Journal of Algebraic Systems
Vol. 12, No. 2, (2025), pp 327-346

ADMISSIBLE (REES) EXACT SEQUENCES AND FLAT ACTS

E. NAFARIEH TALKHOONCHEH, M. SALIMI*, H. RASOULI, E. TAVASOLI AND A. TEHRANIAN

Abstract. Let S be a commutative pointed monoid. In this paper, some properties of admissible (Rees) short exact sequences of S-acts are investigated. In particular, it is shown that every admissible short exact sequence of S-acts is Rees short exact. In addition, a characterization of flat acts via preserving admissible short exact sequences is established. As a consequence, we show that for a flat S-act F, the functor $F \otimes_S -$ preserves admissible morphisms. Finally, it is proved that the class of flat S-acts is a subclass of admissibly projective ones.

1. Introduction

Throughout this paper, the term monoid will always mean a commutative, pointed monoid. For a monoid S, the notion of an S-act is defined and well-studied in literature. An S-act is a pointed set together with an action by S. We show the category of S-acts by S-Act_0. In S-Act_0, epimorphisms are not cokernels, and the "First Isomorphism Theorem" is not true in general. So, in [2] and [3], the authors considered admissible morphisms. In this paper, we recall the notion of admissible short exact sequence of S-acts and we investigate some properties of these sequences. The notion of Rees short exact sequence of S-acts is introduced in [1]. Also, the problem of when a

DOI: 10.22044/JAS.2023.12249.1648.
MSC(2010): Primary: 18A30; Secondary: 18B20, 18G05.
Keywords: S-act; Rees exact sequence; Admissible exact sequence; admissibly projective acts.
Received: 6 September 2022, Accepted: 18 March 2023.
*Corresponding author.

Rees short exact sequence of S-acts is left and right split is devoted in [1]. In this paper, we study some properties of Rees short exact sequences, and we show that every admissible short exact sequence of S-acts is Rees exact, and every Rees short exact sequence is exact. Also, in Example 3.13, we present an exact sequence which is not admissible (Rees) exact.

For an S-act X, the functor $X \otimes_S -$ does not preserve admissible morphisms. In Section 4, we characterize flat acts via preserving admissible short exact sequences. Also, we show that for a flat S-act F, the functor $F \otimes_S -$ from S-Act_0 to S-Act_0, preserves admissible morphisms. The notion of admissibly projective S-acts was defined in [3] as a generalization of projective S-acts. Note that any torsion free S-act is admissibly projective, by [3, Proposition 3.3.10]. Also, every projective S-act is admissibly projective, but not vice versa, see [3, Example 3.3.9]. Finally, we show that the class of flat S-acts is a subclass of admissibly projective S-acts.

2. Preliminaries

In this section, we recall some necessary definitions and properties which will be used in the next sections. We follow standard notation and terminology from [7, 3]. Let S be a commutative pointed monoid and let X be a pointed set, i.e., X has a distinguished basepoint denoted 0_X. A *left S-act* is a pointed set together with a left S-action $\cdot : S \times X \to X$ satisfying:

(i) $1 \cdot x = x$, for every $x \in X$.
(ii) $0_S \cdot x = 0_X$ and $s \cdot 0_X = 0_X$, for every $x \in X$ and $s \in S$.
(iii) $(st) \cdot x = s \cdot (t \cdot x)$, for every $s, t \in S$ and $x \in X$.

One may define a *right S-act* in the obvious way. If T is another commutative pointed monoid, a *two-sided (S,T)-act* is a pointed set X that is *both* a left S-act and a right T-act with actions satisfying $(sx)t = s(xt)$, for all $x \in X$, $s \in S$ and $t \in T$. When $S = T$, hence X has both a left and right S-action, X is an (non-commutative) *S-biact*. The action of an S-biact *commutes* when $sx = xs$, for all $s \in S$ and $x \in X$; then S-biacts with a commutative S-action are *commutative*. Throughout this paper, an S-act is a commutative S-biact and these objects are our primary concern.

