Journal of Algebraic Systems
Vol. 12, No. 2, (2025), pp 237-256

A GRAPH ASSOCIATED TO ESPECIAL ESSENTIALITY OF SUBMODULES

M. EBRAHIMI DORCHEH AND S. BAGHERI*

ABSTRACT. Let R be an associative ring with identity. In this paper we associate to every R-module M a simple graph $\Gamma_e(M)$ which we call it the *essentiality graph* of M. The vertices of $\Gamma_e(M)$ are nonzero submodules of M and two distinct vertices K and L are considered to be adjacent if and only if $K \cap L$ is an essential submodule of $K+L$. We investigate the relationship between some module theoretic properties, such as minimality and closedness of submodules of M with some graph theoretic properties of $\Gamma_e(M)$. In general, this graph is not connected. We study some special cases in which $\Gamma_e(M)$ is complete or a union of complete connected components and give some examples illustrating each specific case.

1. Introduction

Let $\mathfrak{F}$ be a family of nonempty sets. The *intersection graph* of $\mathfrak{F}$ is the simple graph whose vertex set is $\mathfrak{F}$ and two distinct sets S and T in $\mathfrak{F}$ are considered to be adjacent, if $S \cap T \neq \phi$. Apparently, the study of this graph goes back to 1945. In Marczewski [9] it has been proven that every simple graph can be realized as an intersection graph (see also [11, Theorem 1]). Two decades later, the intersection graph has been defined for sets with algebraic structures, namely in [3] for semigroups and in [5] for subgroups of a finite group.

The intersection graph of ideals of a ring R was first defined in [4] as a simple graph whose vertex set was in a one to one correspondence with

DOI: 10.22044/JAS.2023.12356.1662.
MSC(2020): Primary: 16D10; Secondary: 13A70, 16D80.
Keywords: Essential submodule; Closed submodule; UC-module; Clique number; Girth.
Received: 23 October 2022, Accepted: 27 January 2023.
*Corresponding author.

the set of all nonzero ideals of R and two distinct nonzero ideals of R are considered to be adjacent if their intersection is nonzero. This kind of intersection graph has been investigated by many authors(see for example [7, 1, 12]). Similar definitions have been given for intersection graphs of subspaces of a vector space and that of submodules of a module (see for example [8], [7] and [2]).

Essentiality of submodules of a module is an important concept in ring and module theory in which there are many interesting results and questions. Therefore it makes sense to assign a graph whose vertex set is all nonzero submodules of a module and adjacency is some how related to the essentiality of submodules.

In this paper, we consider a simple graph which is defined in terms of essentiality and it can be considered as a subgraph of some kinds of the intersection graph of submodules of a module M. We call this new graph the *essentiality graph* of M and is denoted by $\Gamma_e(M)$. Its vertex set is $L^*(M)$ containing all nonzero submodules of M and two distinct submodules K and L are considered as adjacent vertices in $\Gamma_e(M)$, if $K \cap L$ is an essential submodule in $K + L$. Our main purpose in this paper is to study the relations between module theoretic properties of M and graph theoretic properties of $\Gamma_e(M)$. With the given definition, we can see that the set of all nonzero submodules of a module M (the vertex set of $\Gamma_e(M)$) can be partitioned into some connected components and the diameter of each component of this graph is at most 2. Moreover, if there is a cycle in the graph $\Gamma_e(M)$, then its girth is 3. Also, we show that the number of connected components of $\Gamma_e(M)$ is some how related to the number of minimal submodules and the number of closed submodules of M. Moreover, we investigate the clique number and the girth of this graph and determine some properties of the essentiality graph associated to the special $\mathbb{Z}$-module $\mathbb{Z}_m$.

Section 2 is devoted to an investigation of some fundamental properties of $\Gamma_e(M)$ such as connectivity and the diameter of connected components. We will show that the number of minimal submodules and the number of closed submodules of an R-module M are upper and lower bounds for the number of connected components of $\Gamma_e(M)$ respectively (see Lemma 2.8). Furthermore, as a main result in this section we give several equivalent conditions under which the graph $\Gamma_e(M)$ is complete (see Theorem 2.9). As a special case, we show that for a module M with nonzero socle, this is precisely the case when M is cocyclic.

In Section 3, we will study the clique number and the girth of $\Gamma_e(M)$ and specify the girth of the graph completely. Among other things, we characterize nonsemisimple modules M for which the girth of $\Gamma_e(M)$ is infinite.

In Section 4, we investigate the cases in which every connected component of $\Gamma_e(M)$ is complete. We also examine a special case of these modules, namely the $\mathbb{Z}$-modules $\mathbb{Z}_m$ and characterize the graph associated to these modules. Moreover, we find the number of connected components of $\Gamma_e(\mathbb{Z}_m)$ in terms of the number of prime divisors of m. As another main result, we show that the graph $\Gamma_e(M)$ is a union of complete connected components if and only if M is a UC-module. It is well known that every cocyclic module contains a unique simple submodule, but the converse is not true. We give an example of a module containing a unique simple submodule which is not cocyclic and its essentiality graph is not connected and even it has an incomplete connected component (see Example 4.7).

1.1. **Some preliminaries from module theory.** We recall that a submodule K of an R-module M is called *essential* (*large*) in M, if for every nonzero submodule $L \subseteq M$, we have $K \cap L \neq 0$. In this case M is called an *essential extension* of K and it is denoted by $K \leqslant_e M$. The set of all essential submodules of M is denoted by $\varepsilon(M)$. If M is an essential submodule of an injective module E, then E is called an *injective hull* of M and is usually denoted by $E(M)$. A submodule K is called a *closed* submodule in M if K has no proper essential extensions in M, i.e. whenever L is a submodule of M such that K is essential in L, then $K = L$ and to show this, we write $K \leqslant_c M$. An R-module M is said to be a *uniform* module if $M \neq 0$ and every nonzero submodule of M is essential in M. A submodule K is said to be an *irreducible* submodule of M if $K \neq M$, and there do not exist submodules K_1 and K_2 of M such that $K \subsetneqq K_1, K \subsetneqq K_2$ and $K = K_1 \cap K_2$. An R-module M is called *self-injective* if $Mf \subseteq M$ for every endomorphism f of $E(M)$. M is called π-*injective* if $Mf \subseteq M$ for every idempotent endomorphism f of $E(M)$. M is called *direct injective*, if for every direct summand X of M, every monomorphism $X \to M$ splits. M is said to be *continuous* if it is π-injective and direct injective. An R-module M is nonsingular module if

$$Z(M) = \{m \in M | ann(m) \leqslant_e R\} = 0$$

and M is called a *UC-module* if every submodule of M has a unique closure (maximal essential extention) in M.

Let K be a submodule of the R-module M. A submodule $K' \subseteq M$ is called an (intersection) *complement* of K in M if it is maximal in the set of submodules $L \subseteq M$ with $K \cap L = 0$.

A submodule K of M is called a *complement* (in M) if there exists a submodule N of M such that K is a complement of N in M. It is well known that K is a complement in M if and only if it is closed in M. If K'' is a complement of K' in M with $K \subseteq K''$, then K'' is called a *double complement* of K in M and K' is a complement of K'' in M. In this case, K'' is a maximal essential extension of K in M and K'' is called a *closure* of K in M. In case M is self-injective, we have $M = K' \oplus K''$ (see [17, 17.7]). An R-module M is called cocyclic if there is an element $m_0 \in M$ with the property: every R-homomorphism $g : M \to L$ with $m_0 \notin Ke(g)$ is a monomorphism. This is also equivalent to say: M is an essential extension of a simple module (see [17, 14.8]).

