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GENERALIZED FORMAL LOCAL COHOMOLOGY MODULES

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Abstract. Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M and N two finitely generated R-modules. In this paper, we introduce the concept of generalized formal local cohomology modules. We define i-th generalized formal local cohomology module of M and N with respect to $\mathfrak{a}$ by $\mathfrak{F}^i_{\mathfrak{a}}(M, N) := \varprojlim_n \mathrm{H}^i_{\mathfrak{m}}(M, N/\mathfrak{a}^n N)$ for $i \geq 0$. We prove several results concerning vanishing and finiteness properties of these modules.

1. Introduction

Throughout this paper, $(R, \mathfrak{m})$ is a commutative Noetherian local ring with identity, $\mathfrak{a}$ is an ideal of R and M and N are two finitely generated R-modules. Recall that the i-th local cohomology module of M with respect to $\mathfrak{a}$ is denoted by $\mathrm{H}^i_{\mathfrak{a}}(M)$. For basic facts about local cohomology refer to [6]. Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. For each $i \geq 0$;

$$\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_n \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$$

is called the i-th formal local cohomology of M with respect to $\mathfrak{a}$. The basic properties of formal local cohomology modules are found in [2], [5] and [15].

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A generalization of local cohomology functor has been given by J. Herzog in [12]. The i-th generalized local cohomology module of M and N with respect to $\mathfrak{a}$ is denoted by $\mathrm{H}^i_{\mathfrak{a}}(M,N) := \varinjlim_n \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N)$. Here, by using the concept of generalized local cohomology modules, we introduce the generalized formal local cohomology modules. We define $\mathfrak{F}^i_{\mathfrak{a}}(M,N) := \varprojlim_n \mathrm{H}^i_{\mathfrak{m}}(M, N/\mathfrak{a}^n N)$ for any integer $i \geq 0$ and call it the i-th generalized formal local cohomology module of M and N with respect to $\mathfrak{a}$. Then, we study some properties of generalized formal local cohomology modules.

In Section 2, we investigate vanishing, Artinianness and attached prime ideals of generalized formal local cohomology modules. Among other things, we will prove that for two finitely generated R-modules M and N with $\mathrm{pd}_R M < \infty$ we have $\mathfrak{F}^i_{\mathfrak{a}}(M,N) = 0$ for all $i > \dim R$, $\mathfrak{F}^{\dim R}_{\mathfrak{a}}(M,N)$ is Artinian and

$$\mathrm{Att}_R(\mathfrak{F}^{\dim R}_{\mathfrak{a}}(M,N)) = \mathrm{Att}_R(\mathrm{H}^{\dim R}_{\mathfrak{m}}(M,N)) \cap V(\mathfrak{a}).$$

Also, if M and N are two finitely generated R-modules such that $\mathrm{pd}_R M = d < \infty$ and $\dim N = n < \infty$, then $\mathfrak{F}^{d+n}_{\mathfrak{a}}(M,N)$ is an Artinian R-module and

$$\mathrm{Att}_R(\mathfrak{F}^{d+n}_{\mathfrak{a}}(M,N)) = \mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N)) \cap V(\mathfrak{a}).$$

Recall that, the formal grade of M with respect to $\mathfrak{a}$ is defined to be the least integer i such that $\mathfrak{F}^i_{\mathfrak{a}}(M) \neq 0$ and it is denoted by $\mathrm{fgrade}(\mathfrak{a}, M)$, [15]. The cohomological dimension of M and N with respect to $\mathfrak{a}$ is defined by $\mathrm{cd}_{\mathfrak{a}}(M,N) := \sup\{i : \mathrm{H}^i_{\mathfrak{a}}(M,N) \neq 0\}$, [1]. We define the formal grade of M and N with respect to $\mathfrak{a}$ by

$$\mathrm{fgrade}(\mathfrak{a}, M, N) := \inf\{i : \mathfrak{F}^i_{\mathfrak{a}}(M,N) \neq 0\}$$

and the formal dimension of M and N with respect to $\mathfrak{a}$ by

$$\mathrm{fd}(\mathfrak{a}, M, N) := \sup\{i : \mathfrak{F}^i_{\mathfrak{a}}(M,N) \neq 0\}.$$

Here, we show that if $\mathrm{pd}_R M < \infty$ and $\dim N < \infty$, then

$$\mathrm{fd}(\mathfrak{a}, M, N) = \mathrm{cd}_{\mathfrak{m}}(M, N/\mathfrak{a}N).$$

In Section 3, we investigate the 0-th generalized formal local cohomology module $\mathfrak{F}^0_{\mathfrak{a}}(M,N)$. At first, we prove that

$$\mathfrak{F}^0_{\mathfrak{a}}(M,N) \cong \mathrm{Hom}_R(M, \mathfrak{F}^0_{\mathfrak{a}}(N))$$

and by using it we prove that $\mathfrak{F}^0_{\mathfrak{a}}(M,N)$ is a finitely generated $\hat{R}$-module and

$$\mathrm{Ass}_{\hat{R}}(\mathfrak{F}^0_{\mathfrak{a}}(M,N)) = \{\mathfrak{p} \in \mathrm{Ass}_{\hat{R}}(\mathrm{Hom}_{\hat{R}}(\hat{M}, \hat{N})) : \dim(\hat{R}/(\mathfrak{a}\hat{R}, \mathfrak{p})) = 0\}.$$

In Section 4, we give a duality result for generalized formal local cohomology modules over a Cohen-Macaulay local ring. We show that, if $(R, \mathfrak{m})$ is a d-dimensional Cohen-Macaulay local ring and M and N are two finitely generated R-modules such that $\operatorname{pd}_R M < \infty$ then for any integer $i \geq 0$ we have

$$\mathfrak{F}^i_{\mathfrak{a}}(M, N) \cong \operatorname{Hom}_{\widehat{R}}(\mathrm{H}^{d-i}_{\mathfrak{a}\widehat{R}}(\widehat{N}, \widehat{M} \otimes_{\widehat{R}} w_{\widehat{R}}), E_R(R/\mathfrak{m})),$$

where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R, $w_{\widehat{R}}$ is the canonical module of $\widehat{R}$ and $E_R(R/\mathfrak{m})$ denotes the injective hull of $R/\mathfrak{m}$. Then by using this result, we prove that if $(R, \mathfrak{m})$ is a d-dimensional Cohen-Macaulay local ring and M and N are two finitely generated R-modules such that $\operatorname{pd}_R M < \infty$ and $\operatorname{pd}_R N < \infty$ then

$$\operatorname{fgrade}(\mathfrak{a}, M, N) + \operatorname{cd}_{\mathfrak{a}}(N, M) = \dim R.$$

2. Vanishing and Artinianness Results

In this section, we obtain some results about vanishing, Artinianness and attached prime ideals of generalized formal local cohomology modules. We begin with the following main definition.

Definition 2.1. Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M and N be two finitely generated R-modules. For each $i \geq 0$, we define the i-th generalized formal local cohomology module of M and N with respect to $\mathfrak{a}$ by

$$\mathfrak{F}^i_{\mathfrak{a}}(M, N) := \varprojlim_n \mathrm{H}^i_{\mathfrak{m}}(M, N/\mathfrak{a}^n N).$$

It should be noted that the formal local cohomology modules of N with respect to $\mathfrak{a}$ is defined by $\mathfrak{F}^i_{\mathfrak{a}}(N) = \varprojlim_n \mathrm{H}^i_{\mathfrak{m}}(N/\mathfrak{a}^n N)$. Clearly, $\mathfrak{F}^i_{\mathfrak{a}}(R, N) = \mathfrak{F}^i_{\mathfrak{a}}(N)$ for all $i \geq 0$ and any R-module N.

Recall that, the arithmetic rank of the ideal $\mathfrak{a}$, denoted by $\operatorname{ara}(\mathfrak{a})$, is the least number of elements of R required to generate an ideal which has the same radical as $\mathfrak{a}$. We need the following known results for generalized local cohomology.

