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SOME RESULTS ON ORDERED AND UNORDERED FACTORIZATION OF A POSITIVE INTEGER

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Abstract. A well-known enumerative problem is to count the number of ways a positive integer n can be factorised as $n = n_1 \times n_2 \times \cdots \times n_k$, where $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k > 1$. In this paper, we give some recursive formulas for the number of ordered/unordered factorizations of a positive integer n such that each factor is at least ℓ. In particular, by using elementary techniques, we give an explicit formula in the cases where $k = 2, 3, 4$.

1. Introduction

Let $\mathcal{F}(n, k, \ell)$ be the number of unordered factorizations of a positive integer n into exactly k parts such that each factor is at least ℓ. We denote the number of all unordered factorizations of a positive integer n by $\mathcal{F}(n)$. So, $\mathcal{F}(n)$ is the number of ways a positive integer n can be written as a product $n = n_1 \times n_2 \times \cdots \times n_k$. Clearly, $\mathcal{F}(n) = \sum_{k=1}^{n} \mathcal{F}(n; k, 2)$. Let $p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$ be the prime decomposition of a positive integer n. Then there is a bijection between $\mathcal{F}(n, k, \ell)$ and the number of partitions of the multiset

$$\{\underbrace{p_1, \ldots, p_1}_{\beta_1-\text{times}}, \underbrace{p_2, \ldots, p_2}_{\beta_2-\text{times}}, \ldots, \underbrace{p_r, \ldots, p_r}_{\beta_r-\text{times}}\}$$

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into k unlabelled blocks such that each block has at least ℓ elements. For example, $\mathcal{F}(2^3 \times 3^4 \times 5^6, 6, 2)$ is the number of partitions of the multiset

$$\{2, 2, 2, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5\}$$

into 6 unlabelled blocks such that each block has at least two elements. Using Mathematica software, one observes that $\mathcal{F}(2^3 \times 3^4 \times 5^6) = 11220$. See [1, 8] for further results on partitions of a multiset. The sequence $\{\mathcal{F}(n)\}$ is listed as the sequence A001055 in the *On-Line Encyclopedia of Integer Sequences* [7]. The Dirichlet generating function for the sequence $\{\mathcal{F}(n)\}$ is

$$\prod_{k=2}^{\infty} \frac{1}{1 - k^{-s}} = \sum_{n=1}^{\infty} \frac{\mathcal{F}(n)}{n^s}.$$

For the positive integers $\ell, k \geqslant 1$, let $\mathcal{H}(n; k, \ell)$ denote the number of *ordered factorizations* of a positive integer n into exactly k parts such that each factor is at least ℓ. We use $\mathcal{H}(n)$ to represent the number of all ordered factorizations of the positive integer n. The sequences $\mathcal{H}(n)$ and $\mathcal{F}(n)$ are analogous to that of compositions and partitions of a positive integer n. Clearly, $\mathcal{H}(n) = \sum_{k=1}^{n} \mathcal{H}(n; k, 2)$.

Let $\rho(n)$ denote the number of partitions $n = n_1 + n_2 + \cdots + n_k$ of a positive integer n where we assume that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k > 0$. The integers $n_1, n_2, \ldots, n_k$ are called the *parts* of the partitions. For example $\rho(4)$ corresponds to $1+1+1+1$, $1+1+2$, $1+3$, $2+2$, and 4. It is important to note that if $n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$, where $p_1, p_2, \ldots, p_r$ are distinct prime numbers and $\beta_i \in \mathbb{N}$ for $1 \leqslant i \leqslant r$, then $\mathcal{F}(n)$ and $\mathcal{H}(n)$ depend only on the positive integers $\beta_1, \beta_2, \ldots, \beta_r$. Using this fact, for special choices of n, the values of $\mathcal{F}(n)$ and $\mathcal{H}(n)$ can be determined in closed form. For instance, if a positive integer n is a prime power $n = p^r$, then $\mathcal{F}(n) = \rho(r)$, and $\mathcal{H}(n) = 2^{r-1}$. Also, if $n = p_1 \times p_2 \times \cdots \times p_r$ is square-free, then $\mathcal{F}(n) = \sum_{i=1}^{r} \left\{ {r \atop i} \right\}$, and $\mathcal{H}(n) = \sum_{i=1}^{r} i! \left\{ {r \atop i} \right\}$, where $\left\{ {r \atop i} \right\}$ is the *Stirling number of the second kind.*

