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COMMON NEIGHBOURHOOD SPECTRUM AND ENERGY OF COMMUTING CONJUGACY CLASS GRAPH

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Abstract. In this paper, we compute the common neighbourhood (abbreviated as CN) spectrum and the common neighbourhood energy of commuting conjugacy class graph of several families of finite non-abelian groups. As a consequence of our results, we show that the commuting conjugacy class graphs of the groups D_{2n}, T_{4n}, SD_{8n}, $U_{(n,m)}$, U_{6n}, V_{8n}, $G(p, m, n)$ and some families of groups whose central quotient is isomorphic to D_{2n} or $\mathbb{Z}_p \times \mathbb{Z}_p$, for some prime p, are CN-integral but not CN-hyperenergetic.

1. Introduction

Let G be a finite non-abelian group. The commuting conjugacy class graph of G, denoted by $\Gamma(G)$, is defined as a simple undirected graph whose vertex set is the set of conjugacy classes of the non-central elements of G and two vertices a^G and b^G are adjacent if there exist some elements $a' \in a^G$ and $b' \in b^G$ such that $a'b' = b'a'$. The study of commuting conjugacy class graphs of groups was initiated by Herzog, Longobardi and Maj [10] in the year 2009. In 2016, Mohammadian et al.[11] have characterized finite groups such that their commuting conjugacy class graphs are triangle-free. Later on Salahshour and Ashrafi [16, 15], obtained structures of commuting conjugacy class graphs of several families of finite CA-groups.

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Salahshour [14] also described $\Gamma(G)$ for the groups whose central quotient is isomorphic to a dihedral group. In [3], Bhowal and Nath have characterized certain finite groups such that $\Gamma(G)$ is hyperenergetic, L-hyperenergetic and Q-hyperenergetic. They also have characterized certain finite groups such that $\Gamma(G)$ is planar, toroidal, double-toroidal and triple-toroidal in [2].

Let $\mathcal{G}$ be a simple graph with vertex set $V(\mathcal{G}) := \{v_i : i = 1, 2, \ldots, n\}$. The common neighbourhood (abbreviated as CN) of two distinct vertices v_i and v_j, denoted by $C(v_i, v_j)$, is the set of all vertices other than v_i and v_j, which are adjacent to both v_i and v_j. The common neighbourhood matrix of $\mathcal{G}$, denoted by $\mathrm{CN}(\mathcal{G})$, is defined as

$$(\mathrm{CN}(\mathcal{G}))_{i,j} = \begin{cases} |C(v_i, v_j)|, & \text{if } i \neq j \\ 0, & \text{if } i = j. \end{cases}$$

The set of all eigenvalues of $\mathrm{CN}(\mathcal{G})$ with multiplicities, denoted by CN-spec$(\mathcal{G})$, is called the common neighbourhood spectrum (abbreviated as CN-spectrum) of $\mathcal{G}$. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the eigenvalues of $\mathrm{CN}(\mathcal{G})$ with multiplicities $\alpha_1, \alpha_2, \ldots, \alpha_k$ respectively then we write CN-spec$(\mathcal{G}) = \{(\lambda_1)^{\alpha_1}, (\lambda_2)^{\alpha_2}, \ldots, (\lambda_k)^{\alpha_k}\}$. A graph $\mathcal{G}$ is called CN-integral if CN-spec$(\mathcal{G})$ contains only integers. The common neighbourhood energy (abbreviated as CN-energy) of a graph $\mathcal{G}$, denoted by $\mathrm{E_{CN}}(\mathcal{G})$, is defined as

$$\mathrm{E_{CN}}(\mathcal{G}) = \sum_{i=1}^{n} \alpha_i |\lambda_i|.$$

A graph $\mathcal{G}$ is called CN-hyperenergetic if $\mathrm{E_{CN}}(\mathcal{G}) > \mathrm{E_{CN}}(K_n)$, where $n = |V(\mathcal{G})|$. If $\mathrm{E_{CN}}(\mathcal{G}) = \mathrm{E_{CN}}(K_n)$ then $\mathcal{G}$ is called CN-borderenergetic. In 2011, Alwardi, Soner and Gutman [1] introduced the concepts of CN-spectrum and CN-energy of a graph. Fasfous et al. [9] and Nath et al. [12] have computed CN-spectrum and CN-energy of commuting graphs of finite non-abelian groups, respectively. Various spectra and energies (spectrum, Laplacian spectrum, Signless Laplacian spectrum and their corresponding energies) of commuting graphs of finite groups are computed in [4, 7, 5, 6, 8, 17]. Recently, Rather et al. [13] have investigated the A_α-matrix and the A_α-spectrum for commuting graphs of dihedral, semidihedral and dicyclic groups.

In this paper, we compute the common neighbourhood spectrum and the common neighbourhood energy of commuting conjugacy class graph for several families of finite non-abelian groups. As a consequence of our results, we show that the commuting conjugacy class graphs of the groups D_{2n}, T_{4n}, SD_{8n}, $U_{(n,m)}$, U_{6n}, V_{8n}, $G(p, m, n)$

and some families of groups whose central quotient is isomorphic to D_{2n} or $\mathbb{Z}_p \times \mathbb{Z}_p$, for some prime p, are CN-integral but not CN-hyperenergetic.

2. Computation of CN-spectrum and CN-energy

In this section, we compute the CN-spectrum and the CN-energy of commuting conjugacy class graph for several families of finite groups. We write $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \ldots \cup \mathcal{G}_n$ to denote that a graph $\mathcal{G}$ has n components namely $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n$. Also, lK_m denotes the disjoint union of l copies of the complete graph K_m on m vertices. By [9, Theorem 1] and [12, Theorem 2] we get the following result which is very useful in computing the CN-spectrum and the CN-energy of commuting conjugacy class graphs of the groups considered in this paper.

Theorem 2.1. *Let $\mathcal{G} = l_1K_{m_1} \cup l_2K_{m_2} \cup l_3K_{m_3}$, where $l_iK_{m_i}$ denotes the disjoint union of l_i copies of the complete graphs K_{m_i} on m_i vertices for $i = 1, 2, 3$. Then*

$$\begin{aligned}\text{CN-spec}(\mathcal{G}) &= \{(-(m_1-2))^{l_1(m_1-1)}, ((m_1-1)(m_1-2))^{l_1}, \\ &\qquad (-(m_2-2))^{l_2(m_2-1)}, ((m_2-1)(m_2-2))^{l_2}, \\ &\qquad (-(m_3-2))^{l_3(m_3-1)}, ((m_3-1)(m_3-2))^{l_3}\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{E_{CN}}(\mathcal{G}) &= 2l_1(m_1-1)(m_1-2) + 2l_2(m_2-1)(m_2-2) \\ &\quad +2l_3(m_3-1)(m_3-2).\end{aligned}$$

2.1. **Certain 2-generated finite groups.** In this subsection, we consider dihedral groups, dicyclic groups, semidihedral groups along with some other 2-generated finite groups and compute the CN-spectrum and the CN-energy of their commuting conjugacy class graphs.

Theorem 2.2. *If G is the dihedral group*

$$D_{2n} = \langle x, y : \; x^n = y^2 = 1, \; yxy^{-1} = x^{-1} \rangle$$

then the CN-spectrum and the CN-energy of $\Gamma(G)$ are given by

$$\begin{aligned}&\text{CN-spec}(\Gamma(G)) \\ &= \begin{cases} \{(-\frac{1}{2}(n-5))^{\frac{1}{2}(n-3)}, (\frac{1}{4}(n-3)(n-5))^1, (0)^1\}, & \text{if } 2 \nmid n \\ \{(-\frac{1}{2}(n-6))^{\frac{1}{2}(n-4)}, (\frac{1}{4}(n-4)(n-6))^1, (0)^2\}, & \text{if } 2 \mid n \end{cases}\end{aligned}$$

and

$$\mathrm{E_{CN}}(\Gamma(G)) = \begin{cases} \frac{1}{2}(n-3)(n-5), & \text{if } 2 \nmid n \\ \frac{1}{2}(n-4)(n-6), & \text{if } 2 \mid n. \end{cases}$$

Proof. By [15, Proposition 2.1], we have

$$\Gamma(G)=\Gamma(D_{2n})=\begin{cases} K_{\frac{n-1}{2}}\cup K_1, & \text{if } 2\nmid n\\ K_{\frac{n}{2}-1}\cup 2K_1, & \text{if } 2\mid n \text{ and } \frac{n}{2} \text{ is even}\\ K_{\frac{n}{2}-1}\cup K_2, & \text{if } 2\mid n \text{ and } \frac{n}{2} \text{ is odd.}\end{cases}$$

Now, applying Theorem 2.1, we get

$$\text{CN-spec}(\Gamma(G))$$
$$=\begin{cases} \{(-(\frac{n-1}{2}-2))^{(\frac{n-1}{2}-1)}, ((\frac{n-1}{2}-1)(\frac{n-1}{2}-2))^1, (0)^1\}, & \text{if } 2\nmid n\\ \{(-(\frac{n}{2}-3))^{(\frac{n}{2}-2)}, ((\frac{n}{2}-2)(\frac{n}{2}-3))^1, (0)^2\}, & \text{if } 2\mid n\end{cases}$$

and

$$\text{E}_{\text{CN}}(\Gamma(G))=\begin{cases} 2(\frac{n-1}{2}-1)(\frac{n-1}{2}-2), & \text{if } 2\nmid n\\ 2(\frac{n}{2}-2)(\frac{n}{2}-3), & \text{if } 2\mid n.\end{cases}$$