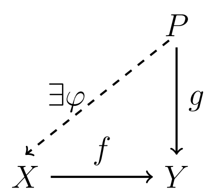
For S-acts X and Y, a function $f : X \to Y$ is called *S-act morphism*, or simply *homomorphism*, when $f(0_X) = 0_Y$ and $f(sx) = sf(x)$, for every $s \in S$ and $x \in X$. The category of S-acts together with their S-act morphisms will be denoted by S-Act_0. The image of an S-act morphism $f : X \to Y$ is the subset

$$\mathrm{Im}(f) = \{y \in Y \mid \exists x \in X, f(x) = y\}$$

together with the induced S-action. Since $0 = f(0)$ and

$$sy = sf(x) = f(sx)$$

for $y = f(x)$, this is indeed an S-subact of Y. Recall that a subset $U \neq \emptyset$ of an S-act X is said to be a *set of generating elements* or a *generating set* of X if every element $x \in X$ can be presented as $x = us$ for some $u \in U$, $s \in S$. A set U of generating elements of S-act X is said to be a *basis* of X if for every element $x \in X$ there exist a unique $u \in U$ and $s \in S$ such that $x = us$, i.e., if $x = u_1s_1 = u_2s_2$, then $u_1 = u_2$ and $s_1 = s_2$. An S-act X is called *free*, when X has a basis. An S-act P is called *projective*, when it satisfies the following universal lifting property, for any epimorphism $f : X \to Y$:



Meaning that, given any epimorphism $f : X \to Y$ of S-acts and any homomorphism $g : P \to Y$ of S-acts, there exists $\varphi : P \to X$ such that $g = f \circ \varphi$. Also, recall that an S-act F is *flat* if the functor $F \otimes_S -$ from S-Act_0 to S-Act_0, preserves monomorphisms. Note that every free S-act is projective by [7, Proposition 2.3.4] and every projective S-act is flat by [7, Proposition 3.17.5] and [7, Lemma 3.9.2].

Proposition 2.1. *Let I be an ideal of the monoid S, and let F be a flat S-act. Then $\alpha : F \otimes_S I \to IF$, defined by $f \otimes i \mapsto if$, for every $f \in F$ and $i \in I$, is an isomorphism.*

Proof. Let $\gamma : F \otimes_S S \to F$ such that $\gamma(f \otimes s) = fs$, for every $f \in F$ and $s \in S$. By [7, Proposition 2.5.13], γ is isomorphism. For the inclusion $\beta : I \to S$, the S-homomorphism $1_F \otimes \beta : F \otimes_S I \to F \otimes_S S$ is monomorphism, by assumption. Hence $\gamma(1_F \otimes \beta) : F \otimes_S I \to F$ is a monomorphism and its image is IF. Therefore, $\alpha = \gamma(1_F \otimes \beta) : F \otimes_S I \to IF$ is isomorphism.

□

Let T be a multiplicatively closed subset of a monoid S. Define $T^{-1}S$ to be a monoid with elements s/t, $s \in S$ and $t \in T$, where $s/t = s'/t'$ if there is an element $u \in T$ such that $u(st') = u(s't)$. The multiplication in $T^{-1}S$ is induced by S, $(s/t)(s'/t') = ss'/tt'$. Note that $(1/t)(t/1) = 1$, so that any element of T becomes a unit in $T^{-1}S$. The monoid $T^{-1}S$ is called the *monoid of fractions of S with respect to T* or the *localization of S at T*. Let X be an S-act. Define $T^{-1}X$, the *localization of X at T*, to be the $(T^{-1}S)$-act with elements x/t, $x \in X$ and $t \in T$, where $x/t = x'/t'$ when $u(t'x) = u(tx')$ for some $u \in T$. The action of $T^{-1}S$ on $T^{-1}X$ is simply $(s/t)(x/t') = sx/tt'$. We refer the readers to [4] for more details about localization. Also, we recall that an S-act X is *faithful* when $sx = tx$ for all $x \in X$ implies $s = t$. In the following, we investigate faithfully flat property of $T^{-1}X$.