Let $\mathcal{F}(n; \{\beta_1, \ldots, \beta_r\}, \ell)$ be the number of unordered factorizations of a positive integer n as $n = n_1^{\beta_1} \times \cdots \times n_r^{\beta_r}$ such that $\beta_1 + \cdots + \beta_r = k$ and $\ell \leqslant n_1 < \cdots < n_r$. Also, let $\mathcal{H}(n; \{\beta_1, \ldots, \beta_r\}, \ell)$ be the number of ordered factorizations of a positive integer n as $n = n_1^{\beta_1} \times \cdots \times n_r^{\beta_r}$ such that $n_i \geqslant \ell$, $\{n_1, \ldots n_k\} = \{n'_1, \ldots, n'_r\}$ and $\beta_j = |\{i : n_i = n'_j\}|$ for each $1 \leqslant i, j \leqslant r$. For example, $\mathcal{F}(n; \{1, 1, 2\}, \ell)$ is the number of unordered factorizations of a positive integer n as xyz^2 with $x > y > z \geqslant \ell$, and

$\mathcal{H}(n;\{1,1,2\},\ell) = 2!\mathcal{F}(n;\{1,1,2\},\ell)$. It is easy to see that

$$\mathcal{F}(n;\{\beta_1,\ldots,\beta_r\},\ell) = \frac{(\beta_1+\cdots+\beta_r)!}{\beta_1!\cdots\beta_r!}\mathcal{H}(n;\{\beta_1,\ldots,\beta_r\},\ell). \quad (1.1)$$

See [2, 3, 4, 6] for further results on factorization partitions including results on bounds and asymptotic behaviours of $\mathcal{F}(n)$ as well as algorithms for calculating their values.

The goal of this paper is to give some recursive formulas for $\mathcal{F}(n)$ and $\mathcal{H}(n)$. Also, we apply elementary techniques to obtain $\mathcal{F}(n)$ and $\mathcal{H}(n)$ when $n = 2, 3, 4$. We give a new proof for the following formula given by *MacMahon* in 1893 (see [5]):

$$\mathcal{H}(n;k,2) = \sum_{i=0}^{k-1}(-1)^i\binom{k}{i}\prod_{j=1}^{n}\binom{\beta_j+k-i-1}{k-i-1}.$$

Finally, we present several propositions involving the partition function $\rho(n)$.

2. A Recursive Formula

In this section, we give some recursive formula for $\mathcal{F}(n;k,\ell)$ and $\mathcal{H}(n;k,\ell)$. Let $n = n_1^{\beta_1}\times\cdots\times n_r^{\beta_r}$ be a positive integer, where $\beta_i \in \mathbb{N}$. By above notations, we can write

$$\mathcal{F}(n;k,\ell) = \sum_{\substack{\beta_1+\cdots+\beta_r=k;\\ \beta_1<\cdots<\beta_r,}} \mathcal{F}(n;\{\beta_1,\ldots,\beta_r\},\ell); \quad (2.1)$$

and

$$\mathcal{H}(n;k,\ell) = \sum_{\beta_1+\cdots+\beta_r=k} \mathcal{H}(n;\{\beta_1,\ldots,\beta_r\},\ell). \quad (2.2)$$

Proposition 2.1. *Let $n > 1$ and k, ℓ be positive integers. Then*

$$\mathcal{F}(n;k,\ell) = \sum_{i=\max\{k-s,1\}}^{\min\{k,s\}} \mathcal{F}(n;i,\ell+1),$$

where $s \leq k$ is the largest positive integer for which ℓ^s divides n.