Hence, the result follows on simplification. □

Theorem 2.3. *The CN-spectrum and the CN-energy of commuting conjugacy class graph of the dicyclic group*

$$T_{4n}=\langle x,y:\ x^{2n}=1, x^n=y^2, y^{-1}xy=x^{-1}\rangle$$

are given by

$$\text{CN-spec}(\Gamma(T_{4n}))=\{(-(n-3))^{(n-2)}, ((n-2)(n-3))^1, (0)^2\}$$

and

$$\text{E}_{\text{CN}}(\Gamma(T_{4n}))=2(n-2)(n-3).$$

Proof. By [15, Proposition 2.2], we have

$$\Gamma(T_{4n})=\begin{cases} K_{n-1}\cup 2K_1, & \text{if } 2\mid n\\ K_{n-1}\cup K_2, & \text{if } 2\nmid n.\end{cases}$$

Applying Theorem 2.1, we get

$$\text{CN-spec}(\Gamma(T_{4n}))=\{(-(n-3))^{(n-2)}, ((n-2)(n-3))^1, (0)^2\}$$

and $\text{E}_{\text{CN}}(\Gamma(T_{4n}))=2(n-2)(n-3)$. □

Theorem 2.4. *The CN-spectrum and the CN-energy of commuting conjugacy class graph of the semidihedral group*

$$SD_{8n}=\langle x,y:\ x^{4n}=y^2=1, yxy=x^{2n-1}\rangle$$

are given by

$$\text{CN-spec}(\Gamma(SD_{8n}))$$
$$=\begin{cases} \{(-(2n-3))^{(2n-2)}, ((2n-2)(2n-3))^1, (0)^2\}, & \textit{if } 2\mid n\\ \{(-(2n-4))^{(2n-3)}, ((2n-3)(2n-4))^1, (-2)^3, (6)^1\}, & \textit{if } 2\nmid n\end{cases}$$

and

$$\mathrm{E_{CN}}(\Gamma(SD_{8n})) = \begin{cases} 2(2n-2)(2n-3), & \textit{if } 2 \mid n \\ 2(2n-3)(2n-4)+12, & \textit{if } 2 \nmid n. \end{cases}$$

Proof. By [15, Proposition 2.5], we have

$$\Gamma(SD_{8n}) = \begin{cases} K_{2n-1} \cup 2K_1, & \text{if } 2 \mid n \\ K_{2n-2} \cup K_4, & \text{if } 2 \nmid n. \end{cases}$$

Now, applying Theorem 2.1, we get

$$\text{CN-spec}(\Gamma(SD_{8n}))=\begin{cases} \{(-((2n-1)-2))^{((2n-1)-1)}, \\ (((2n-1)-1)((2n-1)-2))^1, (0)^2\}, \text{ if } 2 \mid n \\ \\ \{(-((2n-2)-2))^{((2n-2)-1)}, \\ (((2n-2)-1)((2n-2)-2))^1, \\ (-(4-2))^{(4-1)}, ((4-1)(4-2))^1\}, \text{ if } 2 \nmid n \end{cases}$$

and

$$\mathrm{E_{CN}}(\Gamma(V_{8n}))$$
$$= \begin{cases} 2((2n-1)-1)((2n-1)-2) + 2\times 2(1-1)(1-2), & \text{if } 2 \mid n \\ 2((2n-2)-1)((2n-2)-2) + 2(4-1)(4-2), & \text{if } 2 \nmid n. \end{cases}$$

Hence, we get the required result on simplification. □

Theorem 2.5. *The CN-spectrum and the CN-energy of commuting conjugacy class graph of the group*

$$U_{(n,m)} = \langle x, y : \ x^{2n} = y^m = 1, \ x^{-1}yx = y^{-1}\rangle$$

are given by

$$\text{CN-spec}(\Gamma(U_{(n,m)})) = \begin{cases} \{(-(n-2))^{2(n-1)}, (n^2-3n+2)^2, \\ (-(\frac{1}{2}(mn-2n-4))^{\frac{1}{2}(mn-2n-2)}, \\ (\frac{1}{4}(mn-2n-2)(mn-2n-4))^1\}, \ \textit{if } 2 \mid m \\ \\ \{(-(n-2))^{(n-1)}, (n^2-3n+2)^1, \\ (-\frac{1}{2}(mn-n-4))^{\frac{1}{2}(mn-n-2)}, \\ (\frac{1}{4}(mn-n-2)(mn-n-4))^1\}, \quad \textit{if } 2 \nmid m \end{cases}$$

and

$$\mathrm{E_{CN}}(\Gamma(U_{(n,m)})) = \begin{cases} 4(n^2-3n+2)+\frac{1}{2}(mn-2n-2)(mn-2n-4), & \text{if } 2 \mid m \\ 2(n^2-3n+2)+\frac{1}{2}(mn-n-2)(mn-n-4), & \text{if } 2 \nmid m. \end{cases}$$

Proof. By [15, Proposition 2.3], we have

$$\Gamma(U_{(n,m)}) = \begin{cases} 2K_n \cup K_{n(\frac{m}{2}-1)}, & \text{if } 2 \mid m \\ K_n \cup K_{n(\frac{m-1}{2})}, & \text{if } 2 \nmid m. \end{cases}$$

Now, applying Theorem 2.1, we get

$$\text{CN-spec}(\Gamma(U_{(n,m)})) = \begin{cases} \{(-(n-2))^{2(n-1)}, ((n-1)(n-2))^2, \\ (-(n(\frac{m}{2}-1)-2))^{(n(\frac{m}{2}-1)-1)}, \\ ((n(\frac{m}{2}-1)-1)(n(\frac{m}{2}-1)-2))\}, \quad \text{if } 2 \mid m \\ \\ \{(-(n-2))^{(n-1)}, ((n-1)(n-2))^1, \\ (-(n(\frac{m-1}{2})-2))^{(n(\frac{m-1}{2})-1)}, \\ ((n(\frac{m-1}{2})-1)(n(\frac{m-1}{2})-2)\}, \quad \text{if } 2 \nmid m. \end{cases}$$

and

$$\mathrm{E_{CN}}(\Gamma(U_{(n,m)})) = \begin{cases} 2 \times 2(n-1)(n-2) \\ +2(n(\frac{m}{2}-1)-1))(n(\frac{m}{2}-1)-2), \text{ if } 2 \mid m \\ \\ 2(n-1)(n-2) \\ +2(n(\frac{m-1}{2})-1))(n(\frac{m-1}{2})-2), \text{ if } 2 \nmid m. \end{cases}$$

Hence, the result follows on simplification. □

Corollary 2.6. *The CN-spectrum and the CN-energy of commuting conjugacy class graph of the group*

$$U_{6n} = \langle x, y : \; x^{2n} = y^3 = 1, \; x^{-1}yx = y^{-1} \rangle$$

are given by

$$\text{CN-spec}(\Gamma(U_{6n})) = \{(-(n-2))^{2(n-1)}, ((n-1)(n-2))^2\}$$

and $\mathrm{E_{CN}}(\Gamma(U_{6n})) = 4(n-1)(n-2)$.

Proof. The result follows from Theorem 2.5 noting that $U_{(n,3)} = U_{6n}$. □

Theorem 2.7. *The CN-spectrum and the CN-energy of commuting conjugacy class graph of the group*

$$V_{8n} = \langle x, y : \ x^{2n} = y^4 = 1, \ yx = x^{-1}y^{-1}, \ y^{-1}x = x^{-1}y\rangle$$

are given by

$$\text{CN-spec}(\Gamma(V_{8n})) = \begin{cases} \{(-(2n-4))^{(2n-3)}, ((2n-3)(2n-4))^1, (0)^4\}, & \textit{if } 2 \mid n \\ \{(-(2n-3))^{(2n-2)}, ((2n-2)(2n-3))^1, (0)^2\}, & \textit{if } 2 \nmid n \end{cases}$$

and

$$\mathrm{E_{CN}}(\Gamma(V_{8n})) = \begin{cases} 2(2n-3)(2n-4), & \textit{if } 2 \mid n \\ 2(2n-2)(2n-3), & \textit{if } 2 \nmid n. \end{cases}$$

Proof. By [15, Proposition 2.4], we have

$$\Gamma(V_{8n}) = \begin{cases} K_{2n-2} \cup 2K_2, & \text{if } 2 \mid n \\ K_{2n-1} \cup 2K_1, & \text{if } 2 \nmid n. \end{cases}$$

Now, applying Theorem 2.1, we get

$$\text{CN-spec}(\Gamma(V_{8n})) = \begin{cases} \{(-((2n-2)-2))^{((2n-2)-1)}, \\ (((2n-2)-1)((2n-2)-2))^1, \\ (0)^2, (0)^2\}, \quad \text{if } 2 \mid n \\ \\ \{(-((2n-1)-2))^{((2n-1)-1)}, \\ (((2n-1)-1)((2n-1)-2))^1, (0)^2\}, \quad \text{if } 2 \nmid n \end{cases}$$

and

$$\mathrm{E_{CN}}(\Gamma(V_{8n})) = \begin{cases} 2((2n-2)-1)((2n-2)-2), & \text{if } 2 \mid n \\ 2((2n-1)-1)((2n-1)-2), & \text{if } 2 \nmid n. \end{cases}$$