Proposition 2.2. *Let X be a flat S-act, and let T be a multiplicatively closed subset of S such that T acts injectively on X. Then $T^{-1}X$ is faithfully flat $(T^{-1}S)$-act, provided that X is faithful.*

Proof. By [5, Theorem 2.3], $T^{-1}X$ is a flat $(T^{-1}S)$-act. Also, $T^{-1}X$ is a faithful $(T^{-1}S)$-act by [4, Lemma 1.3]. So, we get the result. ☐

3. Admissible and Rees exact sequences

The (co)kernel of an S-act morphism $f : X \to Y$ is defined as the (co)equalizer of the diagram

$$X \underset{*}{\overset{f}{\rightrightarrows}} Y,$$

where the map $*$ is defined by $*(x) = 0_Y$ for all $x \in X$. One can see that the kernel of f, denoted by $\mathrm{Ker}(f)$, is the subset $f^{-1}(0)$ of X, and the cokernel of f, denoted by $\mathrm{Coker}(f)$, is the quotient of Y by the equivalence relation defined as $y \sim y'$ if and only if $y = y'$ or $y, y' \in \mathrm{Im}(f)$. We denote this quotient by $Y/\mathrm{Im}(f)$. This means that the quotient Y/Z for any S-subact Z of Y exists as it is the cokernel of the inclusion map $i : Z \to Y$. All kernels and cokernels exist in S-Act_0 but we do not have f is injective when $\mathrm{Ker}(f) = 0$ and the First Isomorphism Theorem does not hold in general. So, we consider admissible morphisms which are defined as follows:

Definition 3.1. [3] An S-homomorphism $f : X \to Y$ is called *admissible* whenever the surjection $f : X \to f(X)$ is a cokernel. In this case, $\mathrm{Ker}(f) = 0$ implies that f is injective.

In the following, we collect some properties of admissible morphisms from [2], and [3] which will be used in the next sections.

Proposition 3.2. *The following statements hold.*

(i) *Let X and Y be S-acts. Then all injections $X \hookrightarrow Y$ are clearly admissible.*
(ii) *An S-homomorphism $f : X \to Y$ is admissible if and only if $f|_{X\setminus \mathrm{Ker}(f)}$ is an injection.*
(iii) *An admissible morphism is an injection if and only if it has trivial kernel.*
(iv) *Admissible morphisms have "First Isomorphism Theorem", i.e. if $f : X \to Y$ is an admissible S-homomorphism, then $X/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$.*
(v) *Let $f : X \to Y$ be an admissible S-homomorphism. Then, there exists an admissible monomorphism $g : X \to P$ and an admissible epimorphism $h : P \to Y$ such that $f = h \circ g$.*
(vi) *Let $f : X \to Y$ be an admissible S-homomorphism. Then, there exists an admissible epimorphism $h : X \to P$ and an admissible monomorphism $g : P \to Y$ such that $f = g \circ h$.*
(vii) *The composition of admissible S-homomorphisms is admissible.*

A sequence

$$\cdots \longrightarrow X_{n+1} \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

of S-homomorphisms is *admissible* when every morphisms in the sequence is admissible. The sequence is *exact* when $\mathrm{Im}(f_{i+1}) = \mathrm{Ker}(f_i)$, for all i. An *admissible short exact sequence* is an admissible exact sequence of the form

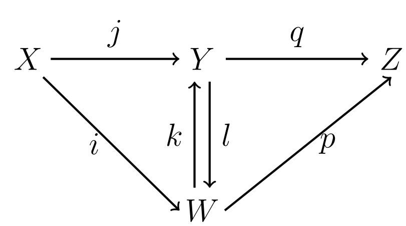
$$0 \to X' \to X \to X'' \to 0.$$

In this case, Proposition 3.2 implies that

(i) $X' \to X$ is an injection,
(ii) $X \to X''$ is a surjection and $X/X' \cong X''$.

Remark 3.3. Let X be an S-act. Since functors $- \otimes_S X$ and $X \otimes_S -$ do not preserve monomorphisms, we should not expect any of these functors preserve admissible morphisms.

Proposition 3.4. *Consider the following commutative diagram with admissible exact row of S-acts and S-homomorphisms:*



Let k be an isomorphism with inverse l. Then $X \xrightarrow{i} W \xrightarrow{p} Z$ is an admissible exact sequence.