Proof. Let

$$E = \{(n_1,n_2,\ldots,n_k) : n = n_1\times n_2\times\cdots\times n_k, \ell \leqslant n_1 \leqslant\cdots\leqslant n_k\}$$

and

$$E_i = \{(n_1,n_2,\ldots,n_k) \in E : n_1 = n_2 = \cdots = n_i = \ell, n_{i+1} \neq \ell\},$$

for $i = 0, 1, \ldots, \min\{k-1, s\}$. Then $E = \cup_{i=0}^{\min\{k-1,s\}} E_i$ and the union is disjoint. Thus,

$$\begin{aligned}\mathcal{F}(n; k, \ell) &= \sum_{i=0}^{\min\{k-1,s\}} |E_i| = \sum_{i=0}^{\min\{k-1,s\}} \mathcal{F}(n; k-i, \ell+1) \\ &= \sum_{i=\max\{k-s,1\}}^{\min\{k,s\}} \mathcal{F}(n; i, \ell+1),\end{aligned}$$

as required. □

Corollary 2.2. *Let $n > 1$ and k, ℓ be positive integers. Then*

$$\mathcal{F}(n; k, 1) = \sum_{i=1}^{k} \mathcal{F}(n; i, 2).$$

Lemma 2.3. *Let n, k and ℓ be positive integers. Then*

$$\mathcal{F}(n; k, \ell) = \sum_{\ell \leqslant d | n} \mathcal{F}\left(\frac{n}{d}; k-1, d\right).$$

Proof. For positive integer n we define $\mathcal{F}(1; k.\ell) = 1$ and $\mathcal{F}(n; 0, \ell) = 0$. Let $n = n_1 \times n_2 \times \cdots \times n_k$ be a unordered factorizations of n such that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant \ell > 0$. Then $\frac{n}{n_1} = n_2 \times n_3 \times \cdots \times n_k$ is the factorization partitions of $\frac{n}{n_1}$ in $k-1$ factor such that each factor is at least ℓ, and the number of all such factorization partition is $\mathcal{F}\left(\frac{n}{n_1}, k, \ell\right)$. Since n_1 was unspecified, therefore, we can write such factorization partition for any divisor $d \leqslant \ell$ of n. Summation over all such divisor d of n gives the proof. □

Lemma 2.4. *Let n, k and ℓ be positive integers. Then*

$$\mathcal{H}(n; k, \ell) = \sum_{\ell \leqslant d | n} \mathcal{H}\left(\frac{n}{d}; k-1, d\right).$$

3. An Explicit Formula For the Cases $k = 1, 2, 3, 4$ and $\ell = 1, 2$

We now intend to give an explicit formula for the functions $\mathcal{H}(n; k, \ell)$ and $\mathcal{F}(n; k, \ell)$ when $k = 1, 2, 3, 4$ and $\ell = 1, 2$. In the following proposition, we denote the number of natural divisors of n by $\tau(n)$. Moreover,

$$\varepsilon_i(n) = \begin{cases} 1 & \text{if } \sqrt[i]{n} \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.1. *Let $n > 1$ be a positive integer. Then*

i. $\mathcal{F}(n;1,1) = \mathcal{F}(n,1,2) = 1$,

ii. $\mathcal{F}(n;2,1) = \lceil \frac{\tau(n)}{2} \rceil$ *and* $\mathcal{F}(n;2,2) = \lceil \frac{\tau(n)}{2} \rceil - 1$.

Proof. The equalities in (i) are obvious. To prove (ii), we note that $\mathcal{H}(n;2,1)$ is the number of ways to write n as xy, where x is a natural divisor of n. Thus, $\mathcal{H}(n;2,1) = \tau(n)$. Now if n is not a perfect square then $\tau(n)$ is even and so $\mathcal{F}(n;2,1) = \frac{\tau(n)}{2} = \lceil \frac{\tau(n)}{2} \rceil$ and if n is a perfect square then $\mathcal{F}(n;2,1) = \frac{\tau(n)-1}{2} + 1 = \lceil \frac{\tau(n)}{2} \rceil$. Using Corollary 2.2, we now have $\mathcal{F}(n;2,2) = \mathcal{F}(n;2,1) - 1$. □