1.2. **Some definitions and terminologies from graph theory.** Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For two arbitrary distinct vertices $x, y \in V(G)$ the length of the shortest path from x to y is denoted by $d(x, y)$ and we write $d(x, y) = \infty$ if there is no such paths. A graph G is called *connected* if $d(x, y)$ is finite for all pairs of vertices in $V(G)$. The *diameter* of G is:

$$diam(G) = Sup\{d(x, y) | x \neq y \text{ are vertices in } G\}.$$

A *clique* in G is a complete subgraph of G and the maximum cardinality of cliques in G is called the *clique number* of G. The *girth* of G, denoted by girth(G), is the least length of cycles in G (if there is one). If there is no cycles in G, we write girth$(G) = \infty$.

The reader is refered to [17], [6] and [13] for undefined concepts in rings and module theory and to [15] and [16] for unmentioned things about graph theory.

2. Some properties of $\Gamma_e(M)$

We assign to every module M a simple graph $\Gamma_e(M)$, called *essentiality graph* of M, whose vertex set is $L^*(M)$ and two distinct submodules $K, L \in L^*(M)$ are defined to be adjacent if $K \cap L \leqslant_e K + L$. In this section, we study the fundamental features of this graph in terms of some module theoretic properties of the module M. At first, we give three equivalent conditions for adjacency in $\Gamma_e(M)$.

Lemma 2.1. *If L and N are nonzero submodules of an R-module M, then the following statements are equivalent:*

(1) *L and N are adjacent in $\Gamma_e(M)$;*
(2) *There are injective hulls such as $E(N)$ and $E(L)$ of N and L such that $E(N) = E(L)$;*

(3) *L and N have a common essential extension.*

Proof. (1) $\Rightarrow$ (2) If L and N are adjacent in $\Gamma_e(M)$, then $L \cap N$ is an essential submodule of $L+N$. Therefore we have $L \leqslant_e L+N$, $N \leqslant_e L+N$ and for an injective hull $E(L+N)$ of $L+N$ there are injective hulls $E(L)$ and $E(N)$ of L and N (respectively) such that $E(L)=E(L+N)=E(N)$.

(2) $\Rightarrow$ (3) If $E(L) = E(N)$, then $E(L)$ is a common essential extension of L and N.

(3) $\Rightarrow$ (1) If L and N are essential in M', then $N \cap L \leqslant_e M'$ and $N \cap L \leqslant N+L \leqslant M'$. Therefore $N \cap L \leqslant_e N+L$, and thus L and N are adjacent in $\Gamma_e(M)$. $\square$

Theorem 2.2. *In $\Gamma_e(M)$ the diameter of a connected component is at most 2.*

Proof. Let N and L be two arbitrary nonadjacent vertices in a connected component of $\Gamma_e(M)$ and consider the path:

$$N - L_1 - L_2 - \cdots - L_n - L$$

Then $N \cap L_1 \leqslant_e L_1$ and taking intersection with L_2, we obtain

$$N \cap L_1 \cap L_2 \leqslant_e L_1 \cap L_2 \leqslant_e L_2.$$

Now by induction we have $N \cap L_1 \cap \ldots \cap L_n \cap L \leqslant_e L$ and using a similar argument, we have $L \cap L_1 \cap \ldots \cap L_n \cap N \leqslant_e N$. Therefore $N \cap L \leqslant_e N$ and $N \cap L \leqslant_e L$ which yields the path $N - N \cap L - L$ and thus $d(N, L) \leqslant 2$. $\square$

Using the proof of Theorem 2.2, we obtain the following:

Corollary 2.3. *Let the submodules A and B be nonadjacent vertices in a connected component of $\Gamma_e(M)$. Then $A \cap B \neq 0$ and we have the path $A - A \cap B - B$ in this graph.*

Lemma 2.4. *Let A and B be distinct vertices in a connected component $\mathcal{C}$ of $\Gamma_e(M)$. Then A and B are adjacent in $\mathcal{C}$ if and only if $A+B \in \mathcal{C}$.*

Proof. If A and B are adjacent in $\mathcal{C}$, then $A \cap B \leqslant_e A, B \leqslant_e A+B$ and so $A+B \in \mathcal{C}$.

Conversely, if $A+B \in \mathcal{C}$, then by Corollary 2.3, we have the path $A+B-(A+B)\cap A - A$. But $(A+B) \cap A = A$ and we can conclude that $A \leqslant_e A+B$. Similarly, $B \leqslant_e A+B$ and $A+B$ is a common essential extension of A and B. $\square$

We sometimes use the following result of [19] which describes the submodules of the direct sum of two modules. It is a simple modification of a result for groups known as Goursat's lemma.

Lemma 2.5. [19, Lemma 4.1] *Let U and W be R-modules. Then:*

(1) *There is a bijective map Ψ from the set of all R-submodules of $U \times W$ to the set of all quintuples $(U_1, U_2, \theta, W_1, W_2)$ where $U_2 \leqslant U_1 \leqslant U$ and $W_2 \leqslant W_1 \leqslant W$ and $\theta : U_1/U_2 \to W_1/W_2$ is an isomorphism of R-modules.*

(2) *Ψ sends an R-submodule M of $U \times W$ to the quintuple*

$$(p_1(M), k_1(M), \theta_M, p_2(M), k_2(M)),$$

where

$$\begin{aligned} p_1(M) &= \{u \in U : \exists w \in W, (u, w) \in M\}, \\ k_1(M) &= \{u \in U : (u, 0) \in M\}, \\ p_2(M) &= \{w \in W : \exists u \in U, (u, w) \in M\}, \\ k_2(M) &= \{w \in W : (0, w) \in M\}, \end{aligned}$$

and, for any $(u, w) \in p_1(M) \times p_2(M)$, the isomorphism θ_M sends $u + k_1(M)$ to $w + k_2(M)$ if and only if $(u, w) \in M$. Furthermore, the three R-modules $p_1(M)/k_1(M), p_2(M)/k_2(M)$ and $M/(k_1(M) \times k_2(M))$ are isomorphic.

(3) *For any quintuple $(U_1, U_2, \theta, W_1, W_2)$ where $U_2 \leqslant U_1 \leqslant U$ and $W_2 \leqslant W_1 \leqslant W$ and $\theta : U_1/U_2 \to W_1/W_2$ is an isomorphism of R-modules, the inverse of Ψ sends $(U_1, U_2, \theta, W_1, W_2)$ to the module $\{(u, w) \in U_1 \times W_1 : \theta(u + U_2) = w + W_2\}$.*

Example 2.6. Applying Lemma 2.5, we can determine the submodules of the $\mathbb{Z}$-module $M = \mathbb{Z}_2 \times \mathbb{Z}_4$. One can see that the graph associated to this module can be presented as four components in which one of the components is incomplete and its diameter is 2.

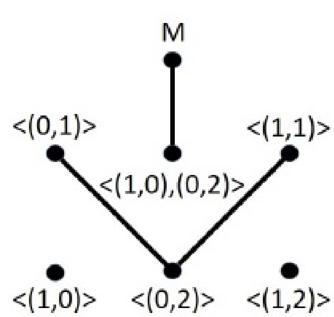

FIGURE 1. $\Gamma_e({}_{\mathbb{Z}}(\mathbb{Z}_2 \times \mathbb{Z}_4))$

For a given graph G, the set of all connected components of G is denoted by $\mathcal{C}(G)$. Also, for a vertex v in G, we denote by $\mathcal{C}_v^G$ the connected component of the graph G containing v.