Theorem 2.2. *Let M and N be two finitely generated R-modules such that* $\operatorname{pd}_R M < \infty$.

i) ([8, Theorem 3.1]*) If* $\dim R = d < \infty$ *then* $\mathrm{H}^i_{\mathfrak{a}}(M, N) = 0$ *for all* $i > d$,

ii) ([17, Theorem 2.5]*)* $\mathrm{H}^i_{\mathfrak{a}}(M, N) = 0$ *for all* $i > \operatorname{pd}_R M + \operatorname{ara}(\mathfrak{a})$,

iii) ([17, Theorem 3.7]*) Suppose* $\dim N < \infty$. *Then* $\mathrm{H}^i_{\mathfrak{a}}(M, N) = 0$ *for all* $i > \operatorname{pd}_R M + \dim(M \otimes_R N)$.

In the following, we give some vanishing results.

Proposition 2.3. *Let $\mathfrak{a}$ be an ideal of a d-dimensional local ring $(R, \mathfrak{m})$ and M and N two finitely generated R-modules such that $\operatorname{pd}_R M < \infty$. Then*

i) $\mathfrak{F}^i_{\mathfrak{a}}(M, N) = 0$ *for all* $i > d$,

ii) $\mathfrak{F}^i_{\mathfrak{a}}(M, N) = 0$ *for all* $i > \operatorname{pd}_R M + \operatorname{ara}(\mathfrak{m})$,

iii) If $\dim N/\mathfrak{a}N < \infty$ *then*

$$\mathfrak{F}^i_{\mathfrak{a}}(M, N) = 0 \text{ for all } \ i > \operatorname{pd}_R M + \dim(M \otimes_R N/\mathfrak{a}N).$$

Proof. (i), (ii): For all $n \in \mathbb{N}_0$, by Theorem 2.2(i),

$$\mathrm{H}^i_{\mathfrak{m}}(M, N/\mathfrak{a}^n N) = 0$$

for all $i > d$ and by Theorem 2.2(ii) $\mathrm{H}^i_{\mathfrak{m}}(M, N/\mathfrak{a}^n N) = 0$ for all $i > \operatorname{pd}_R M + \operatorname{ara}(\mathfrak{m})$. Thus, the result follows from the above definition.

(iii) Note that, $\dim(M \otimes_R N/\mathfrak{a}^k N) = \dim(M \otimes_R N/\mathfrak{a}N)$ for all $k \in \mathbb{N}$. Thus by Theorem 2.2(iii), $\mathrm{H}^i_{\mathfrak{m}}(M, N/\mathfrak{a}^k N) = 0$ for all $i > \operatorname{pd}_R M + \dim(M \otimes_R N/\mathfrak{a}N)$ and $k \geq 0$. Therefore we conclude that

$$\mathfrak{F}^i_{\mathfrak{a}}(M, N) = \varprojlim_k \mathrm{H}^i_{\mathfrak{m}}(M, N/\mathfrak{a}^k N) = 0,$$

for all $i > \operatorname{pd}_R M + \dim(M \otimes_R N/\mathfrak{a}N)$, as required. □

Lemma 2.4. *Let $\mathfrak{a}$ be an ideal of a d-dimensional local ring $(R, \mathfrak{m})$, and let M and N be two finitely generated R-modules such that $\operatorname{pd}_R M < \infty$. Then $\mathfrak{F}^d_{\mathfrak{a}}(M, N)$ is a homomorphic image of $\mathrm{H}^d_{\mathfrak{m}}(M, N)$, and so the following assertions hold:*

i) $\mathfrak{F}^d_{\mathfrak{a}}(M, N)$ *is Artinian,*

ii) $\operatorname{Supp}_R \mathfrak{F}^d_{\mathfrak{a}}(M, N) \subseteq \operatorname{Supp}_R \mathrm{H}^d_{\mathfrak{m}}(M, N)$,

iii) $\operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M, N) \subseteq \operatorname{Att}_R \mathrm{H}^d_{\mathfrak{m}}(M, N)$.

Proof. For all $n \geq 0$, there is the following short exact sequence

$$0 \to \mathfrak{a}^n N \to N \to N/\mathfrak{a}^n N \to 0$$

that implies the following long exact sequence

$$\cdots \to \mathrm{H}^d_{\mathfrak{m}}(M, N) \to \mathrm{H}^d_{\mathfrak{m}}(M, N/\mathfrak{a}^n N) \to \mathrm{H}^{d+1}_{\mathfrak{m}}(M, \mathfrak{a}^n N) \to \cdots .$$

By Theorem 2.2(i) $\mathrm{H}^{d+1}_{\mathfrak{m}}(M, \mathfrak{a}^n N) = 0$ and by [10, Theorem 2.2] the R-modules of the above long exact sequence are Artinian. Thus we obtain an epimorphism $\mathrm{H}^d_{\mathfrak{m}}(M, N) \to \mathrm{H}^d_{\mathfrak{m}}(M, N/\mathfrak{a}^n N) \to 0$ of Artinian R-modules. Therefore [14, Lemma 2.3] implies that the sequence

$$\varprojlim_n \mathrm{H}^d_{\mathfrak{m}}(M, N) \to \varprojlim_n \mathrm{H}^d_{\mathfrak{m}}(M, N/\mathfrak{a}^n N) \to 0,$$

is exact. Thus we obtain the exact sequence

$$\mathrm{H}^d_{\mathfrak{m}}(M,N) \to \mathfrak{F}^d_{\mathfrak{a}}(M,N) \to 0.$$

By [10, Theorem 2.2] $\mathrm{H}^d_{\mathfrak{m}}(M,N)$ is Artinian. Hence $\mathfrak{F}^d_{\mathfrak{a}}(M,N)$ is homomorphic image of an Artinian R-module, and so is Artinian, as required. □

Theorem 2.5. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$, and let M and N be two finitely generated R-modules such that $\operatorname{pd}_R M < \infty$. Let $d = \dim R$. Then*

$$\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M,N)) = \operatorname{Att}_R(\mathrm{H}^d_{\mathfrak{m}}(M,N)) \cap V(\mathfrak{a}).$$

Proof. From Theorem 2.2(i), it follows that $\mathrm{H}^d_{\mathfrak{m}}(M,-)$ is a right exact functor. Thus for all $k \in \mathbb{N}$ we have

$$\begin{aligned}\mathrm{H}^d_{\mathfrak{m}}(M,N/\mathfrak{a}^kN) &\cong \mathrm{H}^d_{\mathfrak{m}}(M,R)\otimes_R N/\mathfrak{a}^kN\\ &\cong \mathrm{H}^d_{\mathfrak{m}}(M,N)\otimes_R R/\mathfrak{a}^k\\ &\cong \mathrm{H}^d_{\mathfrak{m}}(M,N)/\mathfrak{a}^k\,\mathrm{H}^d_{\mathfrak{m}}(M,N).\end{aligned}$$

On the other hand, $\mathrm{H}^d_{\mathfrak{m}}(M,N)$ is Artinian and so there exists an integer t such that $\mathfrak{a}^k\,\mathrm{H}^d_{\mathfrak{m}}(M,N) = \mathfrak{a}^t\,\mathrm{H}^d_{\mathfrak{m}}(M,N)$ for all $k > t$. Therefore

$$\begin{aligned}\mathfrak{F}^d_{\mathfrak{a}}(M,N) &= \varprojlim_k \mathrm{H}^d_{\mathfrak{m}}(M,N/\mathfrak{a}^kN)\\ &\cong \varprojlim_k \mathrm{H}^d_{\mathfrak{m}}(M,N)/\mathfrak{a}^k\,\mathrm{H}^d_{\mathfrak{m}}(M,N)\\ &\cong \mathrm{H}^d_{\mathfrak{m}}(M,N)/\mathfrak{a}^t\,\mathrm{H}^d_{\mathfrak{m}}(M,N).\end{aligned}$$

Hence $\mathfrak{F}^d_{\mathfrak{a}}(M,N) \cong \mathrm{H}^d_{\mathfrak{m}}(M,N)\otimes_R R/\mathfrak{a}^t$. Now, the result follows by [14, Proposition 5.2]. □

Corollary 2.6. *Let $\mathfrak{a}$ be an ideal of a d-dimensional Cohen-Macaulay local ring $(R,\mathfrak{m})$, and let M and N be two finitely generated R-modules such that $\operatorname{pd}_R M < \infty$. Then*

$$\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M,N)) = \operatorname{Ass}_R M \cap \operatorname{Supp}_R(N) \cap \mathrm{V}(\mathfrak{a}),$$

and we have

i) $\mathfrak{F}^d_{\mathfrak{a}}(M,N) = 0$ if and only if $\operatorname{Ass}_R M \cap \operatorname{Supp}_R(N) \cap \mathrm{V}(\mathfrak{a}) = \emptyset$,

ii) $\mathfrak{F}^d_{\mathfrak{a}}(M,N)$ is fintely generated if and only if $\operatorname{Hom}_R(N/\mathfrak{a}N, M)$ is Artinian.