Theorem 3.2. *Let $n > 1$ be a positive integer and $p_1^{\beta_1} \cdots p_r^{\beta_r}$ be its prime decomposition. Then*

i. *the number of factorizations of a positive integer n into three factor such that each factor is at least 1 is given by*

$$\mathcal{F}(n;3,1) = \frac{1}{6}\prod_{j=1}^{r}\binom{\beta_j+2}{2} + \frac{1}{2}\prod_{j=1}^{r}\lfloor\frac{\beta_j+2}{2}\rfloor + \frac{\varepsilon_3(n)}{3}.$$

ii. *the number of factorizations of positive integer n into three factors such that each factor $\geqslant 2$ is given by*

$$\mathcal{F}(n;3,2) = \mathcal{F}(n;3,1) - \lceil\frac{\tau(n)}{2}\rceil.$$

Proof. We have

$$\begin{aligned}\mathcal{H}(n;3,1) &= \mathcal{H}(n;\{1,1,1\},1) + \mathcal{H}(n;\{1,2\},1) + \mathcal{H}(n;\{3\},1)\\ &= 6\mathcal{F}(n;\{1,1,1\},1) + 3\mathcal{F}(n;\{1,2\},1) + \mathcal{F}(n;\{3\},1).\end{aligned}$$

We know that $\mathcal{F}(n;\{1,2\},1)$ is the number of ways to write n as xy^2, where $x \neq y$. This is equal to the number of y's such that $y^2 \mid n$ reduced by the number of ways such that $\frac{n}{y^2} = y$, in which the later is equal to $\varepsilon_3(n)$. The number of y's such that $y^2 \mid n$ is $\prod_{j=1}^{r}\lfloor\frac{\beta_j+2}{2}\rfloor$. Moreover, $\mathcal{F}(n;\{3\},1) = \varepsilon_3(n)$. Thus

$$\begin{aligned}\mathcal{F}(n;\{1,1,1\},1) &= \frac{1}{6}\Big(\mathcal{H}(n;3,1) - 3\big(\prod_{j=1}^{r}\lfloor\frac{\beta_j+2}{2}\rfloor - \varepsilon_3(n)\big) - \varepsilon_3(n)\Big)\\ &= \frac{1}{6}\mathcal{H}(n;3,1) - \frac{1}{2}\prod_{j=1}^{r}\lfloor\frac{\beta_j+2}{2}\rfloor + \frac{1}{3}\varepsilon_3(n)\\ &= \frac{1}{6}\prod_{j=1}^{r}\binom{\beta_j+2}{2} - \frac{1}{2}\prod_{j=1}^{r}\lfloor\frac{\beta_j+2}{2}\rfloor + \frac{\varepsilon_3(n)}{3}.\end{aligned}$$

Therefore, we must have

$$\begin{aligned}\mathcal{F}(n;3,1) &= \mathcal{F}(n;\{1,1,1\},1)+\mathcal{F}(n;\{1,2\},1)+\mathcal{F}(n;\{3\},1)\\ &= \frac{1}{6}\prod_{j=1}^{r}\binom{\beta_j+2}{2}+\frac{1}{2}\prod_{j=1}^{r}\lfloor\frac{\beta_j+2}{2}\rfloor+\frac{\varepsilon_3(n)}{3}.\end{aligned}$$

This proves item (i).

Using Corollary 2.2, we have

$$\mathcal{F}(n;3,2)=\mathcal{F}(n;3,1)-\mathcal{F}(n;2,2)-1=\mathcal{F}(n;3,1)-\lceil\frac{\tau(m)}{2}\rceil$$

which proves item (ii). □

The following lemma is also easy to prove.

Lemma 3.3. *Let $n,\ell>1$ be a positive integer and $p_1^{\beta_1}\cdots p_r^{\beta_r}$ be the prime decomposition of n. Then*

i. $\sum_{d|n}\tau(d)=\prod_{j=1}^{r}\binom{\beta_j+2}{2}$

ii. $\sum_{d|n}\varepsilon_\ell(d)=\prod_{j=1}^{r}\lfloor\frac{\beta_j+\ell}{\ell}\rfloor$.