Hence, the result follows on simplification. □

Theorem 2.8. *The CN-spectrum and the CN-energy of commuting conjugacy class graph of the group*

$$G(p, m, n) = \langle x, y : \ x^{p^m} = y^{p^n} = [x, y]^p = 1, [x, [x, y]] = [y, [x, y]] = 1\rangle$$

are given by

$$\begin{aligned}&\text{CN-spec}(\Gamma(G(p,m,n)))\\&=\Big\{(-(p^{m+n-1}-p^{m+n-2}-2))^{2(p^{m+n-1}-p^{m+n-2}-1)},\\&\quad((p^{m+n-1}-p^{m+n-2}-1)(p^{m+n-1}-p^{m+n-2}-2))^{2},\\&\quad(-(p^{m}-p^{m-1}-2))^{(p^{n}-p^{n-1})(p^{m}-p^{m-1}-1)},\\&\quad((p^{m}-p^{m-1}-1)(p^{m}-p^{m-1}-2))^{(p^{n}-p^{n-1})}\Big\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{E}_{\mathrm{CN}}(\Gamma(G(p,m,n)))&=4(p^{m+n-1}-p^{m+n-2}-1)(p^{m+n-1}-p^{m+n-2}-2)\\&\quad+2(p^{n}-p^{n-1})(p^{m}-p^{m-1}-1)(p^{m}-p^{m-1}-2).\end{aligned}$$

Proof. By [15, Proposition 2.6], we have

$$\begin{aligned}&\Gamma(G(p,m,n))\\&=K_{p^{m-1}(p^{n}-p^{n-1})}\cup K_{p^{n-1}(p^{m}-p^{m-1})}\cup(p^{n}-p^{n-1})K_{p^{m-n}(p^{n}-p^{n-1})}.\end{aligned}$$

Now, applying Theorem 2.1, we get

$$\begin{aligned}&\text{CN-spec}(\Gamma(G(p,m,n)))\\&=\Big\{(-(p^{m-1}(p^{n}-p^{n-1})-2))^{(p^{m-1}(p^{n}-p^{n-1})-1)},\\&\quad((p^{m-1}(p^{n}-p^{n-1})-1)(p^{m-1}(p^{n}-p^{n-1})-2))^{1},\\&\quad(-(p^{n-1}(p^{m}-p^{m-1})-2))^{(p^{n-1}(p^{m}-p^{m-1})-1)},\\&\quad((p^{n-1}(p^{m}-p^{m-1})-1)(p^{n-1}(p^{m}-p^{m-1})-2))^{1},\\&\quad(-(p^{m-n}(p^{n}-p^{n-1})-2))^{(p^{n}-p^{n-1})(p^{m-n}(p^{n}-p^{n-1})-1)},\\&\quad((p^{m-n}(p^{n}-p^{n-1})-1)(p^{m-n}(p^{n}-p^{n-1})-2))^{(p^{n}-p^{n-1})}\Big\}\end{aligned}$$

and

$$\begin{aligned}&\mathrm{E}_{\mathrm{CN}}(\Gamma(G(p,m,n)))\\&=2(p^{m-1}(p^{n}-p^{n-1})-1)(p^{m-1}(p^{n}-p^{n-1})-2)\\&\quad+2(p^{n-1}(p^{m}-p^{m-1})-1)(p^{n-1}(p^{m}-p^{m-1})-2)\\&\quad+2(p^{n}-p^{n-1})(p^{m-n}(p^{n}-p^{n-1})-1)(p^{m-n}(p^{n}-p^{n-1})-2).\end{aligned}$$

Hence, the result follows. □

2.2. **Certain groups with given central quotients.** In this subsection, we mainly consider finite groups whose central quotients are isomorphic to a group of order p^2, p^3 or a dihedral group of order $2n$ and compute the CN-spectrum and the CN-energy of their commuting conjugacy class graphs.

Theorem 2.9. *Let G be a non-abelian finite group with centre $Z = Z(G)$ and $\frac{G}{Z} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. Then*

$$\text{CN-spec}(\Gamma(G)) = \left\{(-(n-2))^{(p+1)(n-1)}, ((n-1)(n-2))^{(p+1)}\right\}$$

and

$$\mathrm{E}_{\mathrm{CN}}(\Gamma(G)) = 2(p+1)(n-1)(n-2),$$

where $n = \frac{(p-1)|Z|}{p}$.

Proof. By [16, Theorem 3.1], we have

$$\Gamma(G) = (p+1)K_n, \quad \text{where } n = \frac{(p-1)|Z|}{p}.$$

Hence, the result follows from Theorem 2.1. □

Corollary 2.10. *If G is a non-abelian p-group of order p^n and $|Z(G)| = p^{n-2}$, p is prime and $n \geq 3$, then*

$$\begin{aligned}\text{CN-spec}(\Gamma(G)) =&\Big\{(-(p^{n-2}-p^{n-3}-2))^{(p+1)(p^{n-2}-p^{n-3}-1)},\\ &((p^{n-2}-p^{n-3}-1)(p^{n-2}-p^{n-3}-2))^{(p+1)}\Big\}\end{aligned}$$

and

$$\mathrm{E}_{\mathrm{CN}}(\Gamma(G)) = 2(p+1)(p^{n-2}-p^{n-3}-1)(p^{n-2}-p^{n-3}-2).$$

Proof. We have $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $n = \frac{(p-1)|Z(G)|}{p} = (p-1)p^{n-3}$. Hence, the result follows from Theorem 2.9. □

Theorem 2.11. *Let G be a non-abelian group with centre Z such that $|\frac{G}{Z}| = p^3$, for a prime p. Then one of the following is satisfied:*

(a) *If $\frac{G}{Z}$ is abelian then*

(i)

$$\begin{aligned}\text{CN-spec}(\Gamma(G)) =&\Big\{(-(m-2))^{(m-1)}, ((m-1)(m-2))^{1},\\ &(-(n-2))^{p^2(n-1)}, ((n-1)(n-2))^{p^2}\Big\}\end{aligned}$$

and $\mathrm{E}_{\mathrm{CN}}(\Gamma(G)) = 2(m-1)(m-2) + 2p^2(n-1)(n-2)$ when $\Gamma(G) = K_m \cup p^2K_n$.

(ii) $\text{CN-spec}(\Gamma(G)) = \left\{(-(n-2))^{(p^2+p+1)(n-1)}, ((n-1)(n-2))^{(p^2+p+1)}\right\}$ *and*

$$\text{E}_{\text{CN}}(\Gamma(G)) = 2(p^2+p+1)(n-1)(n-2)$$

when $\Gamma(G) = (p^2+p+1)K_n$.

Here $m = \frac{(p^2-1)|Z|}{p}$ *and* $n = \frac{(p-1)|Z|}{p^2}$.

(b) *If* $\frac{G}{Z}$ *is non-abelian then*

(i)

$$\begin{aligned}\text{CN-spec}(\Gamma(G)) = \big\{&(-(m-2))^{(m-1)}, ((m-1)(m-2))^1,\\ &(-(n_1-2))^{kp(n_1-1)}, ((n_1-1)(n_1-2))^{kp},\\ &(-(n_2-2))^{(p-k)(n_2-1)}, ((n_2-1)(n_2-2))^{(p-k)}\big\}\end{aligned}$$

and

$$\begin{aligned}\text{E}_{\text{CN}}(\Gamma(G)) = &\, 2(m-1)(m-2) + 2kp(n_1-1)(n_1-2)\\ &+2(p-k)(n_2-1)(n_2-2)\end{aligned}$$

when $\Gamma(G) = K_m \cup kpK_{n_1} \cup (p-k)K_{n_2}$.

(ii)

$$\begin{aligned}\text{CN-spec}(\Gamma(G)) = \big\{&(-(n_1-2))^{(kp+1)(n_1-1)}, ((n_1-1)(n_1-2))^{(kp+1)},\\ &(-(n_2-2))^{(p+1-k)(n_2-1)}, ((n_2-1)(n_2-2))^{(p+1-k)}\big\}\end{aligned}$$

and

$$\begin{aligned}\text{E}_{\text{CN}}(\Gamma(G)) = &\, 2(kp+1)(n_1-1)(n_1-2)\\ &+2(p+1-k)(n_2-1)(n_2-2)\end{aligned}$$

when $\Gamma(G) = (kp+1)K_{n_1} \cup (p+1-k)K_{n_2}$.

(iii)

$$\begin{aligned}\text{CN-spec}(\Gamma(G)) = \big\{&(-(m-2))^{(m-1)}, ((m-1)(m-2))^1,\\ &(-(n_2-2))^{p(n_2-1)}, ((n_2-1)(n_2-2))^p\big\}\end{aligned}$$

and $\text{E}_{\text{CN}}(\Gamma(G)) = 2(m-1)(m-2) + 2p(n_2-1)(n_2-2)$ *when* $\Gamma(G) = K_m \cup pK_{n_2}$.