Proof. By [9, Corollary 3.4] $\operatorname{Att}_R(\mathrm{H}^d_{\mathfrak{m}}(M,N)) = \operatorname{Ass}_R M \cap \operatorname{Supp}_R(N)$. Thus by Theorem 2.5,

$$\begin{aligned}\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M,N)) &= \operatorname{Att}_R(\mathrm{H}^d_{\mathfrak{m}}(M,N)) \cap V(\mathfrak{a})\\ &= \operatorname{Ass}_R M \cap \operatorname{Supp}_R(N) \cap \mathrm{V}(\mathfrak{a}).\end{aligned}$$

Thus, assertion (i) follows because for an Artinian R-module A, $A = 0$ if and only if $\mathrm{Att}_R A = \emptyset$. Now, we prove the assertion (ii). Assume that $\mathfrak{F}^d_{\mathfrak{a}}(M,N)$ is a finitely generated R-module. By Lemma 2.4, $\mathfrak{F}^d_{\mathfrak{a}}(M,N)$ is Artinian and so by [6, 7.2.12] we conclude that $\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M,N)) \subseteq \{\mathfrak{m}\}$. On the other hand,

$$\begin{aligned}\mathrm{Ass}_R \mathrm{Hom}_R(N/\mathfrak{a}N, M) &= \mathrm{Ass}_R M \cap \mathrm{Supp}_R(N/\mathfrak{a}N)\\ &= \mathrm{Ass}_R M \cap \mathrm{Supp}_R(N) \cap \mathrm{V}(\mathfrak{a})\\ &= \mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M,N)).\end{aligned}$$

Hence $\mathrm{Ass}_R \mathrm{Hom}_R(N/\mathfrak{a}N, M) \subseteq \{\mathfrak{m}\}$. Moreover, $\mathrm{Hom}_R(N/\mathfrak{a}N, M)$ is a finitely generated R-module and so $\mathrm{Hom}_R(N/\mathfrak{a}N, M)$ is Artinian. Conversely, assume that $\mathrm{Hom}_R(N/\mathfrak{a}N, M)$ is Artinian. Thus we have $\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M,N)) = \mathrm{Ass}_R(\mathrm{Hom}_R(N/\mathfrak{a}N, M)) \subseteq \{\mathfrak{m}\}$. Since $\mathfrak{F}^d_{\mathfrak{a}}(M,N)$ is Artinian and $\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M,N)) \subseteq \{\mathfrak{m}\}$, [6, 7.2.12] implies that $\mathfrak{F}^d_{\mathfrak{a}}(M,N)$ is finitely generated, as required. □

In the following result, we find a relation between formal dimension and cohomological dimension of two finitely generated R-modules.

Theorem 2.7. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$, and let M and N be two finitely generated R-modules such that $\mathrm{pd}_R M < \infty$ and $\dim N < \infty$. Then $\mathrm{fd}(\mathfrak{a}, M, N) = \mathrm{cd}_{\mathfrak{m}}(M, N/\mathfrak{a}N)$.*

Proof. Let $u := \mathrm{cd}_{\mathfrak{m}}(M, N/\mathfrak{a}N)$. We must show that $\mathfrak{F}^u_{\mathfrak{a}}(M,N) \neq 0$ and $\mathfrak{F}^i_{\mathfrak{a}}(M,N) = 0$ for all $i > u$.

For any integer n, the short exact sequence

$$0 \to \mathfrak{a}N/\mathfrak{a}^n N \to N/\mathfrak{a}^n N \longrightarrow N/\mathfrak{a}N \to 0,$$

induces the exact sequence

$$\mathrm{H}^u_{\mathfrak{m}}(M, N/\mathfrak{a}^n N) \to \mathrm{H}^u_{\mathfrak{m}}(M, N/\mathfrak{a}N) \to \mathrm{H}^{u+1}_{\mathfrak{m}}(M, \mathfrak{a}N/\mathfrak{a}^n N).$$

Since $\mathrm{Supp}_R(\mathfrak{a}N/\mathfrak{a}^n N) \subseteq \mathrm{Supp}_R(N/\mathfrak{a}^n N) = \mathrm{Supp}_R(N/\mathfrak{a}N)$ by [1, Theorem B] we conclude that

$$\mathrm{cd}_{\mathfrak{m}}(M, \mathfrak{a}N/\mathfrak{a}^n N) \leq \mathrm{cd}_{\mathfrak{m}}(M, N/\mathfrak{a}N) = u$$

and so $\mathrm{H}^{u+1}_{\mathfrak{m}}(M, \mathfrak{a}N/\mathfrak{a}^n N) = 0$ for all $n \in \mathbb{N}$. Therefore we have an epimorphism

$$\varprojlim_{n\geq 1} \mathrm{H}^u_{\mathfrak{m}}(M, N/\mathfrak{a}^n N) \to \varprojlim_{n\geq 1} \mathrm{H}^u_{\mathfrak{m}}(M, N/\mathfrak{a}N) \to 0,$$

and so there exists an epimorphism $\mathfrak{F}^u_{\mathfrak{a}}(M,N) \to \mathrm{H}^u_{\mathfrak{m}}(M, N/\mathfrak{a}N) \to 0$. By assumption $\mathrm{H}^u_{\mathfrak{m}}(M, N/\mathfrak{a}N) \neq 0$ and we get $\mathfrak{F}^u_{\mathfrak{a}}(M,N) \neq 0$. On the other hand, let $j > u$ be an integer. Since

$$\mathrm{Supp}_R(N/\mathfrak{a}^n N) = \mathrm{Supp}_R(N/\mathfrak{a}N)$$

we have $\mathrm{cd}_{\mathfrak{m}}(M, N/\mathfrak{a}^n N) = \mathrm{cd}_{\mathfrak{m}}(M, N/\mathfrak{a}N) = u$ for all $n \in \mathbb{N}$. Thus $\mathrm{H}^j_{\mathfrak{m}}(M, N/\mathfrak{a}^n N) = 0$ for all $n \in \mathbb{N}$. Therefore $\mathfrak{F}^j_{\mathfrak{a}}(M, N) = 0$, as required. □

In the next result, we obtain [15, Theorem 4.5] by Theorem 2.7.

Corollary 2.8. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and N be a finitely generated R-module such that $\dim N < \infty$. Then*

$$\sup\{i : \mathfrak{F}^i_{\mathfrak{a}}(N) \neq 0\} = \dim(N/\mathfrak{a}N).$$

Proof. By Grothendieck's Non-vanishing Theorem

$$\mathrm{cd}_{\mathfrak{m}}(R, N/\mathfrak{a}N) = \mathrm{cd}_{\mathfrak{m}}(N/\mathfrak{a}N) = \dim(N/\mathfrak{a}N).$$

Hence, the assertion is immediate by Theorem 2.7 and putting $M = R$. □

Corollary 2.9. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$, and M, N be two finitely generated R-modules such that $\mathrm{pd}_R M < \infty$, $\dim N < \infty$. If $\mathrm{H}^{\mathrm{pd}_R M + \dim N/\mathfrak{a}N}_{\mathfrak{m}}(M, N/\mathfrak{a}N) \neq 0$ then $\mathfrak{F}^{\mathrm{pd}_R M + \dim N/\mathfrak{a}N}_{\mathfrak{a}}(M, N) \neq 0$.*