Lemma 2.7. *The connected component of $\Gamma_e(M)$ containing M is complete.*

Proof. If M is a semisimple module, then $|\mathcal{C}_M^{\Gamma_e(M)}| = 1$ and the assertion trivially holds. Let M be a nonsemisimple module and N be a nontrivial submodule of M in $\mathcal{C}_M^{\Gamma_e(M)}$ which is not adjacent to M. Then using Theorem 2.2, we have a path $N - L - M$ in the connected component of $\Gamma_e(M)$ containing M, where L is a nonzero (essential) submodule of M. Therefore, $N \cap L \leqslant_e N \leqslant_e N + L \leqslant_e M$ implies the adjacence of N and M which is a contradiction. Hence M is a common essential extension of every pair of vertices in $\mathcal{C}_M^{\Gamma_e(M)}$ and thus $\mathcal{C}_M^{\Gamma_e(M)}$ is complete. □

Lemma 2.8. *Every connected component of $\Gamma_e(M)$ contains at most one minimal submodule and at least one closed submodule of M.*

Proof. If N and L are two minimal submodules in the connected component $\mathcal{C}$ of $\Gamma_e(M)$ and $N - N \cap L - L$ is a path between them, then by minimality of N and L, we have $N = N \cap L = L$. For the last part of the assertion, we note that if $A \in \mathcal{C}$, then A'' is a closed submodule and $A \leqslant_e A''$, by [17, 17.7]. Therefore, $A'' \in \mathcal{C}$. □

Let M' be an essential extension of M. The following theorem shows that the cardinal number of the set containing all connected components in $\Gamma_e(M)$ is equal to the cardinal number of the set of all connected components in $\Gamma_e(M')$.

Theorem 2.9. *If M is a nonzero submodule of M', then $M \leqslant_e M'$ if and only if there exist a one to one correspondence*

$$\phi : \mathcal{C}(\Gamma_e(M)) \longrightarrow \mathcal{C}(\Gamma_e(M'))$$

such that every $\mathcal{C} \in \mathcal{C}(\Gamma_e(M))$ is the induced subgraph of $\phi(\mathcal{C})$.

Proof. If $M \leqslant M'$ and $L, N \leqslant M$, then L is adjacent to N in $\Gamma_e(M)$ if and only if $L \cap N \leqslant_e L + N \leqslant M \leqslant M'$. Therefore $\Gamma_e(M)$ is an induced subgraph of $\Gamma_e(M')$.

If $M \leqslant_e M'$ and $0 \neq K \leqslant M'$, then $K \cap M \leqslant_e K$ and so K is adjacent to $K \cap M$, i.e $K \in \mathcal{C}_{K\cap M}^{\Gamma_e(M')}$. Let $0 \neq L \leqslant M$ and L be adjacent to K in $\Gamma_e(M')$. Then $L \cap K \leqslant_e L + K$ and $L \cap K \leqslant M \cap K$ imply that $L \cap K \leqslant_e K$, $M \cap K \leqslant_e K$ and $L \cap K \leqslant M \cap K$. Therefore, we have the path $L - L \cap K - K \cap M$ in $\Gamma_e(M)$ and thus $L \in \mathcal{C}_{K\cap M}^{\Gamma_e(M)}$. Hence

K is added at only one component of $\mathcal{C}(\Gamma_e(M))$ and $\mathcal{C}_{K\cap M}^{\Gamma_e(M)}$ is induced subgraph of $\mathcal{C}_{K\cap M}^{\Gamma_e(M')}$. We define

$$\phi : \mathcal{C}(\Gamma_e(M)) \longrightarrow \mathcal{C}(\Gamma_e(M'))$$

with $\phi(\mathcal{C}_N^{\Gamma_e(M)}) = \mathcal{C}_N^{\Gamma_e(M')}$ for every $0 \neq N \leqslant M$. If $\mathcal{C} \in \mathcal{C}(\Gamma_e(M'))$ and K is a vertex in $\mathcal{C}$, then $\mathcal{C} = \phi(\mathcal{C}_{K\cap M}^{\Gamma_e(M)})$ and ϕ is surjective. If $\phi(\mathcal{C}_{N_1}^{\Gamma_e(M)}) = \phi(\mathcal{C}_{N_2}^{\Gamma_e(M)})$ for submodules N_1 and N_2 of M, then

$$N_1 - N_1 \cap N_2 - N_2$$

is a path in $\Gamma_e(M')$ and thus in $\Gamma_e(M)$. Therefore $\mathcal{C}_{N_1}^{\Gamma_e(M)} = \mathcal{C}_{N_2}^{\Gamma_e(M)}$ and ϕ is injective.

On the other hand, let K be a nonzero submodule of M' such that $K \in \phi(\mathcal{C}) \in \mathcal{C}(\Gamma_e(M'))$, where $\mathcal{C}$ is a (nonempty) connected component of $\Gamma_e(M)$. If $L \in \mathcal{C}$ for $L \leqslant M$, then $L \in \phi(\mathcal{C})$ and thanks to Corollary 2.3, $K \cap L \neq 0$ and thus $K \cap M \neq 0$. Therefore, $M \leqslant_e M'$. □

Corollary 2.10. *The cardinal number of the connected components of $\Gamma_e(M)$ is equal to the cardinal number of the connected components of $\Gamma_e(E(M))$.*

Corollary 2.11. *If $\mathcal{C}$ is a connected component of diameter 2 in $\Gamma_e(M)$ and $M \leqslant_e M'$, then the extension of $\mathcal{C}$ in $\Gamma_e(M')$ is also of diameter 2.*

The following theorem gives us the conditions which are equivalent to the completeness of $\Gamma_e(M)$.

Theorem 2.12. *If M is an R-module, the following statements are equivalent:*

(1) *$\Gamma_e(M)$ is complete;*
(2) *$\Gamma_e(M)$ is connected;*
(3) *M is a uniform module;*
(4) *$E(M)$ is indecomposable;*
(5) *$\Gamma_e(E(M))$ is complete;*
(6) *(0) is an irreducible submodule of M;*
(7) *There is an irreducible left ideal I of R such that*

$$E(M) \simeq E(R/I).$$

If M is π-injective module, then $(1)-(7)$ are equivalent to:

(8) *M is indecomposable.*

If M is continuous, then $(1)-(8)$ are equivalent to:

(9) *$End_R(M)$ is a local ring.*

Proof. (1) ⇒ (2) and (3) ⇔ (6) are obvious.

(2) ⇒ (1) is clear by lemma 2.7.

(1) $\Leftrightarrow$ (3) If $\Gamma_e(M)$ is complete, then M is a common essential extension of any pair of nonzero submodules.

(3) $\Rightarrow$ (4) Let A and B are two nonzero submodules of $E(M)$, then $A \cap M \neq 0 \neq B \cap M$ and since $A \cap M$ and $B \cap M$ are nonzero submodules of M, $0 \neq (A \cap M) \cap (B \cap M) \subseteq A \cap B$. Therefore, every nonzero submodule of $E(M)$ is essential and thus $E(M)$ is indecomposable.

(4) $\Rightarrow$ (3) For every nonzero submodule N of M, $E(N)$ is an injective submodule of $E(M)$ and thus it is a direct summand of $E(M)$. Since $E(M)$ is indecomposable, we have $E(N) = E(M)$. Hence $N \leqslant_e E(M)$ and thus $N \leqslant_e M$.

(3) $\Rightarrow$ (5) If K is a nonzero submodule of $E(M)$, then $K \cap M$ is a nonzero submodule of M and therefore it is essential in M and since $M \leqslant_e E(M)$, we have $K \cap M \leqslant_e E(M)$ and thus $K \leqslant_e E(M)$.

(5) $\Rightarrow$ (1) M is a submodule of $E(M)$ and thus $\Gamma_e(M)$ is an induced subgraph of $\Gamma_e(E(M))$.

(4) $\Leftrightarrow$ (7) see [13, Page 49].

(3) $\Leftrightarrow$ (8) see [6, Page 14].

(3) $\Leftrightarrow$ (9) see [6, Page 15]. $\square$

Example 2.13. As it can be seen in [17, 14.8], an R-module N is cocyclic if and only if it is an essential extension of a simple R-module. Thus for every cocyclic R-module N, the essentiality graph $\Gamma_e(N)$ is complete. In particular, for every simple R-module S, the graph $\Gamma_e(E(S))$ associated to the injective hull $E(S)$ of S is complete.

Corollary 2.14. *Let M be an R-module with $Soc(M) \neq 0$. Then $\Gamma_e(M)$ is complete if and only if M is cocyclic.*

Proof. If M is cocyclic, then $\Gamma_e(M)$ is complete by Example 2.13.