Proof. Let $w \in \mathrm{Ker}(p)$. Then $qk(w) = p(w) = 0$. Hence,

$$k(w) \in \mathrm{Ker}(q) = \mathrm{Im}(j).$$

So, there exists $x \in X$ such that $j(x) = k(w)$. Therefore,

$$i(x) = lj(x) = lk(w) = w.$$

Hence $w \in \mathrm{Im}(i)$ which implies that $\mathrm{Ker}(p) \subseteq \mathrm{Im}(i)$. Now, suppose that $w \in \mathrm{Im}(i)$. Then there exists $x \in X$ such that $i(x) = w$, and $j(x) \in \mathrm{Im}(j) = \mathrm{Ker}(q)$. Therefore,

$$qk(w) = qk(i(x)) = q(ki)(x) = qj(x) = 0,$$

and so $w \in \mathrm{Ker}(p)$ which implies the exactness. Also, the admissible property of i and p follows from Proposition 3.2. □

Proposition 3.5. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{k} T$ be an admissible exact sequence of S-acts and S-homomorphisms. Then*

$$0 \to \mathrm{Coker}(f) \xrightarrow{\alpha} Z \xrightarrow{\beta} \mathrm{Ker}(k) \to 0$$

is an admissible short exact sequence such that $\alpha([y]_{\mathrm{Im}(f)}) = g(y)$ and $\beta(z) = h(z)$, for every $y \in Y$ and $z \in Z$.

Proof. Let $[y_1]_{\mathrm{Im}(f)} = [y_2]_{\mathrm{Im}(f)} \in \mathrm{Coker}(f)$. Then

$$y_1 = y_2 \text{ or } y_1, y_2 \in \mathrm{Im}(f).$$

If $y_1 = y_2$, then $g(y_1) = g(y_2)$. Otherwise, $y_1, y_2 \in \mathrm{Im}(f) = \mathrm{Ker}(g)$. Hence, $g(y_1) = 0 = g(y_2)$ and α is well-defined. Also, for every $s \in S$, $\alpha([y]_{\mathrm{Im}(f)}s) = \alpha([ys]_{\mathrm{Im}(f)}) = g(ys) = g(y)s = \alpha([y]_{\mathrm{Im}(f)})s$. So, α is an S-homomorphism. Now, suppose that $\alpha([y_1]_{\mathrm{Im}(f)}) = \alpha([y_2]_{\mathrm{Im}(f)})$. Then $g(y_1) = g(y_2)$. If $g(y_1) = g(y_2) = 0$, then $y_1, y_2 \in \mathrm{Ker}(g) = \mathrm{Im}(f)$, and so $[y_1]_{\mathrm{Im}(f)} = [y_2]_{\mathrm{Im}(f)}$. Otherwise, $g(y_1) = g(y_2) \neq 0$. Since g is admissible, we get that $y_1 = y_2$, and so $[y_1]_{\mathrm{Im}(f)} = [y_2]_{\mathrm{Im}(f)}$. Therefore, α is a monomorphism and so α is admissible by Proposition 3.2. It is evident that β is an epimorphism. Also, it is easy to check that $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$. For completing the proof, we must show that β is admissible. Suppose that $z_1, z_2 \in Z$ such that $\beta(z_1) = \beta(z_2) \neq 0$. Then $h(z_1) = h(z_2) \neq 0$ which implies that $z_1 = z_2$, as desired. □

Proposition 3.6. *Consider the following commutative diagram with admissible exact rows of S-acts and S-homomorphisms:*

$$\begin{array}{ccccccc} & & X & \xrightarrow{i} & Y & \xrightarrow{p} & Z \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ 0 & \longrightarrow & X' & \xrightarrow{j} & Y' & \xrightarrow{q} & Z' \end{array}$$

Then the sequence

$$\mathrm{Ker}(\alpha) \xrightarrow{\bar{i}} \mathrm{Ker}(\beta) \xrightarrow{\overline{p}} \mathrm{Ker}(\gamma)$$

is admissible exact, where $\bar{i}(x) = i(x)$ and $\overline{p}(y) = p(y)$ for every $x \in \mathrm{Ker}(\alpha)$ and $y \in \mathrm{Ker}(\beta)$.