Proof. By Theorem 2.2(iii) we have $\mathrm{H}^i_{\mathfrak{m}}(M, N/\mathfrak{a}N) = 0$ for all $i > \mathrm{pd}_R M + \dim N/\mathfrak{a}N$. Thus, assumption implies that

$$\mathrm{cd}_{\mathfrak{m}}(M, N/\mathfrak{a}N) = \mathrm{pd}_R M + \dim N/\mathfrak{a}N.$$

Now, by Theorem 2.7, it follows that $\mathrm{fd}(\mathfrak{a}, M, N) = \mathrm{pd}_R M + \dim N/\mathfrak{a}N$. Therefore $\mathfrak{F}^{\mathrm{pd}_R M + \dim N/\mathfrak{a}N}_{\mathfrak{a}}(M, N) \neq 0$, as required. □

Lemma 2.10. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$. Let M and N be two finitely generated R-modules such that $\mathrm{pd}_R M = d < \infty$ and $\dim N = n < \infty$. Then $\mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N)$ is an Artinian R-module and*

i) $\mathrm{Supp}_R \mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N) \subseteq \mathrm{Supp}_R \mathrm{H}^{d+n}_{\mathfrak{m}}(M, N)$,

ii) $\mathrm{Att}_R \mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N) \subseteq \mathrm{Att}_R \mathrm{H}^{d+n}_{\mathfrak{m}}(M, N)$.

Proof. Let $k \geq 0$. The short exact sequence

$$0 \to \mathfrak{a}^k N \to N \to N/\mathfrak{a}^k N \to 0$$

induces the exact sequence

$$\mathrm{H}^{d+n}_{\mathfrak{m}}(M, N) \to \mathrm{H}^{d+n}_{\mathfrak{m}}(M, N/\mathfrak{a}^k N) \to \mathrm{H}^{d+n+1}_{\mathfrak{m}}(M, \mathfrak{a}^k N).$$

Since $\mathrm{pd}_R M + \dim \mathfrak{a}^k N < \mathrm{pd}_R M + \dim N + 1$, by Theorem 2.2(iii) $\mathrm{H}^{d+n+1}_{\mathfrak{m}}(M, \mathfrak{a}^k N) = 0$ and so we have the following exact sequence of Artinian R-modules

$$\mathrm{H}^{d+n}_{\mathfrak{m}}(M, N) \to \mathrm{H}^{d+n}_{\mathfrak{m}}(M, N/\mathfrak{a}^k N) \to 0,$$

and by passing to the inverse limit we get

$$\mathrm{H}^{d+n}_{\mathfrak{m}}(M, N) \to \mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N) \to 0.$$

But by [10, Theorem 2.2] $\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N)$ is Artinian and so $\mathfrak{F}^{d+n}_{\mathfrak{a}}(M,N)$ as a homomorphic image of an Artinian module is Artinian and also we have $\operatorname{Supp}_R \mathfrak{F}^{d+n}_{\mathfrak{a}}(M,N) \subseteq \operatorname{Supp}_R \mathrm{H}^{d+n}_{\mathfrak{m}}(M,N)$ and

$$\operatorname{Att}_R \mathfrak{F}^{d+n}_{\mathfrak{a}}(M,N) \subseteq \operatorname{Att}_R \mathrm{H}^{d+n}_{\mathfrak{m}}(M,N).$$

□

Proposition 2.11. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$. Let M and N be two finitely generated R-modules such that $\operatorname{pd}_R M = d < \infty$ and $0 < \dim N = n < \infty$. If $\mathfrak{F}^{d+n}_{\mathfrak{a}}(M,N) \neq 0$ then it is not finitely generated.*

Proof. By assumption $\mathfrak{F}^{d+n}_{\mathfrak{a}}(M,N) \neq 0$ and so $\operatorname{Att}_R(\mathfrak{F}^{d+n}_{\mathfrak{a}}(M,N)) \neq \emptyset$. On the other hand, by the previous lemma and [11, Proposition 2.2],

$$\begin{aligned} \operatorname{Att}_R \mathfrak{F}^{d+n}_{\mathfrak{a}}(M,N) &\subseteq \operatorname{Att}_R \mathrm{H}^{d+n}_{\mathfrak{m}}(M,N) \\ &\subseteq \operatorname{Att}_R \mathrm{H}^{n}_{\mathfrak{m}}(N) \\ &= \{\mathfrak{p} \in \operatorname{Ass}(N) |\dim R/\mathfrak{p} = n\}. \end{aligned}$$

Assume that $\mathfrak{F}^{d+n}_{\mathfrak{a}}(M,N) \neq 0$ is a finitely generated R-module. Since by lemma 2.10 $\mathfrak{F}^{d+n}_{\mathfrak{a}}(M,N)$ is Artinian, [6, Corollary 7.2.12] implies that

$$\mathfrak{m} \in \operatorname{Att}_R(\mathfrak{F}^{d+n}_{\mathfrak{a}}(M,N)) \subseteq \{\mathfrak{p} \in \operatorname{Ass}(N) |\dim R/\mathfrak{p} = n > 0\},$$

which is a contradiction. □

Theorem 2.12. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$, and let M and N be two finitely generated R-modules such that $d = \operatorname{pd}_R M < \infty$ and $l = \dim N/\mathfrak{a}N < \infty$. Then*

$$\mathfrak{F}^{d+l}_{\mathfrak{a}}(M,N) \cong \operatorname{Ext}^d_R(M, \mathfrak{F}^{l}_{\mathfrak{a}}(N)).$$

Proof. By [11, Proposition 2.2],

$$\mathrm{H}^{d+l}_{\mathfrak{m}}(M, N/\mathfrak{a}N) \cong \operatorname{Ext}^d_R(M, \mathrm{H}^{l}_{\mathfrak{m}}(N/\mathfrak{a}N)).$$

Since $\dim N/\mathfrak{a}^k N = \dim N/\mathfrak{a}N = l$ for all $k \in \mathbb{N}$, we have

$$\mathrm{H}^{d+l}_{\mathfrak{m}}(M, N/\mathfrak{a}^k N) \cong \operatorname{Ext}^d_R(M, \mathrm{H}^{l}_{\mathfrak{m}}(N/\mathfrak{a}^k N)),$$

for all $k \in \mathbb{N}$ and so

$$\varprojlim_k \mathrm{H}^{d+l}_{\mathfrak{m}}(M, N/\mathfrak{a}^k N) \cong \operatorname{Ext}^d_R(M, \varprojlim_k \mathrm{H}^{l}_{\mathfrak{m}}(N/\mathfrak{a}^k N)).$$

Therefore, we have

$$\mathfrak{F}^{d+l}_{\mathfrak{a}}(M,N) \cong \operatorname{Ext}^d_R(M, \mathfrak{F}^{l}_{\mathfrak{a}}(N)),$$

as required. □

Corollary 2.13. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$, and let M and N be two finitely generated R-modules such that $d = \operatorname{pd}_R M < \infty$ and $l = \dim N/\mathfrak{a}N < \infty$. If $\mathfrak{F}^l_{\mathfrak{a}}(N)$ is an Artinian R-module then $\mathfrak{F}^{d+l}_{\mathfrak{a}}(M,N)$ is an Artinian R-module.*

Proof. Since $\mathfrak{F}^l_{\mathfrak{a}}(N)$ is an Artinian R-module, by [3, Lemma 2.1], $\operatorname{Ext}^d_R(M,\mathfrak{F}^l_{\mathfrak{a}}(N))$ is Artinian and so the result follows by Theorem 2.12. □

Corollary 2.14. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$, and let M, N be two finitely generated R-modules such that $\operatorname{pd}_R M < \infty$ and $\dim N < \infty$. Then $\mathfrak{F}^{\operatorname{pd}_R M+\dim N}_{\mathfrak{a}}(M,N)$ is an Artinian R-module.*

Proof. If $\dim N/\mathfrak{a}N < \dim N$ then by Proposition 2.3(iii) we have $\mathfrak{F}^{\operatorname{pd}_R M+\dim N}_{\mathfrak{a}}(M,N) = 0$ and so $\mathfrak{F}^{\operatorname{pd}_R M+\dim N}_{\mathfrak{a}}(M,N)$ is an Artinian R-module. If $\dim N/\mathfrak{a}N = \dim N$ then since by [5, Lemma 2.2] $\mathfrak{F}^{\dim N}_{\mathfrak{a}}(N)$ is an Artinian R-module, the result follows by Corollary 2.13. □