(iv)

$$\begin{aligned}&\text{CN-spec}(\Gamma(G))\\ &= \left\{(-(n_1-2))^{(p^2+p+1)(n_1-1)}, ((n_1-1)(n_1-2))^{(p^2+p+1)}\right\}\end{aligned}$$

and

$$\text{E}_{\text{CN}}(\Gamma(G)) = 2(p^2+p+1)(n_1-1)(n_1-2)$$

when $\Gamma(G) = (p^2+p+1)K_{n_1}$.

(v)

$$\text{CN-spec}(\Gamma(G)) = \left\{(-(n_1-2))^{(n_1-1)}, ((n_1-1)(n_1-2))^1, (-(n_2-2))^{(p+1)(n_2-1)}, ((n_2-1)(n_2-2))^{(p+1)}\right\}$$

and

$$\mathrm{E_{CN}}(\Gamma(G)) = 2(n_1-1)(n_1-2) + 2(p+1)(n_2-1)(n_2-2)$$

when $\Gamma(G) = K_{n_1} \cup (p+1)K_{n_2}$.

Here $m = \frac{(p^2-1)|Z|}{p}$, $n_1 = \frac{(p-1)|Z|}{p^2}$ *and* $n_2 = \frac{(p-1)|Z|}{p}$, $1 \leq k \leq p$.

Proof. (a) If $\frac{G}{Z}$ is abelian then, by [16, Theorem 3.3], we have

$$\Gamma(G) = K_m \cup p^2 K_n \text{ or } (p^2+p+1)K_n,$$

where $m = \frac{(p^2-1)|Z|}{p}$ and $n = \frac{(p-1)|Z|}{p^2}$.

If $\Gamma(G) = K_m \cup p^2 K_n$ then, using Theorem 2.1, we get

$$\text{CN-spec}(\Gamma(G)) = \left\{(-(m-2))^{(m-1)}, ((m-1)(m-2))^1, (-(n-2))^{p^2(n-1)}, ((n-1)(n-2))^{p^2}\right\}$$

and

$$\mathrm{E_{CN}}(\Gamma(G)) = 2(m-1)(m-2) + 2p^2(n-1)(n-2).$$

If $\Gamma(G) = (p^2+p+1)K_n$ then, using Theorem 2.1, we get

$$\text{CN-spec}(\Gamma(G)) = \left\{(-(n-2))^{(p^2+p+1)(n-1)}, ((n-1)(n-2))^{(p^2+p+1)}\right\}$$

and

$$\mathrm{E_{CN}}(\Gamma(G)) = 2(p^2+p+1)(n-1)(n-2).$$

This completes the proof of part (a).

(b) If $\frac{G}{Z}$ is non-abelian then, by [16, Theorem 3.3], we have $\Gamma(G) = K_m \cup kpK_{n_1} \cup (p-k)K_{n_2}$, $(kp+1)K_{n_1} \cup (p+1-k)K_{n_2}$, $K_m \cup pK_{n_2}$, $(p^2+p+1)K_{n_1}$ or $K_{n_1} \cup (p+1)K_{n_2}$, where $m = \frac{(p^2-1)|Z|}{p}$, $n_1 = \frac{(p-1)|Z|}{p^2}$, $n_2 = \frac{(p-1)|Z|}{p}$, $1 \leq k \leq p$.

If $\Gamma(G) = K_m \cup kpK_{n_1} \cup (p-k)K_{n_2}$ then, using Theorem 2.1, we get

$$\text{CN-spec}(\Gamma(G)) = \left\{(-(m-2))^{(m-1)}, ((m-1)(m-2))^1, (-(n_1-2))^{kp(n_1-1)}, ((n_1-1)(n_1-2))^{kp}, (-(n_2-2))^{(p-k)(n_2-1)}, ((n_2-1)(n_2-2))^{(p-k)}\right\}$$

and

$$\mathrm{E_{CN}}(\Gamma(G)) = 2(m-1)(m-2) + 2kp(n_1-1)(n_1-2) + 2(p-k)(n_2-1)(n_2-2).$$

If $\Gamma(G) = (kp+1)K_{n_1} \cup (p+1-k)K_{n_2}$ then, using Theorem 2.1, we get

$$\begin{aligned}\text{CN-spec}(\Gamma(G)) =\big\{&(-(n_1-2))^{(kp+1)(n_1-1)}, ((n_1-1)(n_1-2))^{(kp+1)},\\ &(-(n_2-2))^{(p+1-k)(n_2-1)}, ((n_2-1)(n_2-2))^{(p+1-k)}\big\}\end{aligned}$$

and

$$\text{E}_{\text{CN}}(\Gamma(G)) = 2(kp+1)(n_1-1)(n_1-2) + 2(p+1-k)(n_2-1)(n_2-2).$$

If $\Gamma(G) = K_m \cup pK_{n_2}$ then, using Theorem 2.1, we get

$$\begin{aligned}\text{CN-spec}(\Gamma(G)) =\big\{&(-(m-2))^{(m-1)}, ((m-1)(m-2))^{1},\\ &(-(n_2-2))^{p(n_2-1)}, ((n_2-1)(n_2-2))^{p}\big\}\end{aligned}$$

and

$$\text{E}_{\text{CN}}(\Gamma(G)) = 2(m-1)(m-2) + 2p(n_2-1)(n_2-2).$$

If $\Gamma(G) = (p^2+p+1)K_{n_1}$ then, using Theorem 2.1, we get

$$\text{CN-spec}(\Gamma(G)) = \big\{(-(n_1-2))^{(p^2+p+1)(n_1-1)}, ((n_1-1)(n_1-2))^{(p^2+p+1)}\big\}$$

and

$$\text{E}_{\text{CN}}(\Gamma(G)) = 2(p^2+p+1)(n_1-1)(n_1-2).$$

If $\Gamma(G) = K_{n_1} \cup (p+1)K_{n_2}$ then, using Theorem 2.1, we get

$$\begin{aligned}\text{CN-spec}(\Gamma(G)) =\big\{&(-(n_1-2))^{(n_1-1)}, ((n_1-1)(n_1-2))^{1},\\ &(-(n_2-2))^{(p+1)(n_2-1)}, ((n_2-1)(n_2-2))^{(p+1)}\big\}\end{aligned}$$

and

$$\text{E}_{\text{CN}}(\Gamma(G)) = 2(n_1-1)(n_1-2) + 2(p+1)(n_2-1)(n_2-2).$$

This completes the proof of part (b). □

Corollary 2.12. *Let G be a non-abelian p-group of order p^n and $|Z| = p^{n-3}$, where p is prime and $n \geq 4$. Then one of the following are satisfied:*

(a) *If $\frac{G}{Z}$ is abelian then*

(i)

$$\begin{aligned}\text{CN-spec}(\Gamma(G)) =\big\{&(-(p^{n-2}-p^{n-4}-2))^{(p^{n-2}-p^{n-4}-1)},\\ &((p^{n-2}-p^{n-4}-1)(p^{n-2}-p^{n-4}-2))^{1},\\ &(-(p^{n-4}-p^{n-5}-2))^{p^2(p^{n-4}-p^{n-5}-1)},\\ &((p^{n-4}-p^{n-5}-1)(p^{n-4}-p^{n-5}-2))^{p^2}\big\}\end{aligned}$$

and

$$\mathrm{E_{CN}}(\Gamma(G)) = 2(p^{n-2} - p^{n-4} - 1)(p^{n-2} - p^{n-4} - 2) + 2p^2(p^{n-4} - p^{n-5} - 1)(p^{n-4} - p^{n-5} - 2)$$

when $\Gamma(G) = K_{p^{n-4}(p^2-1)} \cup p^2 K_{p^{n-5}(p-1)}$.

(ii)

$$\text{CN-spec}(\Gamma(G)) = \big\{(-(p^{n-4} - p^{n-5} - 2))^{(p^2+p+1)(p^{n-4}-p^{n-5}-1)}, ((p^{n-4} - p^{n-5} - 1)(p^{n-4} - p^{n-5} - 2))^{(p^2+p+1)}\big\}$$

and

$$\mathrm{E_{CN}}(\Gamma(G)) = 2(p^2 + p + 1)(p^{n-4} - p^{n-5} - 1)(p^{n-4} - p^{n-5} - 2)$$

when $\Gamma(G) = (p^2 + p + 1)K_{p^{n-5}(p-1)}$.