Proof. Let $F(n)=\sum_{d|n}\tau(d)$ for all $n\geq 1$. To prove (i), we note that F is a multiplicative function as τ is multiplicative. Now the result follows from the fact that

$$F(p^\beta)=\sum_{d|p^\beta}\tau(p^\beta)=\sum_{i=0}^{\beta}(i+1)=\frac{1}{2}(\beta+1)(\beta+2)=\binom{\beta+2}{2}$$

for all prime powers p^β. Part (ii) is proved analogously. □

Theorem 3.4. *Let $n>1$ be a positive integer and $p_1^{\beta_1}\cdots p_r^{\beta_r}$ be its prime decomposition. Then*

i. *the number of factorizations of n into four factors such that each factor $\geqslant 1$ is given by*

$$\begin{aligned}\mathcal{F}(n;4,1) &= \frac{1}{24}\prod_{i=1}^{r}\binom{\beta_i+3}{3}+\frac{1}{3}\prod_{i=1}^{r}\lfloor\frac{\beta_i+3}{3}\rfloor\\ &+\frac{1}{4}(\prod_{i=1}^{r}\lfloor\frac{\beta_i+2}{2}\rfloor(\beta_i-\lfloor\frac{\beta_i-2}{2}\rfloor))\\ &+\frac{\varepsilon_2(m)}{4}\prod_{i=1}^{r}\lfloor\frac{\beta_i+2}{2}\rfloor-\frac{\varepsilon_2(m)}{4}\lceil\frac{\tau(\sqrt{m})}{2}\rceil+\frac{3\varepsilon_4(m)}{8}.\end{aligned}$$

ii. *the number of factorizations of n into four factors such that each factor $\geqslant 2$ is given by*

$$\mathcal{F}(n;4,2)=\mathcal{F}(n;4,1)-\mathcal{F}(n;3,1).$$

Proof. We have

$$\begin{aligned}\mathcal{H}(n;4,1) &= \mathcal{H}(n;\{1,1,1,1\},1)+\mathcal{H}(n;\{1,1,2\},1)\\ &\quad+\mathcal{H}(n;\{1,3\},1)+\mathcal{H}(n;\{2,2\},1)+\mathcal{H}(n;\{4\},1)\\ &= 24\mathcal{F}(n;\{1,1,1,1\},1)+12\mathcal{F}(n;\{1,1,2\},1)\\ &\quad+4\mathcal{F}(n;\{1,3\},1)+6\mathcal{F}(n;\{2,2\},1)+\mathcal{F}(n;\{4\},1).\end{aligned}$$

On the other hand, $\mathcal{F}(n;\{1,1,2\},1)$ is the number of ways to write n as xyz^2, where x,y and z are different positive integers. This is equal to the number of z's such that $z^2 \mid n$ minus the number of ways to write n as xz^3, x^2z^2 or z^4, where $x \neq z$. The number of z's such that $z^2 \mid n$ is equal to $\sum_{z^2|n}\lceil\frac{\tau(\frac{n}{z^2})}{2}\rceil$. Thus,

$$\begin{aligned}\mathcal{F}(n;\{1,1,2\},1) = &\sum_{z^2|n}\lceil\frac{\tau\left(\frac{n}{z^2}\right)}{2}\rceil-\mathcal{F}(n;\{1,3\},1)\\ &-\mathcal{F}(n;\{2,2\},1)-\varepsilon_4(n).\end{aligned}$$

If n is not a perfect square, then $\tau(n)$ is even. Let $z=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ be the prime decompositions of positive integer z, where $0\leqslant\alpha_i\leqslant\beta_i$. So, there must exist an integer i such that α_i is odd. Hence,

$$\begin{aligned}\sum_{z^2|n}\lceil\frac{\tau\left(\frac{n}{z^2}\right)}{2}\rceil &= \sum_{z^2|n}\frac{1}{2}\tau(p_1^{\beta_1-2\alpha_1}\cdots p_r^{\beta_s-2\alpha_r})\\ &=\frac{1}{2}\prod_{i=1}^{r}\sum_{0\leqslant\alpha_i\leqslant\frac{\beta_i}{2}}(\beta_i-2\alpha_i+1)\\ &=\frac{1}{2}(\prod_{i=1}^{r}\lfloor\frac{\beta_i+2}{2}\rfloor(\beta_i-\lfloor\frac{\beta_i-2}{2}\rfloor)).\end{aligned}$$