Proposition 2.15. *Let R be a ring and $(Q_n)_{n\geq 1}$ be an inverse system of R-modules, with maps $\varphi_{mn} : Q_m \to Q_n$ for $m \geq n$. Let $\mathfrak{a}$ be an ideal of R such that $u^k Q_k = 0$ for all $u \in \mathfrak{a}$ and all $k \in \mathbb{N}$. If $\varprojlim_n Q_n$ is non-zero and representable, then $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att}_R(\varprojlim_n Q_n)$.*

Proof. Let $\varprojlim_n Q_n = S_1+S_2+...+S_n$ be a minimal secondary representation of $\varprojlim_n Q_n$ where S_j is $\mathfrak{p}_j$-secondary for $j = 1,2,...,n$. Suppose that there exists an integer $j \in \{1,\dots,n\}$ such that $\mathfrak{a} \nsubseteq \mathfrak{p}_j$ and look for a contradiction. Take an element $u \in \mathfrak{a}\setminus\mathfrak{p}_j$. Since $S_j \neq 0$ there exists an element $0 \neq g = (g_k) \in S_j \subseteq \varprojlim_n Q_n$. Let g_k be the first non-zero component of g. Since $u \notin \mathfrak{p}_j$, we have $uS_j = S_j$. But $u^k S_j \subseteq u^k(\varprojlim_n Q_n)$, and so $S_j \subseteq u^k(\varprojlim_n Q_n)$. As $u^k Q_k = 0$, it follows that the k-th component of each element of $u^k(\varprojlim_n Q_n)$ is zero. But, $g \in u^k(\varprojlim_n Q_n)$ and the k-th component of g is non-zero, which is a contradiction. □

Theorem 2.16. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$, and let M and N two finitely generated R-modules. Let i be a non-negative integer. If $\mathfrak{F}^i_{\mathfrak{a}}(M,N)$ is representable, then $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att}_R(\mathfrak{F}^i_{\mathfrak{a}}(M,N))$.*

Proof. By definition we have $\mathfrak{F}^i_{\mathfrak{a}}(M,N) = \varprojlim_k \operatorname{H}^i_{\mathfrak{m}}(M,N/\mathfrak{a}^k N)$. But, $u^k \operatorname{H}^i_{\mathfrak{m}}(M,N/\mathfrak{a}^k N) = 0$ for all $u \in \mathfrak{a}$ and $k \in \mathbb{N}$. Now, the result follows by proposition 2.15. □

Corollary 2.17. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$, and let M and N two finitely generated R-modules. Let i be a non-negative integer. If $\mathfrak{F}^i_{\mathfrak{a}}(M, N)$ is representable, then $\mathfrak{a} \subseteq \sqrt{(0 : \mathfrak{F}^i_{\mathfrak{a}}(M, N))}$*

Proof. By Theorem 2.16, $\operatorname{Att}_R(\mathfrak{F}^i_{\mathfrak{a}}(M, N)) \subseteq \mathrm{V}(\mathfrak{a})$ and so the result follows by [6, 7.2.11]. □

In the next result, we give the formula for the attached primes of the top generalized formal local cohomology module $\mathfrak{F}^{\operatorname{pd}_R M+\dim N}_{\mathfrak{a}}(M, N)$.

Theorem 2.18. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$. Let M and N be two finitely generated R-modules with $d = \operatorname{pd}_R M < \infty$ and $n = \dim N < \infty$. Then*

$$\operatorname{Att}_R(\mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N)) = \operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{m}}(M, N)) \cap V(\mathfrak{a}).$$

Proof. Let $\dim N/\mathfrak{a}N < \dim N$ then by Proposition 2.3(iii) we have $\mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N) = 0$ and so $\operatorname{Att}_R \mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N) = \emptyset$. But by [11, Theorem 2.3]

$$\operatorname{Att}_R \mathrm{H}^{d+n}_{\mathfrak{m}}(M, N) = \{\mathfrak{p} \in \operatorname{Ass}_R N | \operatorname{cd}_{\mathfrak{m}}(M, R/\mathfrak{p}) = d + n\}.$$

Thus we have

$$\operatorname{Att}_R \mathrm{H}^{d+n}_{\mathfrak{m}}(M, N) \cap V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Ass}_R N \mid \operatorname{cd}_{\mathfrak{m}}(M, R/\mathfrak{p}) = d + n\} \cap V(\mathfrak{a}).$$

We show that $\operatorname{Att}_R \mathrm{H}^{d+n}_{\mathfrak{m}}(M, N) \cap V(\mathfrak{a}) = \emptyset$. If

$$\operatorname{Att} \mathrm{H}^{d+n}_{\mathfrak{m}}(M, N) \cap V(\mathfrak{a}) \neq \emptyset,$$

then there is $\mathfrak{p} \in \operatorname{Ass}_R N$ such that $\operatorname{cd}_{\mathfrak{m}}(M, R/\mathfrak{p}) = d+n$ and $\mathfrak{p} \in V(\mathfrak{a})$. Thus $\mathfrak{p} \in \operatorname{Supp}_R N/\mathfrak{a}N$ and we conclude that $\dim R/\mathfrak{p} \leq \dim N/\mathfrak{a}N$. Moreover, by the hypothesis $\dim N/\mathfrak{a}N < \dim N$ and so by Theorem 2.2(iii) we have $\mathrm{H}^{d+n}_{\mathfrak{m}}(M, R/\mathfrak{p}) = 0$ and this shows that

$$\operatorname{cd}_{\mathfrak{m}}(M, R/\mathfrak{p}) \neq d + n,$$

which is a contradiction. Thus the result follows in this case.

Now suppose that $\dim N/\mathfrak{a}N = \dim N$. By Lemma 2.10(ii) and Theorem 2.16 we have $\operatorname{Att}_R(\mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N)) \subseteq \operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{m}}(M, N)) \cap V(\mathfrak{a})$. We will show that

$$\operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{m}}(M, N)) \cap V(\mathfrak{a}) \subseteq \operatorname{Att}_R(\mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N)).$$

Assume that $\mathfrak{p} \in \operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{m}}(M, N)) \cap V(\mathfrak{a})$. Thus we have

$$\mathfrak{p} \in \{\mathfrak{p} \in \operatorname{Ass}_R N \mid \operatorname{cd}_{\mathfrak{m}}(M, R/\mathfrak{p}) = d + n\} \cap V(\mathfrak{a}).$$

Since $\mathfrak{p} \in \operatorname{Ass}_R N$, by [13, Theorem 6.8] there is a $\mathfrak{p}$-primary submodule of N, say L, such that $\operatorname{Ass}_R(N/L) = \{\mathfrak{p}\}$ and $\mathfrak{p} = \sqrt{(0 : N/L)}$. Thus $\operatorname{Supp}_R R/\mathfrak{p} = \operatorname{Supp}_R N/L$ and so, $\dim R/\mathfrak{p} = \dim N/L$.