(b) *If* $\frac{G}{Z}$ *is non-abelian then*

(i)

$$\begin{aligned}\text{CN-spec}(\Gamma(G)) = \big\{&(-(p^{n-2} - p^{n-4} - 2))^{(p^{n-2}-p^{n-4}-1)},\\ &((p^{n-2} - p^{n-4} - 1)(p^{n-2} - p^{n-4} - 2))^1,\\ &(-(p^{n-4} - p^{n-5} - 2))^{kp(p^{n-4}-p^{n-5}-1)},\\ &((p^{n-4} - p^{n-5} - 1)(p^{n-4} - p^{n-5} - 2))^{kp},\\ &(-(p^{n-3} - p^{n-4} - 2))^{(p-k)(p^{n-3}-p^{n-4}-1)},\\ &((p^{n-3} - p^{n-4} - 1)(p^{n-3} - p^{n-4} - 2))^{(p-k)}\big\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{E_{CN}}(\Gamma(G)) = &\ 2(p^{n-2} - p^{n-4} - 1)(p^{n-2} - p^{n-4} - 2)\\ &+ 2kp(p^{n-4} - p^{n-5} - 1)(p^{n-4} - p^{n-5} - 2)\\ &+ 2(p - k)(p^{n-3} - p^{n-4} - 1)(p^{n-3} - p^{n-4} - 2)\end{aligned}$$

when $\Gamma(G) = K_{p^{n-4}(p^2-1)} \cup kpK_{p^{n-5}(p-1)} \cup (p - k)K_{p^{n-4}(p-1)}$.

(ii)

$$\begin{aligned}\text{CN-spec}(\Gamma(G)) = \big\{&(-(p^{n-4} - p^{n-5} - 2))^{(kp+1)(p^{n-4}-p^{n-5}-1)},\\ &((p^{n-4} - p^{n-5} - 1)(p^{n-4} - p^{n-5} - 2))^{(kp+1)},\\ &(-(p^{n-3} - p^{n-4} - 2))^{(p+1-k)(p^{n-3}-p^{n-4}-1)},\\ &((p^{n-3} - p^{n-4} - 1)(p^{n-3} - p^{n-4} - 2))^{(p+1-k)}\big\}\end{aligned}$$

and

$$\mathrm{E}_{\mathrm{CN}}(\Gamma(G)) = 2(kp+1)(p^{n-4}-p^{n-5}-1)(p^{n-4}-p^{n-5}-2) + 2(p+1-k)(p^{n-3}-p^{n-4}-1)(p^{n-3}-p^{n-4}-2)$$

when $\Gamma(G) = (kp+1)K_{p^{n-5}(p-1)} \cup (p+1-k)K_{p^{n-4}(p-1)}$.

(iii)

$$\text{CN-spec}(\Gamma(G)) = \Big\{(-(p^{n-2}-p^{n-4}-2))^{(p^{n-2}-p^{n-4}-1)}, ((p^{n-2}-p^{n-4}-1)(p^{n-2}-p^{n-4}-2))^{1}, (-(p^{n-3}-p^{n-4}-2))^{p(p^{n-3}-p^{n-4}-1)}, ((p^{n-3}-p^{n-4}-1)(p^{n-3}-p^{n-4}-2))^{p}\Big\}$$

and

$$\mathrm{E}_{\mathrm{CN}}(\Gamma(G)) = 2(p^{n-2}-p^{n-4}-1)(p^{n-2}-p^{n-4}-2) + 2p(p^{n-3}-p^{n-4}-1)(p^{n-3}-p^{n-4}-2)$$

when $\Gamma(G) = K_{p^{n-4}(p^2-1)} \cup pK_{p^{n-4}(p-1)}$.

(iv)

$$\text{CN-spec}(\Gamma(G)) = \Big\{(-(p^{n-4}-p^{n-5}-2))^{(p^2+p+1)(p^{n-4}-p^{n-5}-1)}, ((p^{n-4}-p^{n-5}-1)(p^{n-4}-p^{n-5}-2))^{(p^2+p+1)}\Big\}$$

and $\mathrm{E}_{\mathrm{CN}}(\Gamma(G)) = 2(p^2+p+1)(p^{n-4}-p^{n-5}-1)(p^{n-4}-p^{n-5}-2)$ *when* $\Gamma(G) = (p^2+p+1)K_{p^{n-5}(p-1)}$.

(v)

$$\text{CN-spec}(\Gamma(G)) = \Big\{(-(p^{n-4}-p^{n-5}-2))^{(p^{n-4}-p^{n-5}-1)}, ((p^{n-4}-p^{n-5}-1)(p^{n-4}-p^{n-5}-2))^{1}, (-(p^{n-3}-p^{n-4}-2))^{(p+1)(p^{n-3}-p^{n-4}-1)}, ((p^{n-3}-p^{n-4}-1)(p^{n-3}-p^{n-4}-2))^{(p+1)}\Big\}$$

and

$$\mathrm{E}_{\mathrm{CN}}(\Gamma(G)) = 2(p^{n-4}-p^{n-5}-1)(p^{n-4}-p^{n-5}-2) + 2(p+1)(p^{n-3}-p^{n-4}-1)(p^{n-3}-p^{n-4}-2)$$

when $\Gamma(G) = K_{p^{n-5}(p-1)} \cup (p+1)K_{p^{n-4}(p-1)}$.

Here $1 \leq k \leq p$.

Proof. We have $|\frac{G}{Z}| = p^3$ and $m = \frac{(p^2-1)|Z|}{p} = (p^2-1)p^{n-4}$, $n = n_1 = \frac{(p-1)|Z|}{p^2} = (p-1)p^{n-5}$ and $n_2 = \frac{(p-1)|Z|}{p} = (p-1)p^{n-4}$. Hence, the result follows from Theorem 2.11. □

Corollary 2.13. *Let G be a non-abelian p-group of order p^4. Then*

(a)

$$(\Gamma(G)) = \big\{(-(p^2-p-2))^{(p+1)(p^2-p-1)}, \\ ((p^2-p-1)(p^2-p-2))^{(p+1)}\big\}$$

and

$$\mathrm{E_{CN}}(\Gamma(G)) = 2(p+1)(p^2-p-1)(p^2-p-2)$$

if $\Gamma(G) = (p+1)K_{p(p-1)}$.

(b)

$$\text{CN-spec}(\Gamma(G)) = \big\{(-(p^2-3))^{(p^2-2)}, ((p^2-2)(p^2-3))^1, \\ (-(p-3))^{(p^2-2p)}, ((p-2)(p-3))^p\big\}$$

and

$$\mathrm{E_{CN}}(\Gamma(G)) = 2(p^2-2)(p^2-3) + 2p(p-2)(p-3)$$

if $\Gamma(G) = K_{(p^2-1)} \cup pK_{p-1}$.

Proof. If G is a non-abelian p-group of order p^4 then $|Z(G)| = p$ or p^2. Suppose that $|Z(G)| = p^2$. Then by Corollary 2.10, we get

$$\text{CN-spec}(\Gamma(G)) \\ = \big\{(-(p^2-p-2))^{(p+1)(p^2-p-1)}, ((p^2-p-1)(p^2-p-2))^{(p+1)}\big\}$$

and

$$\mathrm{E_{CN}}(\Gamma(G)) = 2(p+1)(p^2-p-1)(p^2-p-2).$$

In this case, by [16, Corollary 3.2], it follows that $\Gamma(G) = (p+1)K_{p(p-1)}$.

If $|Z(G)| = p$ then, by [16, Corollary 3.5], we have

$$\Gamma(G) = K_{(p^2-1)} \cup pK_{p-1}.$$

Using Corollary 2.12(b)(iii), we get

$$\text{CN-spec}(\Gamma(G)) = \big\{(-(p^2-1-2))^{(p^2-1-1)}, ((p^2-1-1)(p^2-1-2))^1, \\ (-(p-1-2))^{p(p-1-1)}, ((p-1-1)(p-1-2))^p\big\}$$

and

$$\mathrm{E_{CN}}(\Gamma(G)) = 2(p^2-1-1)(p^2-1-2) + 2p(p-1-1)(p-1-2).$$

After some simplification we get our required result. □

Theorem 2.14. *Let G be a non-abelian finite group with center $Z = Z(G)$ and $|Z| = z$. If $\frac{G}{Z} \cong D_{2n}$. Then*

$$\text{CN-spec}(\Gamma(G)) = \begin{cases} \Big\{\Big(-\big(\frac{(n-1)z}{2}-2\big)\Big)^{\left(\frac{(n-1)z}{2}-1\right)}, \Big(\big(\frac{(n-1)z}{2}\big)^2 - \frac{3(n-1)z}{2}+2\Big)^1, \\ \Big(-\big(\frac{z}{2}-2\big)\Big)^{2\left(\frac{z}{2}-1\right)}, \Big(\big(\frac{z}{2}\big)^2-\frac{3z}{2}+2\Big)^2\Big\}, \text{ if } 2 \mid n \\ \Big\{\Big(-\big(\frac{(n-1)z}{2}-2\big)\Big)^{\left(\frac{(n-1)z}{2}-1\right)}, \Big(\big(\frac{(n-1)z}{2}\big)^2 - \frac{3(n-1)z}{2}+2\Big)^1, \\ \big(-(z-2)\big)^{(z-1)}, (z^2-3z+2)^1\Big\}, \text{ if } 2 \nmid n \end{cases}$$

and

$$E_{\text{CN}}(\Gamma(G)) = \begin{cases} \frac{n^2z^2}{2} - nz^2 - 3nz + \frac{3z^2}{2} - 3z + 12, & \text{if } 2 \mid n \\ \frac{n^2z^2}{2} - nz^2 - 3nz + \frac{5z^2}{2} - 3z + 8, & \text{if } 2 \nmid n. \end{cases}$$