Moreover, $\mathcal{F}(n;\{1,3\},1)$ is the number of ways to write n as xz^3, where $x\neq z$. This is equal to the number of z's such that $z^3\mid n$ minus the number of ways to write n as z^4. The number of z's such that $z^3\mid n$ is $\prod_{j=1}^{r}\lfloor\frac{\beta_j+3}{3}\rfloor$. Hence,

$$\mathcal{F}(n;\{1,3\},1)=\prod_{j=1}^{r}\lfloor\frac{\beta_j+3}{3}\rfloor-\varepsilon_4(n).$$

Since n is not a perfect square, $\mathcal{F}(n;\{2,2\},1)=0$. Thus, by 1.1 implies

$$\begin{aligned}\mathcal{F}(n;\{1,1,1,1\},1) &= \frac{1}{24}\big(\mathcal{H}(n;4,1)-12\mathcal{F}(n;\{1,1,2\},1)\\ &\quad -4\mathcal{F}(n;\{1,3\},1)-6\mathcal{F}(n;\{2,2\},1)-\varepsilon_4(n)\big)\\ &= \frac{1}{24}\prod_{i=1}^{r}\binom{\beta_i+3}{3}-\frac{1}{4}(\prod_{i=1}^{r}\lfloor\frac{\beta_i+2}{2}\rfloor(\beta_i-\lfloor\frac{\beta_i-2}{2}\rfloor))\\ &\quad +\frac{1}{3}\mathcal{F}(n;\{1,3\},1)+\frac{1}{4}\mathcal{F}(n;\{2,2\},1)+\frac{11}{24}\varepsilon_4(n)\big)\\ &= \frac{1}{24}\prod_{i=1}^{r}\binom{\beta_i+3}{3}-\frac{1}{4}(\prod_{i=1}^{r}\lfloor\frac{\beta_i+2}{2}\rfloor(\beta_i-\lfloor\frac{\beta_i-2}{2}\rfloor))\\ &\quad +\frac{1}{3}\mathcal{F}(n;\{1,3\},1).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathcal{F}(n;4,1) &= \mathcal{F}(n;\{1,1,1,1\},1)+\mathcal{F}(n;\{1,1,2\},1)\\ &\quad +\mathcal{F}(n;\{1,3\},1)+\mathcal{F}(n;\{2,2\},1)+\mathcal{F}(n;\{4\})\\ &= \frac{1}{24}\prod_{i=1}^{r}\binom{\beta_i+3}{3}+\frac{1}{3}\prod_{i=1}^{r}\lfloor\frac{\beta_i+3}{3}\rfloor\\ &\quad +\frac{1}{4}(\prod_{i=1}^{r}\lfloor\frac{\beta_i+2}{2}\rfloor(\beta_i-\lfloor\frac{\beta_i-2}{2}\rfloor)). \qquad (3.1)\end{aligned}$$

Now let n be a perfect square. Then $\tau(n)$ is odd and we have

$$\begin{aligned}\sum_{z^2|n}\lceil\frac{\tau(\frac{n}{d^2})}{2}\rceil &= \sum_{z^2|n}(\frac{\tau(\frac{n}{d^2})+1}{2})\\ &= \sum_{z^2|n}\frac{1}{2}\tau(p_1^{\beta_1-2\alpha_1}\cdots p_r^{\beta_r-2\alpha_r})+\frac{1}{2}\sum_{z^2|n}1\\ &= \frac{1}{2}\sum_{0\leqslant\alpha_i\leqslant\frac{\beta_i}{2}}\prod_{i=1}^{r}(\beta_i-2\beta_i+1)+\frac{1}{2}\sum_{0\leqslant\alpha_i\leqslant\frac{\beta_i}{2}}1\\ &= \frac{1}{2}\prod_{i=1}^{r}\sum_{0\leqslant\alpha_i\leqslant\frac{\beta_i}{2}}(\beta_i-2\beta_i+1)+\frac{1}{2}\prod_{i=1}^{r}(\lfloor\frac{\beta_i+2}{2}\rfloor)\\ &= \frac{1}{2}(\prod_{i=1}^{r}\lfloor\frac{\beta_i+2}{2}\rfloor(\beta_i-\lfloor\frac{\beta_i-2}{2}\rfloor))+\frac{1}{2}\prod_{i=1}^{r}(\lfloor\frac{\beta_i+2}{2}\rfloor)\end{aligned}$$