We claim that $\dim R/\mathfrak{p} = \dim N$. Let $\dim R/\mathfrak{p} < \dim N$, then

$$\begin{aligned}\operatorname{pd}_R M + \dim(M \otimes_R R/\mathfrak{p}) &\leq \operatorname{pd}_R M + \dim R/\mathfrak{p} \\ &< \operatorname{pd}_R M + \dim N \\ &= d + n,\end{aligned}$$

and by Theorem 2.2(iii), $\mathrm{H}^{d+n}_{\mathfrak{m}}(M, R/\mathfrak{p}) = 0$ and that is a contradiction with $\operatorname{cd}_{\mathfrak{m}}(M, R/\mathfrak{p}) = d + n$, so $\dim N/L = \dim R/\mathfrak{p} = \dim N$. But $\mathfrak{a} \subseteq \mathfrak{p}$ thus $\sqrt{\mathfrak{a}} \subseteq \sqrt{(0 : N/L)}$ and so

$$\sqrt{(0 : (N/L)/\mathfrak{a}(N/L))} = \sqrt{\mathfrak{a} + (0 : N/L)} = \sqrt{(0 : N/L)}.$$

Therefore $\operatorname{Supp}_R(\frac{N/L}{\mathfrak{a}(N/L)}) = \operatorname{Supp}_R N/L = \operatorname{Supp}_R R/\mathfrak{p}$ and so

$$\dim \frac{N/L}{\mathfrak{a}(N/L)} = \dim N/L = \dim N = \dim R/\mathfrak{p}.$$

It follows from [1, Theorem B] that

$$\operatorname{cd}_{\mathfrak{m}}(M, \frac{N/L}{\mathfrak{a}(N/L)}) = \operatorname{cd}_{\mathfrak{m}}(M, R/\mathfrak{p}) = d + n$$

and so we have $\mathrm{H}^{d+n}_{\mathfrak{m}}(M, \frac{N/L}{\mathfrak{a}(N/L)}) \neq 0$. Now by Corollary 2.9 we conclude that $\mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N/L) \neq 0$ and so $\operatorname{Att}_R(\mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N/L)) \neq \emptyset$. By lemma 2.10, $\operatorname{Att}_R(\mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N/L)) \subseteq \operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{m}}(M, N/L))$ and

$$\operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{m}}(M, N/L)) \subseteq \operatorname{Ass}_R(N/L) = \{\mathfrak{p}\}.$$

Thus $\mathfrak{p} \in \operatorname{Att} \mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N/L)$. On the other hand, the short exact sequence $0 \to L \to N \to N/L \to 0$ induces the exact sequence $\mathfrak{F}^{n}_{\mathfrak{a}}(N) \to \mathfrak{F}^{n}_{\mathfrak{a}}(N/L) \to 0$ and by using the functor $\operatorname{Ext}^d_R(M, -)$, we get the exact sequence

$$\operatorname{Ext}^d_R(M, \mathfrak{F}^{n}_{\mathfrak{a}}(N)) \to \operatorname{Ext}^d_R(M, \mathfrak{F}^{n}_{\mathfrak{a}}(N/L)) \to 0$$

and so by Theorem 2.12 we obtain the exact sequence

$$\mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N) \longrightarrow \mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N/L) \longrightarrow 0.$$

Thus $\operatorname{Att} \mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N/L) \subseteq \operatorname{Att} \mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N)$. Now, $\mathfrak{p} \in \operatorname{Att}_R \mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N/L)$ implies that $\mathfrak{p} \in \operatorname{Att}_R \mathfrak{F}^{d+n}_{\mathfrak{a}}(M, N)$, as required. □

3. 0-th Generalized Formal Local Cohomology

In this section, we obtain some results about the 0-th generalized formal local cohomology module.

Lemma 3.1. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$. Let M and N be two finitely generated R-modules and $i \in \mathbb{N}_0$. Then $\mathfrak{F}^i_{\mathfrak{a}}(M,N)$ has a natural structure as an $\widehat{R}$-module and*

$$\mathfrak{F}^i_{\mathfrak{a}}(M,N) \cong \mathfrak{F}^i_{\mathfrak{a}\widehat{R}}(\widehat{M},\widehat{N}).$$

Proof. For any integer k, $\mathrm{H}^i_{\mathfrak{m}}(M,N/\mathfrak{a}^kN)$ is an Artinian R-module, and so we have $\mathrm{H}^i_{\mathfrak{m}}(M,N/\mathfrak{a}^kN) \cong \mathrm{H}^i_{\mathfrak{m}}(M,N/\mathfrak{a}^kN) \otimes_R \widehat{R}$. But by [10, Lemma 2.1 (ii)], $\mathrm{H}^i_{\mathfrak{m}}(M,N/\mathfrak{a}^kN) \otimes_R \widehat{R} \cong \mathrm{H}^i_{\mathfrak{m}\widehat{R}}(\widehat{M},\widehat{N}/\widehat{\mathfrak{a}^k}\widehat{N})$. Thus for any integer k we have

$$\mathrm{H}^i_{\mathfrak{m}}(M,N/\mathfrak{a}^kN) \cong \mathrm{H}^i_{\mathfrak{m}\widehat{R}}(\widehat{M},\widehat{N}/\widehat{\mathfrak{a}^k}\widehat{N}).$$

Now we get the result, by passing to the inverse limit. □

Lemma 3.2. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$. Let M and N be two finitely generated R-modules. Then $\mathfrak{F}^0_{\mathfrak{a}}(M,N) \cong \mathrm{Hom}_R(M,\mathfrak{F}^0_{\mathfrak{a}}(N))$.*

Proof. By [16, Lemma 2.2]

$$\mathrm{H}^0_{\mathfrak{m}}(M,N/\mathfrak{a}^kN) \cong \mathrm{Hom}_R(M,\mathrm{H}^0_{\mathfrak{m}}(N/\mathfrak{a}^kN)),$$

for all $k \geq 0$. Therefore

$$\varprojlim_k \mathrm{H}^0_{\mathfrak{m}}(M,N/\mathfrak{a}^kN) \cong \varprojlim_k \mathrm{Hom}(M,H^0_{\mathfrak{m}}(N/\mathfrak{a}^kN)).$$

Thus

$$\mathfrak{F}^0_{\mathfrak{a}}(M,N) = \varprojlim_k \mathrm{H}^0_{\mathfrak{m}}(M,N/\mathfrak{a}^kN) \cong \mathrm{Hom}_R(M,\varprojlim_k \mathrm{H}^0_{\mathfrak{m}}(N/\mathfrak{a}^kN))$$

and so

$$\mathfrak{F}^0_{\mathfrak{a}}(M,N) \cong \mathrm{Hom}_R(M,\mathfrak{F}^0_{\mathfrak{a}}(N)).$$

□

Theorem 3.3. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$, and let M and N be two finitely generated R-modules. Then $\mathfrak{F}^0_{\mathfrak{a}}(M,N)$ is a finitely generated $\hat{R}$-module.*

Proof. By Lemma 3.1 we can assume that R is complete. By [2, Theorem 2.6] $\mathfrak{F}^0_{\mathfrak{a}}(N)$ is a finitely generated R-module. Thus it follows that $\mathrm{Hom}_R(M,\mathfrak{F}^0_{\mathfrak{a}}(N))$ is a finitely generated R-module. Now the result follows by Lemma 3.2. □

Let $0 = \bigcap_{\mathfrak{p}\in \mathrm{Ass}\, M} Z(\mathfrak{p})$, denotes a minimal primary decomposition of 0 in M and

$$T_{\mathfrak{a}}(M) = \{\mathfrak{p} \in \operatorname{Ass} M : \dim(R/(\mathfrak{a}, \mathfrak{p})) = 0\}.$$

Put $U_M(\mathfrak{a}) = \bigcap_{\mathfrak{p} \in \operatorname{Ass} M \setminus T_{\mathfrak{a}}(M)} Z(\mathfrak{p})$. In [15, Lemma 4.1], Schenzel has proved that $\mathfrak{F}^0_{\mathfrak{a}}(M) \cong U_{\widehat{M}}(\mathfrak{a}\widehat{R})$ and $\operatorname{Ass}_R U_M(\mathfrak{a}) = T_{\mathfrak{a}}(M)$. Now, in the following we give a similar result for $\mathfrak{F}^0_{\mathfrak{a}}(M, N)$.

Theorem 3.4. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$. Let M and N be two finitely generated R-modules. Then*

$$\mathfrak{F}^0_{\mathfrak{a}}(M, N) \cong \operatorname{Hom}_{\widehat{R}}(\widehat{M}, U_{\widehat{N}}(\mathfrak{a}\widehat{R})).$$

Proof. By Lemma 3.1, $\mathfrak{F}^0_{\mathfrak{a}}(M, N) \cong \mathfrak{F}^0_{\mathfrak{a}\widehat{R}}(\widehat{M}, \widehat{N})$ and by Lemma 3.2, $\mathfrak{F}^0_{\mathfrak{a}\widehat{R}}(\widehat{M}, \widehat{N}) \cong \operatorname{Hom}_{\widehat{R}}(\widehat{M}, \mathfrak{F}^0_{\mathfrak{a}\widehat{R}}(\widehat{N}))$. Since by [15, Lemma 4.1], $\mathfrak{F}^0_{\mathfrak{a}\widehat{R}}(\widehat{N}) \cong U_{\widehat{N}}(\mathfrak{a}\widehat{R})$ we conclude that $\mathfrak{F}^0_{\mathfrak{a}}(M, N) \cong \operatorname{Hom}_{\widehat{R}}(\widehat{M}, U_{\widehat{N}}(\mathfrak{a}\widehat{R}))$, as required. □

The following result is a generalization of [15, Corollary 4.2].