Proof. By [14, Theorem 1.2], we have

$$\Gamma(G) = \begin{cases} K_{\frac{(n-1)z}{2}} \cup 2K_{\frac{z}{2}}, & \text{if } 2 \mid n \\ K_{\frac{(n-1)z}{2}} \cup K_z, & \text{if } 2 \nmid n. \end{cases}$$

Now, applying Theorem 2.1, we get

$$\text{CN-spec}(\Gamma(G)) = \begin{cases} \Big\{\Big(-\Big(\frac{(n-1)z}{2}-2\Big)\Big)^{\left(\frac{(n-1)z}{2}-1\right)}, \Big(\Big(\frac{(n-1)z}{2}-2\Big)\Big(\frac{(n-1)z}{2}-1\Big)\Big)^1, \\ \qquad \big(-\big(\frac{z}{2}-2\big)\big)^{2\left(\frac{z}{2}-1\right)}, \big(\big(\frac{z}{2}-2\big)\big(\frac{z}{2}-1\big)\big)^2\Big\}, \text{ if } 2 \mid n \\ \Big\{\Big(-\Big(\frac{(n-1)z}{2}-2\Big)\Big)^{\left(\frac{(n-1)z}{2}-1\right)}, \Big(\Big(\frac{(n-1)z}{2}-2\Big)\Big(\frac{(n-1)z}{2}-1\Big)\Big)^1, \\ \qquad (-(z-2))^{(z-1)}, ((z-2)(z-1))^1\Big\}, \text{ if } 2 \nmid n \end{cases}$$

and

$$E_{\text{CN}}(\Gamma(G)) = \begin{cases} 2(\frac{(n-1)z}{2}-1)(\frac{(n-1)z}{2}-2) + 2 \times 2(\frac{z}{2}-1)(\frac{z}{2}-2), & \text{if } 2 \mid n \\ 2(\frac{(n-1)z}{2}-1)(\frac{(n-1)z}{2}-2) + 2(z-1)(z-2), & \text{if } 2 \nmid n. \end{cases}$$

Hence the result follows on simplification. □

We conclude this section with the following remark.

Remark 2.15. We have $\frac{D_{2n}}{Z(D_{2n})} \cong D_{2\times\frac{n}{2}}$ or D_{2n} according as n is even or odd, $\frac{T_{4n}}{Z(T_{4n})} \cong D_{2n}$ and $\frac{U_{6n}}{Z(U_{6n})} = D_{2\times 3}$. Therefore, Theorem 2.2, Theorem 2.3 and Corollary 2.6 can also be obtained from Theorem 2.14.

3. Some consequences

We begin this section with the following consequence of the results obtained in Section 2.

Theorem 3.1. (a) *If G is isomorphic to D_{2n}, T_{4n}, SD_{8n}, $U_{(n,m)}$, U_{6n}, V_{8n} or $G(p,m,n)$ then commuting conjugacy class graph of G is CN-integral.*

(b) *If G is a finite group and $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ or D_{2n} then commuting conjugacy class graph of G is CN-integral.*

(c) *If G is a finite group and $\frac{G}{Z(G)}$ is of order p^3 then commuting conjugacy class graph of G is CN-integral.*

Theorem 3.2. *Let G be the semidihedral group SD_{8n} $(n \geq 2)$. Then the commuting conjugacy class graph of G is not CN-hyperenergetic.*

Proof. We have $|V(\Gamma(G))| = 2n+1$ or $2n+2$ according as n is even or odd. By Theorem 2.1 and Theorem 2.4, we get

$$\begin{aligned} &\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) \\ &= \begin{cases} 4n(2n-1) - 2(2n-3)(2n-2), & \text{if } 2 \mid n \\ 4n(2n+1) - 2(2n-4)(2n-3) - 12, & \text{if } 2 \nmid n. \end{cases} \end{aligned}$$

Since $n \geq 2$ we have

$$4n(2n-1) - 2(2n-3)(2n-2) = 2(8n-6) > 0$$

and

$$4n(2n+1) - 2(2n-4)(2n-3) - 12 = 2(16n-18) > 0.$$

Therefore, $\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) > \mathrm{E_{CN}}(\Gamma(G))$ and so $\Gamma(G)$ is not CN-hyperenergetic. □

Theorem 3.3. *Let G be the group $U_{(n,m)}$ $(n \geq 2$ and $m \geq 3)$. Then the commuting conjugacy class graph of G is not CN-hyperenergetic.*

Proof. We have $|V(\Gamma(G))| = n + \frac{nm}{2}$ or $\frac{n}{2} + \frac{nm}{2}$ according as m is even or odd. If m is even then, by Theorem 2.1 and Theorem 2.5, we have

$$\begin{aligned}
&\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) \\
&= \frac{1}{2}(2n + nm - 2)(2n + nm - 4) \\
&\quad - 4(n^2 - 3n + 2) - \frac{1}{2}(mn - 2n - 2)(mn - 2n - 4) \\
&= 4(n^2(m-1) - 2) > 0,
\end{aligned}$$

since $n \geq 2$ and $m \geq 3$. If m is odd then, by Theorem 2.1 and Theorem 2.5, we have

$$\begin{aligned}
&\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) \\
&= \frac{1}{2}(n + nm - 2)(n + nm - 4) \\
&\quad - 2(n^2 - 3n + 2) - \frac{1}{2}(mn - n - 2)(mn - n - 4) \\
&= 2n^2(m-1) - 4 \\
&\geq 0,
\end{aligned}$$

since $n \geq 2$ and $m \geq 3$. Hence, the result follows. □

Theorem 3.4. *Let G be the group V_{8n}, where $n \geq 2$. Then the commuting conjugacy class graph of G is not CN-hyperenergetic.*

Proof. We have $|V(\Gamma(G))| = 2n + 2$ or $2n + 1$ according as n is even or odd. By Theorem 2.1 and Theorem 2.7, we get

$$\begin{aligned}
&\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) \\
&= \begin{cases} 4n(2n+1) - 2(2n-3)(2n-4), & \text{if } 2 \mid n \\ 4n(2n-1) - 2(2n-3)(2n-2), & \text{if } 2 \nmid n. \end{cases}
\end{aligned}$$

Since $n \geq 2$ we have

$$4n(2n+1) - 2(2n-3)(2n-4) = 8(4n-3) > 0$$

and

$$4n(2n-1) - 2(2n-3)(2n-2) = 4(4n-3) > 0.$$

Therefore,

$$\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) > \mathrm{E_{CN}}(\Gamma(G))$$

and so $\Gamma(G)$ is not CN-hyperenergetic. □

Theorem 3.5. *The commuting conjugacy class graph of the group $G(p, m, n)$, where p is a prime and $m, n \geq 1$, is not CN-hyperenergetic.*

Proof. We have $|V(\Gamma(G(p,m,n)))| = p^{m+n} - p^{m+n-2}$. Using Theorem 2.1 and Theorem 2.8, we get

$$\begin{aligned} \mathrm{E_{CN}}(K_{|V(\Gamma(G(p,m,n)))|} - \mathrm{E_{CN}}(\Gamma(G(p,m,n))) &= 2p^{2m+n-3} - 6p^{2m+n-2} \\ &+ 6p^{2m+n-1} - 2p^{2m+n} \\ &- 2p^{2m+2n-4} + 8p^{2m+2n-3} \\ &- 8p^{2m+2n-2} + 2p^{2m+2n} \\ &+ 4p^{n-1} - 4p^n - 4 \\ &:= \beta. \end{aligned}$$

Let $S_1 = 8p^{2m+2n-3} + 2p^{2m+2n} - 2p^{2m+2n-4} - 8p^{2m+2n-2}$ and $S_2 = 6p^{2m+n-1} - 6p^{2m+n-2} - 4$. Then

$$S_1 = p^{2m+2n-4}(2p^2(p^2-4) + (8p-2)) > 0,$$

and

$$S_2 = 6p^{2m+n-1}(1-p^{-1}) - 4 = 6p^{2m+n-2}(p-1) - 4 \geq 8,$$

since $p^2 - 4 \geq 0$, $8p - 2 \geq 14$ and $6p^{2m+n-2}(p-1) \geq 12$.

Let $S_3 = 2p^{2m+n-3} + 4p^{n-1} - 4p^n = 2p^{n-3}(p^{2m} + 2p^2 - 2p^3)$. If $m \geq 2$ then $2m - 3 \geq 1 \implies p^{2m-3} - 2 \geq 0$. So,

$$p^{2m} + 2p^2 - 2p^3 = p^3(p^{2m-3} - 2) + 2p^2 \geq 0.$$

Hence $S_3 \geq 0$.

Let $S_4 = S_1 - 2p^{2m+n} = p^{2m+n}(8p^{n-3} + 2p^n - 2p^{n-4} - 8p^{n-2} - 2)$. Then $8p^{n-3} + 2p^n - 2p^{n-4} - 8p^{n-2} - 2 = 2p^{n-4}(p^2(p^2-4) + 4p - 1) - 2 \geq 12$, if $n \geq 4$ (in this case $2p^{n-4} \geq 2$ and $p^2(p^2-4) + 4p - 1 \geq 7$). Therefore $S_4 > 0$. Hence, $\beta = S_2 + S_3 + S_4 > 0$, for all $m \geq 2$ and $n \geq 4$.

We shall now consider the following cases:

Case 1. $m \geq 2$ and $n = 1$.