Using above equtation, we can write

$$\begin{aligned}\mathcal{F}(n;\{1,1,1,1\},1) &= \frac{1}{24}\prod_{i=1}^{r}\binom{\beta_i+3}{3} - \frac{1}{4}(\prod_{i=1}^{r}\lfloor\frac{\beta_i+2}{2}\rfloor(\beta_i-\lfloor\frac{\beta_i-2}{2}\rfloor))\\ &+\frac{1}{2}\prod_{i=1}^{r}(\lfloor\frac{\beta_i+2}{2}\rfloor)+\frac{1}{3}\mathcal{F}(n;\{1,3\},1)\\ &-\frac{1}{4}\mathcal{F}(n;\{2,2\},1)-\frac{1}{24}\varepsilon_4(n).\end{aligned}$$

Furthermore, $\mathcal{F}(n;\{2,2\},1)$ is the number of ways to write n as $x^2y^2 = (xy)^2$, where $x \neq y$. If n is not a perfect square then this number is 0 and if n is a perfect square, then $\sqrt{n} \in \mathbb{N}$. It is easy to see that

$$\begin{aligned}\mathcal{F}(n;\{2,2\},1) &= \varepsilon_2(n)\mathcal{F}(\sqrt{m};\{1,1\},1)\\ &= \varepsilon_2(n)\big(\mathcal{F}(\sqrt{n},\{2\},1)-\varepsilon_2(\sqrt{n})\big)\\ &= \varepsilon_2(n)\big(\lceil\frac{\tau(\sqrt{n})}{2}\rceil-\varepsilon_4(n)\big)\\ &= \varepsilon_2(n)\lceil\frac{\tau(\sqrt{n})}{2}\rceil-\varepsilon_4(n).\end{aligned}$$

This proves item (i). According Corollary 2.2, the item (ii) can be driven from the first assertion. □

We can add some conditions to the problem. For instance, we can think about the number of solutions of the equation $n = n_1 + \ldots + n_k$ with the conditions $\ell_i \leqslant n_i \leqslant \ell'_i$ for $i = 1, \ldots, k$. A straightforward application of the Inclusion-Exclusion principle solves this problem. Using these facts, we proved Lemma 3.5 below.

Lemma 3.5. *Let $n > 1, k$ be positive integers and $p_1^{\beta_1}\cdots p_r^{\beta_r}$ be the prime decomposition of n. Then*

$$\mathcal{H}(n,k,1) = \prod_{j=1}^{r}\binom{\beta_j+k-1}{k-1}$$

and

$$\mathcal{H}(n,k,2) = \sum_{i=0}^{k-1}(-1)^i\binom{k}{i}\prod_{j=1}^{r}\binom{\beta_j+k-i-1}{k-i-1}.$$

Proof. Let

$$A = \{\underbrace{p_1,\ldots,p_1}_{\beta_1-\text{time's}},\underbrace{p_2,\ldots,p_2}_{\beta_2-\text{time's}},\underbrace{\ldots,p_r,\ldots,p_r}_{\beta_r-\text{time's}}\}$$

be a multiset with β_i balls labelled p_i. Clearly, $\mathcal{H}(n,k,1)$ is the number of partitions of the multiset A into k nonempty blocks. The number of partitions of β_j unlabelled balls with labelled p_j into k nonempty labelled blocks is $\binom{\beta_j+k-1}{k-1}$. Thus, the first part is obvious.