Corollary 3.5. *Let $\mathfrak{a}$ be an ideal of local ring $(R, \mathfrak{m})$. Let M and N be two finitely generated R-modules. Then*

$$\operatorname{Ass}_{\hat{R}}(\mathfrak{F}^0_{\mathfrak{a}}(M, N)) = \{\mathfrak{p} \in \operatorname{Ass}_{\hat{R}}(\operatorname{Hom}_{\hat{R}}(\hat{M}, \hat{N})) : \dim_{\hat{R}}(\hat{R}/(\mathfrak{a}\hat{R}, \mathfrak{p})) = 0\}.$$

and so $\mathfrak{F}^0_{\mathfrak{a}}(M, N) = 0$ if and only if $\dim \widehat{R}/(\mathfrak{a}\widehat{R} + \mathfrak{p}) > 0$ for all $\mathfrak{p} \in \operatorname{Ass}_{\widehat{R}} \operatorname{Hom}_{\widehat{R}}(\widehat{M}, \widehat{N})$.

Proof. By Theorem 3.4 and [15, Lemma 4.1] we have

$$\begin{aligned}
\operatorname{Ass}_{\hat{R}}(\mathfrak{F}^0_{\mathfrak{a}}(M, N)) &= \operatorname{Ass}_{\hat{R}}(\operatorname{Hom}_{\widehat{R}}(\widehat{M}, U_{\widehat{N}}(\mathfrak{a}\widehat{R}))) \\
&= \operatorname{Supp}_{\widehat{R}} \widehat{M} \cap \operatorname{Ass}_{\widehat{R}}(U_{\widehat{N}}(\mathfrak{a}\widehat{R})) \\
&= \operatorname{Supp}_{\widehat{R}} \widehat{M} \cap \{\mathfrak{p} \in \operatorname{Ass}_{\widehat{R}} \widehat{N} : \dim_{\hat{R}}(\widehat{R}/(\mathfrak{a}\widehat{R}, \mathfrak{p})) = 0\} \\
&= \{\mathfrak{p} \in \operatorname{Ass}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(\widehat{M}, \widehat{N})) : \dim_{\hat{R}}(\widehat{R}/(\mathfrak{a}\widehat{R}, \mathfrak{p})) = 0\},
\end{aligned}$$

which finishes the proof. □

Lemma 3.6. *Let $\mathfrak{a}$ be an ideal of local ring $(R, \mathfrak{m})$ and, let M, N be two finitely generated R-modules. Then, the two $\widehat{R}$-modules*

$$\operatorname{Tor}^R_j(R/\mathfrak{a}, \mathfrak{F}^i_{\mathfrak{a}}(M, N)) \text{ and } \operatorname{Tor}^R_j(\widehat{R}/\widehat{\mathfrak{a}}, \mathfrak{F}^i_{\widehat{\mathfrak{a}}}(\widehat{M}, \widehat{N}))$$

are isomorphic for all i and j.

Proof. Let $F_\bullet$ be a free resolution of the R-module $R/\mathfrak{a}$. Then, $F_\bullet \otimes_R \widehat{R}$ is a free resolution of the $\widehat{R}$-module $\widehat{R}/\mathfrak{a}\widehat{R}$. Thus, for any $\widehat{R}$-module X and any integer $i \geq 0$ we have

$$\operatorname{Tor}^R_j(R/\mathfrak{a}, X) \cong H_i(F_\bullet \otimes_R X) \cong H_i((F_\bullet \otimes_R \widehat{R}) \otimes_{\widehat{R}} X) \cong \operatorname{Tor}^{\widehat{R}}_j(\widehat{R}/\mathfrak{a}\widehat{R}, X).$$

Now, Lemma 3.1 completes the proof. □

Theorem 3.7. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M be a finitely generated R-module. Then $\operatorname{Tor}_j^R(R/\mathfrak{a}, \mathfrak{F}_{\mathfrak{a}}^0(M,N))$ is Artinian for all j.*

Proof. By Lemma 3.6 we may and do assume that R is complete. By Theorem 3.3, $\mathfrak{F}_{\mathfrak{a}}^0(M,N)$ is finitely generated and by Corollary 3.5,

$$\operatorname{Ass}_R(\mathfrak{F}_{\mathfrak{a}}^0(M,N)) = \{\mathfrak{p} \in \operatorname{Ass}_R(\operatorname{Hom}_R(M,N)) : \dim R/(\mathfrak{a}R, \mathfrak{p}) = 0\}.$$

Thus, for any integer j, we have

$$\operatorname{Supp}_R(\operatorname{Tor}_j^R(R/\mathfrak{a}, \mathfrak{F}_{\mathfrak{a}}^0(M,N))) \subseteq \operatorname{V}(\mathfrak{a}) \cap \operatorname{Supp}_R(\mathfrak{F}_{\mathfrak{a}}^0(M,N)) \subseteq \{\mathfrak{m}\}.$$

Thus, it follows that $\operatorname{Tor}_j^R(R/\mathfrak{a}, \mathfrak{F}_{\mathfrak{a}}^0(M,N))$ has finite length, as required. $\square$

4. A Duality Theorem

In this section, we give a Duality Theorem for generalized formal local cohomology modules over a Cohen-Macaulay local ring $(R, \mathfrak{m})$ and by using it we obtain some results about lower and upper bounds for non-vanishing of generalized formal local cohomology modules. We define the formal grade of M and N with respect to $\mathfrak{a}$ by

$$\operatorname{fgrade}(\mathfrak{a}, M, N) := \inf\{i : \mathfrak{F}_{\mathfrak{a}}^i(M,N) \neq 0\},$$

and the formal dimension of M and N with respect to $\mathfrak{a}$ by

$$\operatorname{fd}(\mathfrak{a}, M, N) := \sup\{i : \mathfrak{F}_{\mathfrak{a}}^i(M,N) \neq 0\}.$$

At first, recall the following Duality Theorem.

Theorem 4.1. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring, and let M and N be two finitely generated R-modules such that $\operatorname{pd}_R M < \infty$. Let $w_{\widehat{R}}$ denotes the canonical module of $\widehat{R}$. Then for all $i \geq 0$,*

$$\operatorname{H}_{\mathfrak{m}}^i(M,N) \cong \operatorname{Hom}_{\widehat{R}}(\operatorname{Ext}_{\widehat{R}}^{d-i}(\widehat{N}, \widehat{M} \otimes_{\widehat{R}} w_{\widehat{R}}), E_R(R/\mathfrak{m})).$$

Proof. Since $\operatorname{H}_{\mathfrak{m}}^i(M,N) \cong \operatorname{H}_{\widehat{\mathfrak{m}}}^i(\widehat{M}, \widehat{N})$, it follows from [16, Theorem 3.5]. $\square$

Theorem 4.2. *Let $\mathfrak{a}$ be an ideal of a d-dimensional Cohen-Macaulay local ring $(R, \mathfrak{m})$, and let M and N be two finitely generated R-modules such that $\operatorname{pd}_R M < \infty$. Then for all $i \geq 0$,*

$$\mathfrak{F}_{\mathfrak{a}}^i(M,N) \cong \operatorname{Hom}_{\widehat{R}}(\operatorname{H}_{\mathfrak{a}\widehat{R}}^{d-i}(\widehat{N}, \widehat{M} \otimes_{\widehat{R}} w_{\widehat{R}}), E_R(R/\mathfrak{m})).$$