Conversely, let S be a simple submodule of M and $\Gamma_e(M)$ be complete. Then M is uniform by Theorem 2.12. Hence for every nonzero submodule L of M, the fact $S \cap L \neq 0$ implies that $S \leqslant_e M$. This means that M is an essential extension of the simple submodule S; i.e. M is cocyclic. $\square$

The following easy example shows that $Soc(M) \neq 0$ is not a necessary condition for $\Gamma_e(M)$ to be complete.

Example 2.15. $\Gamma_e({}_{\mathbb{Z}}\mathbb{Z})$ is complete but ${}_{\mathbb{Z}}\mathbb{Z}$ does not have minimal submodules.

Example 2.16. Let K be the field of fractions of a commutative domain D. Then ${}_DK = E({}_DD)$ and thus $\Gamma_e({}_DK)$ and $\Gamma_e({}_DD)$ are complete graphs because (0) is an irreducible submodule of ${}_DD$.

In [10] Matlis showed that if R is a commutative noetherian ring, then there is a one-to-one correspondence between the prime ideals of R and the indecomposable injective R-modules given by $P \mapsto E(\frac{R}{P})$, where P is a prime ideal of R. Accordingly, the only $\mathbb{Z}$-modules M with complete $\Gamma_e(M)$ are the modules whose injective hulls are isomorphic to $\mathbb{Q}$ or $\mathbb{Z}_{p^\infty}$.

3. The clique number and the girth of $\Gamma_e(M)$

In this section, we investigate the clique number and the girth of the essentiality graph $\Gamma_e(M)$, associated to an R-module M.

In the following lemma, for a submodule K of an R-module M, the set $\varepsilon(K)$ of all essential submodules of K has been determined using that of M.

Lemma 3.1. *For a submodule K of M, $\varepsilon(K) = \{K \cap N : N \in \varepsilon(M)\}$.*

Proof. If $N \leqslant_e M$, then $N \cap K \leqslant_e K$ and thus $N \cap K \in \varepsilon(K)$.

Let $B \leqslant_e K$. We know that $B \oplus B' \leqslant_e M$, where B' is a complement of B in M. According to the modular law, we have

$$(B \oplus B') \cap K = B \oplus (B' \cap K).$$

Since $B \leqslant_e K$, we have $B' \cap K = 0$ and hence $(B \oplus B') \cap K = B$. Now just put $N = B \oplus B'$. □

Corollary 3.2. *For every submodule K of M we have $|\varepsilon(K)| \leqslant |\varepsilon(M)|$.*

Remark 3.3. Let A and B be submodules of M with $A \leqslant_e B$. If B'' is a double complement of B in M and A'' be a double complement of A in B'', then $A'' \leqslant B''$ and $A \leqslant_e B \leqslant_e B''$ imply that $A'' \leqslant_e B''$ and A'' is a closed submodule of M, by [6, Page 6]. Therefore, $A'' = B''$, by closedness of B'' in M . This means that in case $A \leqslant_e B$, every double complement of B is equal to a double complement of A in M.

Lemma 3.4. *Every maximal clique in $\Gamma_e(M)$ is of the form $\varepsilon(N)$, where N is a closed submodule of M.*

Proof. Let N be a closed submodule of M. Then N is a common essential extension for every pair of submodules in $\varepsilon(N)$ and thus it forms a clique in $\Gamma_e(M)$. Now if $\varepsilon(N) \cup \{B\}$ is also a clique, then adjacence of B with N and closedness of N imply that $B \leqslant_e N$. This means that $\varepsilon(N)$ is a maximal clique.

Conversely, let $\sum$ be a maximal clique in a connected component $\mathcal{C}$ of $\Gamma_e(M)$ with $X \in \sum$. Then every vertex $Y \in \sum$ is adjacent to X. This implies that $X \leqslant_e X + Y$ and $Y \leqslant_e X + Y$. As a consequence of Remark 3.3, for every double complement $(X + Y)''$ of $X + Y$, there exists a double complement X'' of X and also a double complement Y'' of Y such that $X'' = (X+Y)'' = Y''$. This means that for these double complements X'' and Y'', we have $\sum = \varepsilon(X'') = \varepsilon(Y'')$.

□

Lemma 3.5. *Suppose that $\mathcal{C}$ is a connected component of $\Gamma_e(M)$. Then $\mathcal{C}$ is complete if and only if $\mathcal{C} = \varepsilon(N)$, where N is a closed submodule of M. In this case N is the unique closed submodule of M in $\mathcal{C}$.*

Proof. If $\mathcal{C}$ is a complete connected component of $\Gamma_e(M)$, then $\mathcal{C}$ is a maximal clique and and $\mathcal{C} = \varepsilon(N)$ for some closed submodule N of M, by Lemma 3.4.

Conversely, if $\mathcal{C} = \varepsilon(N)$, for a submodule N of M, then N is a common essential extension of any two vertices in $\mathcal{C}$ and thus $\mathcal{C}$ is complete.

If $N \leqslant_e B$, then $B \in \mathcal{C} = \varepsilon(N)$ and hence $B \leqslant_e N$. Therefore $B = N$ and thus N is closed. Now let L be a closed submodule of M in $\mathcal{C} = \varepsilon(N)$. Then $L \leqslant_e N$ and thus $L = N$ by closedness of L. This means that N is the unique closed submodule of M in $\mathcal{C}$. □

Corollary 3.6. *If $\Gamma_e(M)$ is a union of complete connected components, then every connected component $\mathcal{C}$ of $\Gamma_e(M)$ contains precisely one closed submodule N of M and $\mathcal{C} = \varepsilon(N)$.*

Corollary 3.7. *The clique number of $\Gamma_e(M)$ is $|\varepsilon(M)|$.*

Proof. By Lemma 2.7, $\varepsilon(M)$ is a complete connected component of $\Gamma_e(M)$. Since M is closed, $\varepsilon(M)$ is also a maximal clique in $\Gamma_e(M)$. If $\sum$ is a maximal clique in $\Gamma_e(M)$, then $\mathcal{C} = \varepsilon(N)$ for a closed submodule N of M, by Lemma 3.4. Now we have $|\sum| = |\varepsilon(N)| \leqslant |\varepsilon(M)|$, by Corollary 3.2. □

Remark 3.8. A submodule $N$of M is an isolated vertex in $\Gamma_e(M)$ if and only if N is semisimple and closed in M.

If $\mathcal{C}$ is a stellar connected component of $\Gamma_e(M)$, then the girth of $\mathcal{C}$ is infinite. In the following lemma, we show that the girth of $\mathcal{C}$ is 3 or infinite.

Lemma 3.9. *If $\mathcal{C}$ is a non-stellar connected component of $\Gamma_e(M)$ with at least three vertices, then its girth is* 3.

Proof. If $|\mathcal{C}| = 3$, then $\mathcal{C}$ is complete and its girth is 3.

Let $|\mathcal{C}| \geqslant 4$ and $N - L - P - Q$ be a path in $\mathcal{C}$ of distinct submodules. Then N and L have a common essential extension M'. Also, L and P have a common essential extension M'' and M''' can be considered as a common essential extension of P and Q. If $N \neq M'$ and $L \neq M'$, then we have a triangle with vertices N, L and M'. Otherwise

(1) If $N = M'$ and $L = M''$, then $P \leqslant_e L \leqslant_e N$ and we have a triangle.
(2) If $L = M'$ and $P = M''$, then $N \leqslant_e L \leqslant_e P$ and we have a triangle.
(3) If $L = M' = M''$, then $N \leqslant_e L$ and $P \leqslant_e L$ and we have a triangle.
(4) If $L = M'$ and $P = M''$, then if $Q = M'''$ similar to (1) and if $P = M'''$ similar to (3), we have a triangle.