Proof. Since

$$\overline{p}(\bar{i}(x)) = \overline{p}(i(x)) = p(i(x)) = 0,$$

we get that $\mathrm{Im}(\bar{i}) \subseteq \mathrm{Ker}(\overline{p})$. Now, suppose that $y \in \mathrm{Ker}(\overline{p})$. Then $\overline{p}(y) = p(y) = 0$ and $\beta(y) = 0$. So, there exists $x \in X$ such that $i(x) = y$. Therefore, $0 = \beta(y) = \beta(i(x))$, and so $i(x) \in \mathrm{Ker}(\beta)$. On the other hand, $j(\alpha(x)) = \beta(i(x)) = 0$ which implies that $x \in \mathrm{Ker}(\alpha)$. So, $y \in \mathrm{Im}(\bar{i})$. Therefore, $\mathrm{Im}(\bar{i}) = \mathrm{Ker}(\overline{p})$. Let

$$x_1, x_2 \in \mathrm{Ker}(\alpha) \setminus \mathrm{Ker}(\bar{i}).$$

Then

$$\alpha(x_1) = \alpha(x_2) = 0, \; i(x_1) \neq 0$$

and $i(x_2) \neq 0$. Now, if $\bar{i}(x_1) = \bar{i}(x_2)$, then $i(x_1) = i(x_2)$ and since $i|_{X \setminus \mathrm{Ker}(i)}$ is monomorphism, we have $x_1 = x_2$ and so, $\bar{i}$ is admissible. Now, let $y_1, y_2 \in \mathrm{Ker}(\beta) \setminus \mathrm{Ker}(\overline{p})$. Then $\beta(y_1) = \beta(y_2) = 0$, $p(y_1) \neq 0$ and $p(y_2) \neq 0$. Now, if $\overline{p}(y_1) = \overline{p}(y_2)$, then $p(y_1) = p(y_2)$ and since $i|_{Y \setminus \mathrm{Ker}(p)}$ is a monomorphism, we have $y_1 = y_2$ and so, $\overline{p}$ is admissible. □

Proposition 3.7. *Consider the following commutative diagram with admissible exact rows of S-acts and S-homomorphisms:*

$$\begin{array}{ccccccc} X & \xrightarrow{i} & Y & \xrightarrow{p} & Z & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & & & \\ X' & \xrightarrow{j} & Y' & \xrightarrow{q} & Z' & \longrightarrow & 0 \end{array}$$

Let β be an admissible S-homomorphism. Then there exists a unique admissible S-homomorphism $\gamma : Z \longrightarrow Z'$ which commutes the *following diagram.*

$$\begin{array}{ccccccc} X & \xrightarrow{i} & Y & \xrightarrow{p} & Z & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ X' & \xrightarrow{j} & Y' & \xrightarrow{q} & Z' & \longrightarrow & 0 \end{array}$$

Proof. Let $z \in Z$. Then there exists $y \in Y$ such that $p(y) = z$. Define a map $\gamma : Z \to Z'$ by $\gamma(z) = q\beta(y)$. Let $z_1, z_2 \in Z$. Then there exists $y_1, y_2 \in Y$ such that $p(y_1) = z_1$ and $p(y_2) = z_2$. So, $\gamma(z_1) = q\beta(y_1)$ and $\gamma(z_2) = q\beta(y_2)$. If $p(y_1) = p(y_2) = 0$, then

$$y_1, y_2 \in \mathrm{Ker}(p) = \mathrm{Im}(i).$$

Hence, there exists $x_1, x_2 \in X$ such that $i(x_1) = y_1$ and $i(x_2) = y_2$. Therefore, $\beta i(x_1) = \beta(y_1)$. So, $j\alpha(x_1) = \beta(y_1)$. Thus

$$\beta(y_1) \in \mathrm{Im}(j) = \mathrm{Ker}(q)$$

and then $q\beta(y_1) = 0$. So, $\gamma(z_1) = 0$. By the same way, we have $\gamma(z_2) = 0$. Then $\gamma(z_1) = \gamma(z_2)$. Otherwise, $p(y_1) = p(y_2) \neq 0$. Since p is admissible, $p|_{Y \setminus \mathrm{Ker}(p)}$ is a monomorphism. So, $y_1 = y_2$. Then $\gamma(z_1) = q\beta(y_1) = q\beta(y_2) = \gamma(z_2)$. This shows that γ is well-defined. It is routine to check that γ is an S-homomorphism and $\gamma \circ p = q \circ \beta$. Now, we prove that γ is admissible. For this, let $z_1, z_2 \in Z \setminus \mathrm{Ker}(\gamma)$ such that $\gamma(z_1) = \gamma(z_2)$. Hence, there exists $y_1, y_2 \in Y$ such that $p(y_1) = z_1$ and $p(y_2) = z_2$. Also,

