis, $\sum_{\phi_i \in B_i} \phi_i(\tau(W_i))$ is a d.s. submodule of U for each $i \in \mathcal{J}$. As U satisfies summand sum property for d.s. submodules included in $\tau(N)$, we have

$$\begin{aligned}\sum_{\phi \in A} \phi(\tau(W)) &= \sum_{\phi \in A} \phi(\oplus_{i=1}^{n}(\tau(W_i))) \\ &= \sum_{i \in \mathcal{J}} \sum_{\phi_i \in B_i} \phi_i(\tau(W_i)) \\ &\leq^{\oplus} U.\end{aligned}$$

Therefore $\bigoplus_{i \in \mathcal{J}} W_i$ is U-τ-d Baer.

(2) Similar to (1). □

Corollary 5.10. *Let $\mathcal{J}$ be an index set and $\{W_i\}_{i \in \mathcal{J}}$ a class of R-modules. Then, for each $j \in \mathcal{J}$, $\bigoplus_{i \in \mathcal{J}} W_i$ is W_j-τ-d Baer if and only if W_i is W_j-τ-d Baer for all $i \in \mathcal{J}$.*

Proof. It follows from Proposition 5.9 and Theorem 2.8. □

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DUAL RICKART (BAER) MODULES AND PRERADICALS

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مدول‌های دوگان ریکارت (بئر) و پیش رادیکال‌ها

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در این مقاله، با استفاده از مفهوم پیش رادیکال‌ها، مدول‌های دوگان ریکارت (بئر) را معرفی می‌کنیم. نشان می‌دهیم یک مدول مانند W، $\tau - d$ ریکارت است اگر و تنها اگر $W = \tau(W) \oplus L$ که در آن $\tau(W)$، یک مدول دوگان ریکارت می‌باشد. ثابت می‌کنیم W مدولی $\tau - d$ بئر است اگر و تنها اگر W، $\tau - d$ ریکارت باشد و در خاصیت قویاً جمعوند مستقیم جمعی برای زیرمدول‌های $d.s.$ از W که مشمول در $\tau(W)$ هستند، صدق کند. همچنین با استفاده از $\tau(R_R)$، حلقه‌های $\tau - d$ بئر راست را رده‌بندی می‌کنیم.

کلمات کلیدی: پیش رادیکال، مدول دوگان ریکارت، $\tau - d$ مدول ریکارت، $\tau - d$ مدول بئر.