So $\mathrm{girth}(\Gamma_e(M)) = 3$. □

Remark 3.10. An R-module M is a semisimple module if and only if $\Gamma_e(M)$ is a null graph.

Proof. If M is a semisimple module, then every submodule of M is semisimple and so there is no essential inclusion between submodules.

Conversely, if $\Gamma_e(M)$ is a null graph, then M does not have any proper essential submodule which implies that M is semisimple. □

Theorem 3.11. *If M is a nonsemisimple module, then*

$$girth(\Gamma_e(M)) = \infty$$

if and only if $|\varepsilon(M)| = 2$ (module with one proper essential submodule).

Proof. Let M be a nonsemisimple module and $girth(\Gamma_e(M)) = \infty$. Then $|\varepsilon(M)| \geqslant 2$. If $|\varepsilon(M)| \geqslant 3$, then $\mathcal{C}_M^{\Gamma_e(M)}$ has a circle of length 3 which is a contradiction.

Conversely, if $|\varepsilon(M)| = 2$ and N is a common essential extention of the submodules A and B in M with $N \neq A$ and $N \neq B$, then

$$|\varepsilon(M)| \geqslant |\varepsilon(N)| \geqslant 3,$$

and this is a contradiction by Corollary 3.2. Therefore, there is no triangle in $\Gamma_e(M)$ and therefore $girth(\Gamma_e(M)) = \infty$. □

4. The essentiality graph associated to a UC-module

In [14], P. F. Smith called a module M a *UC-module* if every submodule of M has a unique closure (maximal essential extension) in M. He also gave necessary and sufficient conditions for M to be a UC-module. In this section, we specify the essentiality graph associated to a UC-module. As an special case, we examine the $\mathbb{Z}$-module $\mathbb{Z}/m\mathbb{Z}$, which is isomorphic to $\mathbb{Z}_m$.

The following lemma determines when all connected components of $\Gamma_e(M)$ are complete.

Lemma 4.1. *Let M be an R-module. The graph $\Gamma_e(M)$ is a union of complete connected components if and only if the intersection of any two distinct members of the set $\{\varepsilon(N) : 0 \neq N \leqslant_c M\}$ is empty.*

Proof. If M is a uniform module, then $\Gamma_e(M)$ contains only one component, namely $\varepsilon(M)$. Then M is the only nonzero closed submodule of M and the assertion is obvious in this case.

Now let $\Gamma_e(M)$ be a union of complete connected components and $K \in \varepsilon(N) \cap \varepsilon(L)$ where L and N be two distinct closed submodules of M. Then $K \leqslant_e N$, $K \leqslant_e L$ imply that $N, L \in \mathcal{C}_K^{\Gamma_e(M)}$ and $\mathcal{C}_K^{\Gamma_e(M)}$ is a complete connected component of $\Gamma_e(M)$. Therefore, $N + L$ will be a common essential extension of N and L which contradicts the closedness of N and L.

Conversely, let $\mathcal{C}$ be a connected component of $\Gamma_e(M)$. If $|\mathcal{C}| \leqslant 2$, then $\mathcal{C}$ is complete. If $|\mathcal{C}| \geqslant 3$ and K and N are two arbitrary distinct vertices in $\mathcal{C}$, then we have the path $N - N \cap K - K$. Hence, $N \cap K \leqslant_e N \leqslant_e N''$ and $N \cap K \leqslant_e K \leqslant_e K''$. Therefore,

$$N \cap K \in \varepsilon(N'') \cap \varepsilon(K'')$$

and by hypothesis $N'' = K''$ is a common essential extension of N and K. This means that N is adjacent to K and thus $\mathcal{C}$ is complete. $\square$

We show in the following that the essentiality graph associated to the $\mathbb{Z}$-module $\mathbb{Z}_m$, is a union of complete connected components. Furthermore, the closed submodules of $\mathbb{Z}_m$ and the vertices in each component of $\Gamma_e(\mathbb{Z}_m)$ are identified.

In the rest of this section, let $m = p_1^{\alpha_1} \cdots, p_n^{\alpha_n}$, where $n \in \mathbb{N}$, $p_1, \cdots, p_n$ are distinct prime numbers and for every $i \in \{1, \cdots, n\}$, $\alpha_i \in \mathbb{N}$.

Lemma 4.2. *The set of all closed submodules of the $\mathbb{Z}$-module $\mathbb{Z}_m$ is $X = \{< p_1^{t_1} \cdots p_n^{t_n} >: 1 \leqslant i \leqslant n, t_i \in \{0, \alpha_i\}\}$.*

Proof. Without loss of generality, we can consider $A \in X$ of the form $A =< p_1^{\alpha_1} \cdots p_k^{\alpha_k} >$, where $1 \leqslant k \leqslant n$. If $k = n$, then $A = 0$ which is a closed submodule. Let $1 \leqslant k \leqslant n-1$. If B is an extension of A, then $B =< p_1^{t_1} \cdots p_k^{t_k} >$ where $t_i \leqslant \alpha_i$. Now for the submodule $C =< p_1^{t_1} \cdots p_k^{t_k} p_{k+1}^{\alpha_{k+1}} \cdots p_n^{\alpha_n} >$ of B, we have $A \cap C = \{0\}$. This means that A is not essential in B and thus A is a closed submodule of $\mathbb{Z}_m$.

If $A \notin X$, without loss of generality, we can take

$$A =< p_1^{u_1} \cdots p_l^{u_l} p_{l+1}^{\alpha_{l+1}} \cdots p_k^{\alpha_k} >$$

with $1 \leqslant l \leqslant k \leqslant n$ and $\alpha_i \neq u_i \neq 0$ for $1 \leqslant i \leqslant l$. Now, if we take $B =< p_{l+1}^{\alpha_{l+1}} \cdots p_k^{\alpha_k} >$, then we have $A \subsetneqq B$ and every nonzero submodule T of B is of the form

$$T =< p_1^{v_1} \cdot p_l^{v_l} p_{l+1}^{\alpha_{l+1}} \cdots p_k^{\alpha_k} p_{k+1}^{v_{k+1}} ... p_n^{v_n} >,$$

where at least one v_i is nonequal to α_i. Therefore,

$$A \cap T =< p_1^{max\{u_1,v_1\}} \cdots p_l^{max\{u_l,v_l\}} p_{l+1}^{\alpha_{l+1}} \cdots p_k^{\alpha_k} p_{k+1}^{v_{k+1}} \cdots p_n^{v_n} > \neq \{0\}.$$

Thus B is a proper essential extension of A and this means that A is not a closed submodule of $\mathbb{Z}_m$. $\square$

Lemma 4.3. *If A is a closed $\mathbb{Z}$-submodule of $\mathbb{Z}_m$ of the form $A =< p_1^{\alpha_1} \cdots p_k^{\alpha_k} >$, then the set of all essential submodules of A is*

$$\varepsilon(A) = \{< p_1^{\alpha_1} \cdots p_k^{\alpha_k} p_{k+1}^{t_{k+1}} \cdots p_n^{t_n} >: k+1 \leqslant i \leqslant n, 0 \leqslant t_i \leqslant \alpha_i - 1\}.$$

Proof. Let

$$Y_A = \{< p_1^{\alpha_1} \cdots p_k^{\alpha_k} p_{k+1}^{t_{k+1}} \cdots p_n^{t_n} >: k+1 \leqslant i \leqslant n, 0 \leqslant t_i \leqslant \alpha_i - 1\}.$$

Then A is the maximum element of Y_A and

$$B =< p_1^{\alpha_1} \cdots p_k^{\alpha_k} p_{k+1}^{\alpha_{k+1}-1} \cdots p_n^{\alpha_n - 1} >$$

is the minimum element of Y_A with respect to inclusion. We show $B \leqslant_e A$. For this purpose, suppose that C is a nonzero submodule of A. If $C \in Y_A$, then $B \leqslant C$ and $B \cap C \neq \{0\}$. If $C \notin Y_A$ without loss of generality we consider

$$C =< p_1^{\alpha_1} \cdots p_k^{\alpha_k} p_{k+1}^{\alpha_{k+1}} p_{k+2}^{\beta_{k+2}} \cdots p_n^{\beta_n} >$$

wherein $\beta_i \neq \alpha_i$ for at least one $k+2 \leqslant i \leqslant n$ and

$$C \cap B =< p_1^{\alpha_1} \cdots p_k^{\alpha_k} p_{k+1}^{\alpha_{k+1}} p_{k+2}^{r_{k+2}} \cdots p_n^{r_n} >$$

wherein $r_i \neq \alpha_i$ for at least one $k+2 \leqslant i \leqslant n$ and so $C \cap B \neq \{0\}$ and $B \leqslant_e A$.