For the second part, using this fact, $\mathcal{H}(n,k,2)$ is the number of partitions of the multiset A into k blocks such that each block has at least 2 elements. let E_r be the set of all situations in which the blocks r is empty, where $1 \leqslant r \leqslant k$. Then we have

$$|E_{r_1} \cap \cdots \cap E_{r_i}| = \prod_{j=1}^{r} \binom{\beta_j + k - i - 1}{k - i - 1}, \quad 1 \leqslant i \leqslant k-1.$$

Thus, the Inclusion-Exclusion principle implies the result. □

Example 3.6. We evaluate $\mathcal{H}(p^\alpha, k, 2)$. Using Lemma 3.5, $\mathcal{H}(p^\alpha, k, 1)$ is the number of partitions of the multiset A into k nonempty labelled blocks, means $\binom{\alpha+k-1}{k-1}$.

Now, suppose that A_r be the set of all situations in which the the blocks r is empty. Using Lemma 3.5,

$$|A_{r_1} \cap \cdots \cap A_{r_i}| = \binom{\alpha + k - i - 1}{k - i - 1}, \quad 1 \leqslant i \leqslant k-1.$$

By the Inclusion-Exclusion principle we therefore have

$$\begin{aligned}\mathcal{H}(n,k,2) &= \sum_{k=1}^{\alpha}\sum_{i=0}^{k-1}(-1)^i\binom{k}{i}\prod_{j=1}^{n}\binom{\alpha_j+k-i-1}{k-i-1}\\ &= \sum_{k=1}^{\alpha-1}\binom{k-1}{\alpha-1}\\ &= 2^{\alpha-1}.\end{aligned}$$

Now, note that if a positive integer n is a prime power, say $n = p^k$, then $\mathcal{F}(n,k,2) = \rho(n,k)$, where, $\rho(n,k)$ is the number of partitions of positive integer n to exactly k parts.

Corollary 3.7. *Let n be a positive integer. Then, the number of all positive solutions of the equation*

$$x_1 + x_2 + x_3 = n,$$

under the condition $x_1 \leq x_2 \leq x_3$, is

$$\mathcal{F}(p^3, 3, 2) = \frac{1}{6}\binom{n+2}{2} + \frac{1}{2}\lfloor\frac{n+2}{2}\rfloor + \frac{\varepsilon_3(p^n)}{3}.$$

Corollary 3.8. *Let n be a positive integer. Then, the number of all positive solutions of the equation*

$$x_1 + x_2 + x_3 + x_4 = n,$$

under the condition $x_1 \leq x_2 \leq x_3 \leq x_4$, is

$$\mathcal{F}(p^4, 4, 2) = \frac{1}{24}\binom{n+3}{3} + \frac{1}{3}\lfloor\frac{n+3}{3}\rfloor + \frac{1}{4}(\lfloor\frac{n+2}{2}\rfloor(n - \lfloor\frac{n-2}{2}\rfloor)) + \frac{\varepsilon_2(p^n)}{4}\lfloor\frac{n+2}{2}\rfloor - \frac{\varepsilon_2(p^n)}{4}\lceil\frac{\tau(\sqrt{p^n})}{2}\rceil + \frac{3\varepsilon_4(p^n)}{8}.$$

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Journal of Algebraic Systems

SOME RESULTS ON ORDERED AND UNORDERED

FACTORIZATION OF A POSITIVE INTEGER

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برخی نتایج درباره تجزیه مرتب و نامرتب یک عدد صحیح مثبت

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یکی از مسائل مهم شمارشی، محاسبه تعداد راه‌های تجزیه عدد صحیح مثبت n به صورت

$$n = n_1 \times \cdots \times n_k,$$

بطوریکه $n_1 \geq n_2 \geq \cdots \geq n_k > 1$ می‌باشد. در این مقاله، با روش‌های بازگشتی تعداد حالات تجزیه مرتب/نامرتب عدد صحیح مثبت n را برای حالتی که هر بخش تجزیه حداقل برابر با ℓ باشد را بدست آورده‌ایم. به‌ویژه، با استفاده از روش‌های مقدماتی، فرمول صریحی برای حالت‌های $k = 2, 3, 4$ محاسبه کرده‌ایم.

کلمات کلیدی: تابع افراز ضربی، افرازهای مجموعه، تابع افراز، مربع کامل، تابع فی اویلر، تابع تاو.