In this case $\beta = 2p^{2m-1}(p-1)^2(p+1) - 4p > 0$, since $p + 1 \geq 3$ and $2m - 1 \geq 3$.

Case 2. $m \geq 2$ and $n = 2$.

In this case $\beta = 2p^{2m-1}(p^3(p^2-5) + p(7p-4) + 1) - 4(p^2 - p + 1)$. We have $2p^{2m-1} \geq 16$, since $m \geq 2$. If $p = 2$ then $\beta = 13.2^{2m} - 12 > 0$. Suppose that $p \geq 3$. Then $p^2 - 5 \geq 4$, $7p - 4 \geq 17$ and $p^3 \geq p^2$. Hence, $\beta > 0$.

Case 3. $m \geq 2$ and $n = 3$.

In this case

$$\beta = 2p^{2m}(1 + 3p(p^2-1) + 2p^2) + 2p^{2m+4}(p^2-4) - 4p^3 + 4(p^2-1).$$

If $p = 2$ then $\beta = 27.2^{2m+1} - 20 > 0$. If $p \geq 3$ then $p^4(p^2-4) > p^3$ and $2p^{2m} > 4$. Therefore, $2p^{2m+4}(p^2-4) > 4p^3$. Hence $\beta > 0$.

Case 4. $m = 1$ and $n \geq 1$.

In this case

$$\begin{aligned}\beta &= 6p^{n-1} - 4 + 2p^{n+1}(3 - 5p^{-1}) \\ &\quad + p^{n+2}(-2 - 2p^{n-4} + 8p^{n-3} - 8p^{n-2} + 2p^{n}) \\ &= 6p^{n-1} - 4 + 2p^{n+1}(3 - 5p^{-1}) \\ &\quad + p^{n+2}(2p^{n-4}(p^2(p^2 - 4) + 4p - 1) - 2).\end{aligned}$$

Note that $\beta > 0$, if $n \geq 4$. If $n = 1$ then

$$\beta = 2p^4 - 2p^3 - 2p^2 - 2p = 2p(p(p(p-1) - 1) - 1) > 0,$$

since $p - 1 \geq 1$. If $n = 2$ then

$$\begin{aligned}\beta &= 6p - 12p^2 + 14p^3 - 10p^4 + 2p^6 - 4 \\ &= 2p^3(7 - 6p^{-1}) + 2p^6(1 + 3p^{-5} - 5p^{-2}) - 4.\end{aligned}$$

It is clear that $\beta > 0$, if $p \geq 3$. If $p = 2$ then $\beta = 40 > 0$. If $n = 3$ then

$$\beta = (6p^2 - 4) + 2p^4(2 - 5p^{-1} + 3p) + 2p^8(1 - 4p^{-2}) > 0.$$

Hence,

$$\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) > \mathrm{E_{CN}}(\Gamma(G))$$

and so $\Gamma(G)$ is not CN-hyperenergetic. □

Theorem 3.6. *Let G be a finite group with centre Z and $|Z| = z$ such that $\frac{G}{Z} \cong D_{2n}$. Then the commuting conjugacy class graph of G is not CN-hyperenergetic. In particular, if $G \cong D_6$ then $\Gamma(G)$ is CN-borderenergetic.*

Proof. If n is even then $|V(\Gamma(G))| = \frac{(n-1)z}{2} + 2 \times \frac{z}{2} = \frac{nz}{2} + \frac{z}{2}$. By Theorem 2.1 and Theorem 2.14, we get

$$\begin{aligned}\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) &= 2\left(\frac{nz}{2} + \frac{z}{2} - 1\right)\left(\frac{nz}{2} + \frac{z}{2} - 2\right) \\ &\quad - \left\{\frac{n^2z^2}{2} - nz^2 - 3nz + \frac{3z^2}{2} - 3z + 12\right\} \\ &= z^2(2n - 1) - 8 \\ &> 0,\end{aligned}$$

since $n \geq 3$ and $z \geq 2$. If n is odd then $|V(\Gamma(G))| = \frac{(n-1)z}{2} + z = \frac{nz}{2} + \frac{z}{2}$. By Theorem 2.1 and Theorem 2.14, we get

$$\begin{aligned} \mathrm{E}_{\mathrm{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E}_{\mathrm{CN}}(\Gamma(G)) &= 2\left(\frac{nz}{2} + \frac{z}{2} - 1\right)\left(\frac{nz}{2} + \frac{z}{2} - 2\right) \\ &\quad - \left\{\frac{n^2z^2}{2} - nz^2 - 3nz + \frac{5z^2}{2} - 3z + 8\right\} \\ &= 2z^2(n-1) - 4. \end{aligned}$$

We have

$$2z^2(n-1) - 4 \begin{cases} = 0, & \text{for } n = 3, z = 1 \\ > 0, & \text{otherwise.} \end{cases}$$

Hence the result follows. □

In view of Theorem 3.6 and Remark 2.15 we get the following corollary.

Corollary 3.7. *The commuting conjugacy class graph of*

(a) *the dihedral group D_{2m}, where $m \geq 3$, is not CN-hyperenergetic.*
(b) *the dicyclic group T_{4n}, where $n \geq 2$, is not CN-hyperenergetic.*
(c) *the group U_{6n} is not CN-hyperenergetic.*

Theorem 3.8. *Let G be a non-abelian finite group with centre $Z(G)$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. Then the commuting conjugacy class graph of G is not CN-hyperenergetic.*

Proof. We have $|V(\Gamma(G))| = np + n$, where $n = \frac{(p-1)|Z(G)|}{p}$. By Theorem 2.1 and Theorem 2.9, we get

$$\begin{aligned} \mathrm{E}_{\mathrm{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E}_{\mathrm{CN}}(\Gamma(G)) &= 2(np + n - 1)(np + n - 2) \\ &\quad - 2(n-1)(n-2)(p+1) \\ &= 2n^2p + 2p(n^2p - 2) \\ &> 0, \end{aligned}$$

since $p \geq 2$ and n is a positive integer. Hence, $\Gamma(G)$ is not CN-hyperenergetic. □

Corollary 3.9. *Let G be a non-abelian p-group of order p^n and $|Z(G)| = p^{n-2}$, where p is a prime and $n \geq 3$. Then the commuting conjugacy class graph of G is not CN-hyperenergetic.*

Proof. If G is a non-abelian p-group of order p^n and $|Z(G)| = p^{n-2}$ then $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, p is prime and $n \geq 3$. Hence, the result follows from Theorem 3.8. □

Theorem 3.10. *Let G be a non-abelian p-group of order p^4. Then $\Gamma(G)$ is not CN-hyperenergetic.*

Proof. By Theorem 2.1 and Corollary 2.13, we get

$$\begin{aligned}&\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G))\\ &= \begin{cases} 2p(p^2(p-1)(p^2-1)-2) := \mu_1(p), & \text{if } \Gamma(G) = (p+1)K_{p(p-1)} \\ 2p((p-1)+p^2(3p-5)) := \mu_2(p), & \text{if } \Gamma(G) = K_{p^2-1} \cup pK_{p-1} \end{cases}\end{aligned}$$

noting that $|V((p+1)K_{p(p-1)})| = p^3 - p$ and

$$|V(K_{p^2-1} \cup pK_{p-1})| = 2p^2 - p - 1.$$

Case 1. $\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) = \mu_1(p)$.

Since $p \geq 2$ we have $p^2 \geq 4$, $p-1 \geq 1$, $p^2-1 \geq 3$ and so $p^2(p-1)(p^2-1) \geq 12$. Therefore, $p^2(p-1)(p^2-1)-2 \geq 10$. Hence, Therefore, $\mu_1(p) > 0$.

Case 2. $\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) = \mu_2(p)$.

Since $p \geq 2$ we have $p-1 > 0$ and $3p-5 > 0$ and so $\mu_2(p) > 0$. □

Theorem 3.11. *Let G be a non-abelian p-group of order p^n and $|Z(G)| = p^{n-3}$, where p is a prime and $n \geq 4$.*

(a) *If $\frac{G}{Z(G)}$ is abelian then $\Gamma(G)$ is not CN-hyperenergetic.*

(b) *If $\frac{G}{Z(G)}$ is non-abelian then $\Gamma(G)$ is not CN-hyperenergetic.*

Proof. (a) If $\frac{G}{Z(G)}$ is abelian then, by Theorem 2.1 and Corollary 2.12, we get

$$\begin{aligned}&\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G))\\ &= \begin{cases} -4p^2 + 2p^{2n-6}(p^2-2) + 2p^{2n-8}(2p^3(p-2)+4p-1) := \beta_1(p,n), \\ \quad \text{when } \Gamma(G) = K_{p^{n-4}(p^2-1)} \cup p^2K_{p^{n-5}(p-1)} \\ 2p^{2n-7}(p^3-p-1) - (4p^2+4p) + 2p^{2n-9} := \beta_2(p,n), \\ \quad \text{when } \Gamma(G) = (p^2+p+1)K_{p^{n-5}(p-1)} \end{cases}\end{aligned}$$

noting that $|V(K_{p^{n-4}(p^2-1)} \cup p^2K_{p^{n-5}(p-1)})| = 2p^{n-2} - p^{n-3} - p^{n-4}$ and

$$|V((p^2+p+1)K_{p^{n-5}(p-1)})| = p^{n-2} - p^{n-5}.$$

Case 1. $\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) = \beta_1(p,n)$.