The set of submodules of A is

$$L(A) = \{< p_1^{\alpha_1} \cdots p_k^{\alpha_k} p_{k+1}^{t_{k+1}} \cdots p_n^{t_n} >: t_i \in \{0, 1, \ldots, \alpha_i\}\}$$

and the set of nonzero submodules of A is $L^*(A)$. If $T_1 \in L^*(A) - Y_A$ is of the form $T_1 =< p_1^{\alpha_1} \cdots p_k^{\alpha_k} p_{k+1}^{t_{k+1}} \cdots p_n^{t_n} >$, where some t_i is equal to α_i. We can consider $t_i \neq \alpha_i$ for $k+1 \leqslant i \leqslant u < n$ and put $T_2 =< p_1^{\alpha_1} \cdots p_k^{\alpha_k} p_{k+1}^{\alpha_{k+1}} \cdots p_u^{\alpha_u} p_{u+1}^{v_{u+1}} \cdots p_n^{v_n} >$ such that $0 \leqslant v_i \leqslant \alpha_i$ and therefore $T_2 \leqslant A$ and $T_1 \cap T_2 = \{0\}$ so T_1 is not essential in A. $\square$

Theorem 4.4. *$\Gamma_e(\mathbb{Z}_m)$ is a union of complete connected components.*

Proof. Let $N = \langle P_{i_1}^{\alpha_{i_1}} \cdots P_{i_k}^{\alpha_{i_k}} \rangle$ and $L = \langle P_{j_1}^{\alpha_{j_1}} \cdots P_{j_r}^{\alpha_{j_r}} \rangle$ be two arbitrary distinct nonzero closed submodules of $\mathbb{Z}_m$, where $1 \leqslant k \leqslant n-1$, $1 \leqslant r \leqslant n-1$, $\{i_1, \cdots, i_k\}$ and $\{j_1, \cdots, j_r\}$ are distinct subsets of $\{1, \cdots, n\}$. For an arbitrary element

$$s \in \{i_1, \cdots, i_k\} \setminus \{j_1, \cdots, j_r\},$$

the generator element of no essential submodule of L can contain a multiple of $p_s^{\alpha_s}$, by Lemma 4.3. However, the generator element of every essential submodule of N contains some multiples of $p_s^{\alpha_s}$. This means that $\varepsilon(N) \cap \varepsilon(L) = \varnothing$. Thus we can conclude that $\Gamma_e(\mathbb{Z}_m)$ is a union of complete connected components, by Lemma 4.1. □

Now, using Theorem 4.4, we have:

Corollary 4.5. *If $m = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$, then the number of connected components of $\Gamma_e(\mathbb{Z}_m)$ is equal to $2^n - 1$.*

Since the set of all closed submodules of $\mathbb{Z}_m$ is the set

$$X = \{< p_1^{t_1} ... p_n^{t_n} >: t_i \in \{0, \alpha_i\}\}$$

and for $A \in X$ of the form $A =< p_1^{\alpha_1} ... p_k^{\alpha_k} >$ the set of essential submodules of A is

$$\varepsilon(A) = \{< p_1^{\alpha_1} \cdots p_k^{\alpha_k} p_{k+1}^{t_{k+1}} \cdots p_n^{t_n} >: 0 \leqslant t_i \leqslant \alpha_i - 1, \text{ for } k+1 \leqslant i \leqslant n\},$$

and thus $|\varepsilon(A)| = \alpha_{k+1} \ldots \alpha_n$. We can define the bijection map:

$$f : X \to U = \{(\beta_1, \beta_2, \ldots, \beta_n) : \quad \beta_i \in \{0, 1\}\}$$

$$f(< p_1^{t_1} p_2^{t_2} \ldots p_n^{t_n} >) = (\beta_1, \beta_2, \ldots, \beta_n)$$

wherein if $t_i = 0$, then $\beta_i = 1$ and if $t_i = \alpha_i$, then $\beta_i = 0$. Also, the number of submodules of $\mathbb{Z}_m$ is obtained by the following relation:

$$\prod_{i=1}^{n} (\alpha_i + 1) = \sum_{(\beta_1, \ldots, \beta_n) \in U} \alpha_1^{\beta_1} \ldots \alpha_n^{\beta_n}.$$

In the above summation, each summand (except the last 1) is equal to the number of vertices in a (complete) connected component of $\Gamma_e(\mathbb{Z}_m)$ and the frequency of each summand shows the frequency of components of the same cardinality. We illustrate this in the following example.

Example 4.6. In this example, we want to draw the essentiality graph associated to the $\mathbb{Z}$-module $\mathbb{Z}_{360}$. The nonzero closed submodules of $\mathbb{Z}_{360}$ are $\langle 1 \rangle$, $\langle 5 \rangle$, $\langle 8 \rangle$, $\langle 9 \rangle$,$\langle 40 \rangle$,$\langle 45 \rangle$ and $\langle 72 \rangle$, by Lemma 4.2. Now, using Lemma 4.3, we can obtain the connected components of $\Gamma_e({}_{\mathbb{Z}}\mathbb{Z}_{360})$. Note that the vertex set of each connected component is the set of essential submodules of a nonzero closed submodule, by Corollary 3.6. In fact, since $360 = 2^3 \times 3^2 \times 5$, the equalities

$$\begin{aligned}|L(\mathbb{Z}_{360})| &= (3+1)(2+1)(1+1)\\ &= (3\times 2\times 1) + (3\times 2\times 1) + (3\times 1\times 1) + (3\times 1\times 1)\\ &+ (1\times 2\times 1) + (1\times 2\times 1) + (1\times 1\times 1) + 1\times 1\times 1.\end{aligned}$$

show that in $\Gamma_e({}_{\mathbb{Z}}\mathbb{Z}_{360})$ there are two components with 6 vertices, two components with 3 vertices, two components with 2 vertices and a component with 1 vertex. This graph has been drawn in the following figure.

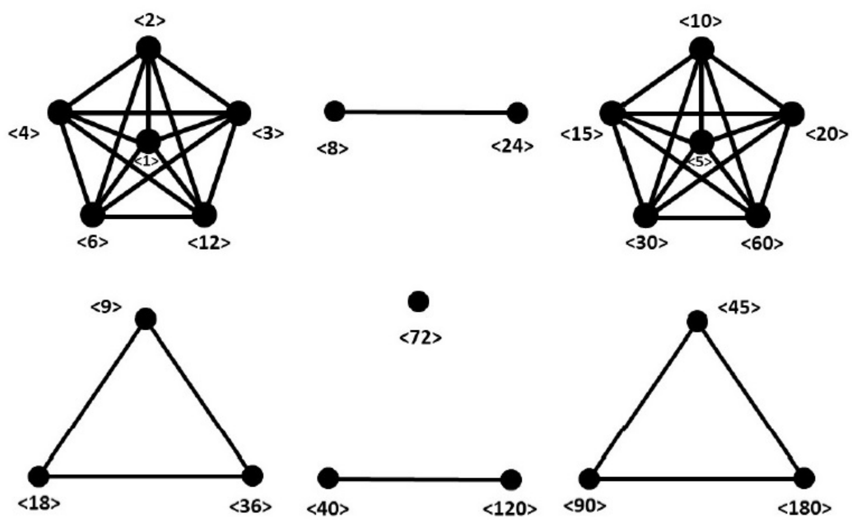

FIGURE 2. $\Gamma_e({}_{\mathbb{Z}}\mathbb{Z}_{360})$

Note that the number of prime divisors of 360 is 3. Thus $\Gamma_e({}_{\mathbb{Z}}\mathbb{Z}_{360})$ has 7 connected components, by Corollary 4.5.