Proof. Let $i \geq 0$ be an integer. By using Theorem 4.1 we have

$$\begin{aligned}\mathfrak{F}^i_{\mathfrak{a}}(M,N) &= \varprojlim_k \mathrm{H}^i_{\mathfrak{m}}(M, N/\mathfrak{a}^k N)\\ &\cong \mathrm{Hom}_{\widehat{R}}(\varinjlim_k \mathrm{Ext}^{d-i}_{\widehat{R}}(\widehat{N}/\widehat{\mathfrak{a}^k}\widehat{N}, \widehat{M}\otimes_{\widehat{R}} w_{\widehat{R}}), E_R(R/\mathfrak{m}))\\ &\cong \mathrm{Hom}_{\widehat{R}}(\mathrm{H}^{d-i}_{\mathfrak{a}\widehat{R}}(\widehat{N}, \widehat{M}\otimes_{\widehat{R}} w_{\widehat{R}}), E_R(R/\mathfrak{m})).\end{aligned}$$

□

Theorem 4.3. *Let $\mathfrak{a}$ be an ideal of a d-dimensional Cohen-Macaulay local ring $(R,\mathfrak{m})$, and let M and N be two finitely generated R-modules such that $\mathrm{pd}_R M < \infty$. Then*

i) $\mathrm{fgrade}(\mathfrak{a}, M, N) = \dim R - \mathrm{cd}_{\mathfrak{a}\widehat{R}}(\widehat{N}, \widehat{M}\otimes_{\widehat{R}} w_{\widehat{R}})$.

ii) $\mathrm{fd}(\mathfrak{a}, M, N) = \dim R - \mathrm{grade}_{\widehat{M}\otimes_{\widehat{R}} w_{\widehat{R}}}(\widehat{N}/\widehat{\mathfrak{a}}\widehat{N})$.

Proof. By Theorem 4.2 we have

i)

$$\begin{aligned}\mathrm{fgrade}(\mathfrak{a}, M, N) &= \inf\{i : \mathfrak{F}^i_{\mathfrak{a}}(M,N) \neq 0\}\\ &= \inf\{i : \mathrm{Hom}_{\widehat{R}}(\mathrm{H}^{d-i}_{\mathfrak{a}\widehat{R}}(\widehat{N}, \widehat{M}\otimes_{\widehat{R}} w_{\widehat{R}}), E_R(R/\mathfrak{m})) \neq 0\}\\ &= d - \sup\{i : \mathrm{H}^i_{\mathfrak{a}\widehat{R}}(\widehat{N}, \widehat{M}\otimes_{\widehat{R}} w_{\widehat{R}}) \neq 0\}\\ &= d - \mathrm{cd}_{\mathfrak{a}\widehat{R}}(\widehat{N}, \widehat{M}\otimes_{\widehat{R}} w_{\widehat{R}}).\end{aligned}$$

ii)

$$\begin{aligned}\mathrm{fd}(\mathfrak{a}, M, N) &= \sup\{i : \mathfrak{F}^i_{\mathfrak{a}}(M,N) \neq 0\}\\ &= \sup\{i : \mathrm{Hom}_{\widehat{R}}(\mathrm{H}^{d-i}_{\mathfrak{a}\widehat{R}}(\widehat{N}, \widehat{M}\otimes_{\widehat{R}} w_{\widehat{R}}), E_R(R/\mathfrak{m})) \neq 0\}\\ &= d - \inf\{i : \mathrm{H}^i_{\mathfrak{a}\widehat{R}}(\widehat{N}, \widehat{M}\otimes_{\widehat{R}} w_{\widehat{R}}) \neq 0\}\\ &= d - \mathrm{grade}_{\widehat{M}\otimes_{\widehat{R}} w_{\widehat{R}}}(\widehat{N}/\widehat{\mathfrak{a}}\widehat{N}).\end{aligned}$$

The last equality follows by [4, Proposition 5.5]. □

Corollary 4.4. *Let $\mathfrak{a}$ be an ideal of a d-dimensional Cohen-Macaulay local ring $(R,\mathfrak{m})$, and let M and N be two finitely generated R-modules such that $\mathrm{pd}_R M < \infty$ and $\mathrm{pd}_R N < \infty$. Then*

$$\mathrm{fgrade}(\mathfrak{a}, M, N) + \mathrm{cd}_{\mathfrak{a}}(N, M) = \dim R$$

Proof. By [7, Theorem 3.3.5 b] $\mathrm{Supp}_{\widehat{R}}\, w_{\widehat{R}} = \mathrm{Spec}\, \widehat{R}$. Thus

$$\mathrm{Supp}_{\widehat{R}}(\widehat{M}\otimes_{\widehat{R}} w_{\widehat{R}}) = \mathrm{Supp}_{\widehat{R}}\, \widehat{M}.$$

On the other hand, $\operatorname{pd}_R N < \infty$ implies that $\operatorname{pd}_{\widehat{R}} \widehat{N} < \infty$ and so by [1, Theorem B] $\operatorname{cd}_{\mathfrak{a}\widehat{R}}(\widehat{N}, \widehat{M} \otimes_{\widehat{R}} w_{\widehat{R}}) = \operatorname{cd}_{\mathfrak{a}\widehat{R}}(\widehat{N}, \widehat{M})$. But by [10, Lemma 2.1 (ii)] it follows that $\operatorname{cd}_{\mathfrak{a}\widehat{R}}(\widehat{N}, \widehat{M}) = \operatorname{cd}_{\mathfrak{a}}(N, M)$. Now the result follows by Theorem 4.3(i). □

Corollary 4.5. *Let $\mathfrak{a}$ be an ideal of a d-dimensional Cohen-Macaulay local ring $(R, \mathfrak{m})$, and let M and N be two finitely generated R-modules such that $\operatorname{pd}_R M < \infty$ and $\operatorname{pd}_R N < \infty$. Then*

$$\operatorname{depth} N \leq \operatorname{fgrade}(\mathfrak{a}, M, N) + \dim M.$$

Proof. Since R is Cohen-Macaulay, by the Auslander-Buchsbaum formula we have $\operatorname{pd}_R N + \operatorname{depth} N = \dim R$. Also, by Corollary 4.4 we have $\operatorname{fgrade}(\mathfrak{a}, M, N) + \operatorname{cd}_{\mathfrak{a}}(N, M) = \dim R$. Thus

$$\operatorname{pd}_R N + \operatorname{depth} N + \dim M = \operatorname{fgrade}(\mathfrak{a}, M, N) + \operatorname{cd}_{\mathfrak{a}}(N, M) + \dim M.$$

Since $\operatorname{cd}_{\mathfrak{a}}(N, M) \leq \operatorname{pd}_R N + \dim M$ by Theorem 2.2(iii), the result follows from the above equality. □

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GENERALIZED FORMAL LOCAL COHOMOLOGY MODULES

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مدول‌های کوهمولوژی موضعی صوری تعمیم یافته

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فرض کنید $\mathfrak{a}$ یک ایده‌آل از حلقه نوتری موضعی $(R, \mathfrak{m})$ و M و N دو R–مدول متناهی مولد باشند. در این مقاله ابتدا مفهوم مدول‌های کوهمولوژی موضعی صوری تعمیم یافته‌ی دو مدول نسبت به یک ایده‌آل را تعریف می‌کنیم. در واقع i–امین مدول کوهمولوژی موضعی صوری تعمیم یافته‌ی دو مدول M و N نسبت به یک ایده‌آل $\mathfrak{a}$ را به صورت زیر تعریف می‌کنیم:

$$\mathfrak{F}^i_{\mathfrak{a}}(M,N) := \varprojlim_n \mathrm{H}^i_{\mathfrak{m}}(M, N/\mathfrak{a}^n N).$$

سپس ضمن بررسی ساختار این مدول‌ها، چندین نتیجه درباره صفر بودن، متناهی مولد بودن و آرتینی بودن این مدول‌ها به‌دست می‌آوریم. همچنین نتایجی درباره اولین و آخرین مدول ناصفر کوهمولوژی موضعی صوری تعمیم یافته به‌دست می‌آوریم.

کلمات کلیدی: کوهمولوژی موضعی صوری، کوهمولوژی موضعی، آرتینی.