In this case $p \geq 2$ and $n \geq 5$. Since $2p^3(p-2)+4p-1 > 0$ and $2p^{2n-6}(p^2-2) \geq 4p^2$ we have $\beta_1(p,n) > 0$.

Case 2. $\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) = \beta_2(p,n)$.

In this case $p \geq 2$ and $n \geq 5$. Since $p^{2n-7} \geq p^2$ and $p^3 - p - 1 > 4$ we have $p^{2n-7}(p^3 - p - 1) > 4p^2$ and so $p^{2n-7}(p^3 - p - 1) > 4p$. Therefore, $2p^{2n-7}(p^3 - p - 1) > 4p + 4p^2$. Hence, $\beta_2(p, n) > 0$.

(b) If $\frac{G}{Z}$ is non-abelian then, by Theorem 2.1 and Corollary 2.12, we get

$$\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) =$$

$$\begin{cases} 2kp^{2n-8}(3 - p^{-1}) + 2kp^{2n-6}(1 - 3p^{-1}) + 2p^{2n-4}(3 - 5p^{-1}) \\ +4k + 2p(p^{2n-8} - 2) + 2p(p^{2n-7} - 2k) := \mu_1(p, n, k), \\ \text{ when } \Gamma(G) = K_{p^{n-4}(p^2-1)} \cup kpK_{p^{n-5}(p-1)} \cup (p - k)K_{p^{n-4}(p-1)} \\ 4(k - 1) + \{2p^{2n-4}(1 - p^{-1}) - 4p^{2n-8}\} + \{2p^{2n-6}(1 - p^{-1}) - 4p\} \\ +\{2kp^{2n-6}(1 - 3p^{-1}) - 4kp\} + 2kp^{2n-8}(3 - p^{-1}) \\ +4p^{2n-9} := \mu_2(p, n.k), \\ \text{ when } \Gamma(G) = (kp + 1)K_{p^{n-5}(p-1)} \cup (p + 1 - k)K_{p^{n-4}(p-1)} \\ 2p^{2n-4}(3 - 5p^{-1}) + 2p(p^{2n-8} + p^{2n-7} - 2) := \mu_3(p, n), \\ \text{ when } \Gamma(G) = K_{p^{n-4}(p^2-1)} \cup pK_{p^{n-4}(p-1)} \\ 2p^{2n-7}(p^3 - p - 1) - (4p^2 + 4p) + 2p^{2n-9} := \mu_4(p, n), \\ \text{ when } \Gamma(G) = (p^2 + p + 1)K_{p^{n-5}(p-1)} \\ 4p^{2n-9} - 4 + 2p^{2n-7}(p - 1) - 4p + 2p^{2n-4}(1 - p^{-1} - 2p^{-4}) \\ := \mu_5(p, n), \text{ when } \Gamma(G) = K_{p^{n-5}(p-1)} \cup (p + 1)K_{p^{n-4}(p-1)} \end{cases}$$

noting that

$$\begin{aligned} &|V(K_{p^{n-4}(p^2-1)} \cup kpK_{p^{n-5}(p-1)} \cup (p - k)K_{p^{n-4}(p-1)})| \\ &= -p^{n-4} - p^{n-3} + 2p^{n-2}, \\ &|V((kp + 1)K_{p^{n-5}(p-1)} \cup (p + 1 - k)K_{p^{n-4}(p-1)})| = p^{n-2} - p^{n-5}, \\ &|V(K_{p^{n-4}(p^2-1)} \cup pK_{p^{n-4}(p-1)})| = -p^{n-4} - p^{n-3} + 2p^{n-2}, \\ &|V((p^2 + p + 1)K_{p^{n-5}(p-1)})| = p^{n-2} - p^{n-5} \end{aligned}$$

and $|V(K_{p^{n-5}(p-1)} \cup (p + 1)K_{p^{n-4}(p-1)})| = p^{n-2} - p^{n-5}$.

Case 1. $\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) = \mu_1(p, n, k)$.

In this case $p \geq 2$ and $n \geq 5$. For $p \geq 3$ and $n \geq 5$ we have $1 \geq 3p^{-1}$, $p^{2n-8} > 2$ and $p^{2n-7} \geq 2k$. Therefore, $\mu_1(p, n, k) > 0$.

If $p = 2$ and $n \geq 5$ then

$$\mu_1(2, n, k) = (8kn - 5) + (5k \times 2^{2n-8} - 4k) + 7 \times 2^{2n-6} > 0,$$

since $k = 1, 2$ and $2^{2n-8} \geq 4$.

Case 2. $\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) = \mu_2(p,n,k)$.

In this case $p \geq 2$ and $n \geq 5$. For $p \geq 5$ we have

$$2p^{2n-4}(1-p^{-1}) - 4p^{2n-8} = 2p^{2n-5}(p-1) - 4p^{2n-8} \geq 0,$$
$$2p^{2n-6}(1-p^{-1}) - 4p = 2p^{2n-7}(p-1) - 4p > 0$$

and

$$2kp^{2n-6}(1-3p^{-1}) - 4kp = 2kp^{2n-7}(p-3) - 4kp \geq 0.$$

Therefore, $\mu_2(p,n,k) > 0$. For $p = 3$ and $n \geq 5$ we have

$$\begin{aligned}\mu_2(3,n,k) =& -16 - 8k + 3^{2n-9}(352 + 16k)\\ =&(352 \times 3^{2n-9} - 16) + (16k3^{2n-9} - 8k).\end{aligned}$$

Since $n \geq 5$, $3^{2n-9} \geq 3$ and so $\mu_2(3,n,k) > 0$. For $p = 2$ and $n \geq 5$ we have

$$\begin{aligned}\mu_2(2,n,k) =& -12 - 4k + 2^{2n-8}k + 18 \times 2^{2n-8}\\ =&(18 \times 2^{2n-8} - 12) + (2^{2n-8}k - 4k).\end{aligned}$$

Since $n \geq 5$, $2^{2n-8} \geq 4$ and so $\mu_2(2,n,k) > 0$.

Case 3. $\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) = \mu_3(p,n)$. In this case $p \geq 2$ and $n \geq 4$. Therefore, $3 - 5p^{-1} > 0$ and $p^{2n-8} + p^{2n-7} - 2 > 0$. Hence, $\mu_3(p,n) > 0$.

Case 4. $\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) = \mu_4(p,n)$.

As shown in Case 2 of part (a), we have $\mu_4(p,n) > 0$ since

$$\mu_4(p,n) = \beta_2(p,n).$$

Case 5. $\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) = \mu_5(p,n)$.

In this case $p \geq 2$ and $n \geq 5$. We have $2p^{2n-4}(1 - p^{-1} - 2p^{-4}) > 0$. Since $p^{2n-9} > 1$ we have $4p^{2n-9} > 4$. Also,

$$2p^{2n-7}(p-1) = (2p)p^{2n-8}(p-1) > 4p$$

since $2p \geq 4$ and $p^{2n-8}(p-1) > p$. Therefore, $\mu_5(p,n) > 0$.

Thus, in all the cases $\mathrm{E_{CN}}(K_{|V(\Gamma(G))|}) - \mathrm{E_{CN}}(\Gamma(G)) > 0$. Hence, $\Gamma(G)$ is not CN-hyperenergetic. □

Concluding Remark: It is observed that the commuting conjugacy class graphs of all the groups considered in this paper are not CN-hyperenergetic. It may be interesting to find a finite group G such that $\Gamma(G)$ is CN-hyperenergetic or to prove that there is no finite group G whose $\Gamma(G)$ is CN-hyperenergetic.

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COMMON NEIGHBOURHOOD SPECTRUM AND ENERGY OF COMMUTING CONJUGACY CLASS GRAPH

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طیف همسایگی مشترک و انرژی همسایگی مشترک گراف کلاس‌های تزویج جابه‌جایی

فردوس ای جنت[۱] و راجات کانتی نات[۲]

[۱,۲]گروه علوم ریاضی، دانشگاه تزپور، سونیتپور، هند

ما در این مقاله، طیف همسایگی مشترک (به اختصار، طیف CN) و انرژی همسایگی مشترک گراف کلاس‌های تزویج جابه‌جایی خانواده‌هایی از گروه‌های ناآبلی متناهی را محاسبه می‌کنیم. یکی از نتایجی که از مطالب این مقاله بدست می‌آید این است که طیف همسایگی مشترک گراف کلاس‌های تزویج جابه‌جایی گروه‌های D_{2n}، T_{4n}، SD_{8n}، $U_{(n,m)}$، U_{6n}، V_{8n}، $G(p,m,n)$ و همچنین خانواده‌ای از گروه‌ها که گروه خارج قسمتی مرکزی آن‌ها با D_{2n} یا برای برخی عدد اول p با $\mathbb{Z}_p \times \mathbb{Z}_p$ یکریخت باشد، یک CN-صحیح است اما CN-هایپر انرژی نمی‌باشد.

کلمات کلیدی: همسایگی مشترک، طیف، انرژی، گراف کلاس‌های تزویج.