As we saw in Theorem 4.4 and in Example 4.6, all connected components of the graph $\Gamma_e(\mathbb{Z}_m)$ are complete. The following example shows that this need not be true even for every $\mathbb{Z}$-module.

Example 4.7. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_2 \times \mathbb{Z}$. According to Lemma 2.5, the set of all nonzero submodules of M is:

$$\begin{aligned}L^*(M) = \{\langle(0,n)\rangle &= \{0\}\times n\mathbb{Z} | n \in \mathbb{N}\}\\ &\cup \{\langle(1,0),(0,n)\rangle = \mathbb{Z}_2 \times n\mathbb{Z} | n \in \mathbb{Z}\}\\ &\cup \{\langle(1,n)\rangle = \mathbb{Z}_{1,n} | n \in \mathbb{N}\}.\end{aligned}$$

Also, it is easy to see that the set of all essential submodules of M is $\varepsilon(M) = \{\mathbb{Z}_2 \times n\mathbb{Z} | n \in \mathbb{N}\}$. By Lemma 3.1, for every positive integer n, the set of all essential submodules of $\mathbb{Z}_{1,n}$ is:

$$\varepsilon(\mathbb{Z}_{1,n}) = \{(\mathbb{Z}_2 \times m\mathbb{Z}) \cap \mathbb{Z}_{1,n} | m \in \mathbb{N}\}.$$

With a few calculations and considering several cases for m and n, it can be seen that

$$\varepsilon(\mathbb{Z}_{1,n}) = \{\{0\} \times 2kn\mathbb{Z}) | k \in \mathbb{N}\} \cup \{\mathbb{Z}_{1,(2k-1)n} | k \in \mathbb{N}\}.$$

This means that if t is an odd multiple of n, then the $\mathbb{Z}$-module $\langle(1, t)\rangle = \mathbb{Z}_{1,t}$ has an essential extension $\mathbb{Z}_{1,n}$. But for every nonnegative integer i, the $\mathbb{Z}$-module $\langle(1, 2^i)\rangle = \mathbb{Z}_{1,2^i}$ has no proper essential extension of the form $\mathbb{Z}_{1,m}$. Moreover, it is not contained in any submodule of the form $\{0\} \times k\mathbb{Z}$ and it can not be essential in a $\mathbb{Z}$-module of the form $\mathbb{Z}_2 \times n\mathbb{Z}$. Therefore, $\langle(1, 2^i)\rangle = \mathbb{Z}_{1,2^i}$ is a closed submodule in M for every nonnegative integer i. Actually, the set of closed submodules of $M = \mathbb{Z}_2 \times \mathbb{Z}$ is:

$$\{\mathbb{Z}_2 \times \{0\}, M = \mathbb{Z}_2 \times \mathbb{Z}, \{0\} \times \mathbb{Z}\} \cup \{\mathbb{Z}_{1,2^i} | i \geqslant 0\}.$$

Now, using Lemma 3.4, the maximal cliques in $\Gamma_e(M)$ are: $\varepsilon(\mathbb{Z}_2 \times \{0\})$, $\mathcal{C}_2 = \varepsilon(M)$, $\varepsilon(\{0\} \times \mathbb{Z})$, and the cliques of the form $\varepsilon(\mathbb{Z}_{1,2^i})$. For nonnegative integers i and j, if we set $t = \mathrm{Max}\{i + 1, j + 1\}$, then

$$\{0\} \times 2^t\mathbb{Z} \in \varepsilon(\mathbb{Z}_{1,2^i}) \cap \varepsilon(\mathbb{Z}_{1,2^j}).$$

Moreover, For every nonnegative integer i, we have:

$$\{0\} \times 2^{i+1}\mathbb{Z} \in \varepsilon(\mathbb{Z}_{1,2^i}) \cap \varepsilon(\{0\} \times \mathbb{Z}).$$

Thus the set $\varepsilon(\{0\} \times \mathbb{Z}) \cup \bigcup_{i=1}^{\infty} \varepsilon(\mathbb{Z}_{1,2^i})$ is a connected component of $\Gamma_e(M)$. This connected component is not complete, because it has more than one closed submodules. In fact, $\Gamma_e(M)$ contains the following three connected components:

- the isolated vertex $\mathcal{C}_1 = \varepsilon(\mathbb{Z}_2 \times \{0\}) = \{\mathbb{Z}_2 \times \{0\}\}$,
- the complete connected component $\mathcal{C}_2 = \varepsilon(M)$,
- incomplete connected component $\mathcal{C}_3 = \varepsilon(\{0\} \times \mathbb{Z}) \cup \bigcup_{i=0}^{\infty} \varepsilon(\mathbb{Z}_{1,2^i})$.

By determining the associated graph to this $\mathbb{Z}$-module, the double complement of each submodule can be determined. In fact, if K is the submodule $\mathbb{Z}_2 \times \{0\}$ of M, then $K'' = K = \mathbb{Z}_2 \times \{0\}$. The double complement of submodules of the form $\mathbb{Z}_2 \times n\mathbb{Z}$ is $M = \mathbb{Z}_2 \times \mathbb{Z}$. In case K is of the form $\mathbb{Z}_{1,n}$, the double complement K'' is $\mathbb{Z}_{1,1}$, if n is odd and $K'' = \mathbb{Z}_{1,2^i}$ if $n = t \times 2^i$, where $i \in \mathbb{N}$ and t is an odd positive integer. For the case $K = \{0\} \times n\mathbb{Z}$, we have $K'' = \{0\} \times \mathbb{Z}$ if n is odd and $K'' \in \{\{0\} \times \mathbb{Z}, \mathbb{Z}_{1,1}, \cdots, \mathbb{Z}_{1,2^{i-1}}\}$ if $n = t \times 2^i$, where t is odd.

In fact, the $\mathbb{Z}$-module $\mathbb{Z}_2\times\mathbb{Z}$ in Example 4.7, contains a unique simple submodule (namely $\mathbb{Z}_2 \times 0$), but it is not cocyclic. This example illustrates a big difference between cocyclic modules and the modules with unique simple submodules. Namely, the essentiality graph associated to every cocyclic module has a unique complete connected component. But that of the module $\mathbb{Z}_2 \times \mathbb{Z}$ is not connected and it consists of an incomplete component.

The following theorem characterizes all modules whose essentiality graphs are a union of complete connected components.

Theorem 4.8. *An R-module M is a UC-module if and only if every connected component of $\Gamma_e(M)$ is complete.*

Proof. If $\mathcal{C}$ is a connected component of $\Gamma_e(M)$ and N_1, N_2 are two distinct closed submodules in $\mathcal{C}$, then we have the path $N_1 - N_1\cap N_2 - N_2$ and thus $N_1 \cap N_2$ has two distinct closures N_1 and N_2 which contradicts the assumption. Therefore $\mathcal{C}$ contains only one closed submodule N and so $\mathcal{C} = \varepsilon(N)$. Hence $\mathcal{C}$ is complete.

Conversely, If $\mathcal{C}$ is a complete connected component of $\Gamma_e(M)$, then by Lemma 3.5, $\mathcal{C} = \varepsilon(N)$, where $N \leqslant_c M$. Therefore, N is the unique closure of the members of $\mathcal{C}$ and so M is a UC-module. □

According to Theorem 4.8, uniform modules, semisimple modules and every $\mathbb{Z}$-module of the form $\mathbb{Z}_m$ are UC-modules and by Theorem 2.9, whenever $N \leqslant M$, then $\Gamma_e(N)$ is an induced subgraph of $\Gamma_e(M)$ and we conclude that the submodules of a UC-module are themselves UC-modules. If M is a nonsingular module and $N \leqslant M$ then,

$$N'' = \{m \in M | (N : m) \leqslant_e R\}$$

is the unique closure of N (see [18]). Therefore, every nonsingular module is a UC-module. The $\mathbb{Z}$-module $\mathbb{Z}_6$ is a UC-module but not nonsingular module ($ann(2) = 3\mathbb{Z} \leqslant_e \mathbb{Z}$).

Let ${}_RM$ be a nonsingular module and $N \leqslant_e M$. In [18], Wong specified the relationship between the closed submodules of N and the closed submodules of M. In the next theorem, we prove this relation for every UC-module with the help of its essentiality graph. First, we have the following two lemmas.

Lemma 4.9. *Let M be a UC-module and $L \leqslant N$ are submodules of M with unique closures L'' and N'' in M, respectively. Then, $L'' \leqslant N''$.*

Proof. If L_1 is the closure of L in N'' then, by [6, page 6], L_1 is closed in M and by uniqueness of closure, we have $L_1 = L$ and thus $L'' \leqslant N''$. □

Lemma 4.10. *If M is a UC-module and $N \leqslant M$ and $K \leqslant_c M$ then, $N \cap K \leqslant_c N$.*

Proof. We consider L as closure of $N \cap K$ in N and L'' as closure of L in M. In this case, L'' is the unique closure of $N \cap K$ in M and by Lemma 4.9. Therefore, $L'' = (N \cap K)'' \leqslant K'' = K$ and thus $L \leqslant K$. But L is the closure of $N \cap K$. Hence $L = N \cap K$ is closed in N. □

Theorem 4.11. *If N is an essential submodule in a UC-module M, then there is a one to one correspondence between the closed submodules of M and the closed submodules of N (closed in N) given by:*
$K \mapsto K \cap N$, where K is closed in M,
$K \mapsto K''$, where K is closed in N and K'' is the closure of K in M.

Proof. $\Gamma_e(M)$ is a union of complete connected components and by Theorem 2.9, there is a one to one correspondence between connected components of $\Gamma_e(M)$ and $\Gamma_e(N)$ and by Lemma 3.5, each connected component of these two graphs has only one closed submodule. Hence, M and N have equally closed submodules.

If $K \leqslant_c M$, then $K \cap N \leqslant_c N$ by Lemma 4.10. □

Acknowledgments

The authors would like to thank Dr. K. Samei, Dr. R. Rezaei and Dr. Y. Tolooei for their useful discussions and valuable suggestions and Dr. M. R. Vedadi who provided us a copy of [14]. They wish also to extend their grateful and thanks to the referee for careful reading the manuscript and the constructive comments.

References

1. S. Akbari, R. Nikandish and M. J. Nikmehr, Some results on the intersection graph of ideals of rings, *J. Algebra Appl.*, **12**(4) (2013), Article ID: 1250200.
2. S. Akbari, H. A. Tavallaee and S. Khalashi Ghezelahmad, Intersection graph of submodules of a module, *J. Algebra Appl.*, **10** (2011), 1–8.
3. J. Bosak, *The graphs of semigroups*, Theory of Graphs and Applications, Academic Press, New York, (1964), 119–125.
4. I. Chakrabarty, S. Ghosh, T. K. Mukherjee and M. K. Sen, Intersection graphs of ideals of rings, *Discrete Math.*, **309**(17) (2009), 5381–5392.
5. B. Csakany, and G. Pollak, The graph of subgroups of a finite group, Czechoslavak Math. J., **19** (1969), 241–247.
6. N. V. Dung, D. V. Huynh, P. F., Smith and R. Wisbauer, *Extending Modules*, Pitman Research Notes in Math., **313**, Longman, Harlow, 1994.
7. S. H. Jafari and N. Jafari Rad, Domination in the intersection graph of rings and modules, *Italian J. Pure Appl. Math.*, **28** (2011), 17–20.

8. N. Jafari Rad and S. H. Jafari, On intersection graphs of subspaces of a vector space, *arXive:1105.0803v1*, 2011.
9. E. Marczewski, Sur deux propriétés des classes d'ensembles, *Fund. Math.*, **33** (1945), 303–307.
10. E. Matlis, Injective Modules Over Noetherian Rings, *Pacific J. Math.*, **8**(3) (1958), 511–528.
11. R. Nikandish and M. J. Nikmehr, The intersection graph of ideals of $\mathbb{Z}_n$ is weakly perfect, *Util. Math.*, **101** (2016), 329–336.
12. Z. S. Pucanovic and Z. Z. Petrovic, Toroidality of intersection graph of ideals of commutative rings, *Graphs Combin.*, **30** (2014), 707–716.
13. D. W. Sharpe and P. Vamos, *Injective Modules*, Cambridge University Press, 1972.
14. P. F. Smith, Modules for which every submodule has unique closure, *Proceedings of the Biennial Ohio-Denison Conference*, (1992), 302–313.
15. D. B. West, *Introduction to Graph Theory*, Second Ed., Prentice Hall Upper Saddle River, 2001.
16. R. J. Wilson, *Introduction to Graph Theory*, Fourth Ed. Addison Wesley Longman Limited, 1996.
17. R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991.
18. E. T. Wong, Atomic Quasi-Injective Modules, *J. Math. Kyoto Univ*, **3**(3) (1964), 295–303.
19. E. Yaraneri, Intersection Graph of a Module, *J. Algebra Appl.*, **12** (2013), Article ID: 1250218.

Mehdi Ebrahimi Dorcheh

Department of Mathematics, Faculty of Mathematical Sciences, Malayer University, P.O. Box 65719-95863, Malayer, Iran.

Email:mi_ebrahimi@yahoo.com

Saeid Bagheri

Department of Mathematics, Faculty of Mathematical Sciences, Malayer University, P.O. Box 65719-95863, Malayer, Iran.

Email:bagheri_saeid@yahoo.com

Journal of Algebraic Systems

A GRAPH ASSOCIATED TO ESPECIAL

ESSENTIALITY OF SUBMODULES

M. EBRAHIMI DORCHEH AND S. BAGHERI

گرافی منسوب به اساسی بودن زیرمدول‌ها

مهدی ابراهیمی درچه[۱] و سعید باقری[۲]

[۱,۲]گروه ریاضی، دانشگاه ملایر، ملایر، ایران

فرض کنید R حلقه‌ای شرکت‌پذیر و یکدار باشد. در این مقاله به هر R-مدول M، یک گراف ساده $\Gamma_e(M)$ نسبت داده و آن را گراف اساسی بودن M می‌نامیم. مجموعه رأس‌های این گراف شامل همه زیرمدول‌های ناصفر M بوده و دو رأس K و L را مجاور در نظر می‌گیریم هرگاه اشتراک آن‌ها، $K \cap L$، در مجموعشان، $K+L$، اساسی باشد. سپس رابطه بین ساختار جبری (زیر مدولی) M از جمله مینیمال بودن و بسته بودن زیرمدول‌ها در M را با ویژگی‌های نظریه گرافی $\Gamma_e(M)$ بررسی می‌کنیم. این گراف درحالت کلی همبند نیست. در واقع نشان می‌دهیم که این گراف همبند است اگر و تنها اگر کامل باشد اگر و تنها اگر مدول M یکنواخت باشد. هچنین حالت خاصی را بررسی می‌کنیم که این گراف اجتماعی از مولفه‌های کامل باشد. به علاوه، تمایز بین این حالات خاص را با ارائه مثال‌هایی نشان می‌دهیم.

کلمات کلیدی: زیرمدول اساسی، زیرمدول بسته، UC-مدول، عدد خوشه‌ای، کمر.