$$q\beta(y_1) = q\beta(y_2) \neq 0,$$

and so $\beta(y_1), \beta(y_2) \in Y' \setminus \mathrm{Ker}(q)$. Hence, $\beta(y_1) = \beta(y_2) \neq 0$, since q is admissible. Therefore, $y_1, y_2 \in Y \setminus \mathrm{Ker}(\beta)$. Since β is admissible, we get that $y_1 = y_2$. Hence $z_1 = z_2$, as desired. For the uniqueness, let $\gamma' : Z \to Z'$ be an admissible S-homomorphism such that $\gamma' \circ p = q \circ \beta$, and let $z \in Z$. Then there exists $y \in Y$ such that $p(y) = z$. Therefore, $\gamma'(z) = \gamma'(p(y)) = \gamma' \circ p(y) = q \circ \beta(y) = \gamma(z)$, which implies that $\gamma = \gamma'$. □

Proposition 3.8. *Consider the following commutative diagram with admissible exact rows of S-acts and S-homomorphisms:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{i} & Y & \xrightarrow{p} & Z \\ & & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ 0 & \longrightarrow & X' & \xrightarrow{j} & Y' & \xrightarrow{q} & Z' \end{array}$$

Then there exists a unique admissible S-homomorphism $\gamma : X \longrightarrow X'$ which commutes the following diagram.

$$\begin{array}{ccccccc}
0 & \longrightarrow & X & \xrightarrow{i} & Y & \xrightarrow{p} & Z \\
 & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\
0 & \longrightarrow & X' & \xrightarrow{j} & Y' & \xrightarrow{q} & Z'
\end{array}$$

Proof. This is proved as the same line as Proposition 3.7. □

In the following, we recall the notion of Rees exact sequences of S-acts as defined in [1].

Let $f : X \to Y$ be an S-homomorphism. Set

$$\mathcal{K}_f = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}, \text{ and}$$

$$\mathcal{L}_{\mathrm{Im}(f)} = (\mathrm{Im}(f) \times \mathrm{Im}(f)) \cup \Delta_Y,$$

where Δ_Y is the identity congruence on Y. It is clear that both $\mathcal{K}_f$ and $\mathcal{L}_{\mathrm{Im}(f)}$ are congruences on X and Y respectively, and $f(X) \cong X/\mathcal{K}_f$ as S-acts. The sequence

$$\cdots \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow \cdots$$

of S-acts is called *Rees exact* at Y if $\mathcal{K}_g = \mathcal{L}_{\mathrm{Im}(f)}$. If the sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \tag{3.1}$$

of S-acts is Rees exact at X, Y and Z, then it is called a *Rees short exact sequence*. Note that if the sequence 3.1 is Rees short exact, then f is a monomorphism and g is an epimorphism. Moreover, let $x \in X$. Then

$$(f(x), f(0)) \in \mathrm{Im}(f) \times \mathrm{Im}(f) \subseteq \mathcal{L}_{\mathrm{Im}(f)} = \mathcal{K}_g.$$

So, $g(f(x)) = g(f(0)) = g(0) = 0$. Then $g \circ f = 0$.

We also use the term "Rees exact sequence" for sequences of the forms

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$$

and

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

being Rees exact at X, Y and Y, Z, respectively.

Remark 3.9. Let X be an S-act, and let $L \to M \to N \to 0$ be a Rees exact sequence of S-acts. Then $X \otimes_S L \to X \otimes_S M \to X \otimes_S N \to 0$ is also a Rees exact sequence of S-acts, by [6, Theorem 3.1].

Proposition 3.10. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an admissible short exact sequence of* S*-acts and* S*-homomorphisms. Then it is Rees short exact.*

Proof. Let $(y_1, y_2) \in \mathcal{K}_g$. Then $g(y_1) = g(y_2)$. If $g(y_1) = g(y_2) = 0$, then $y_1, y_2 \in \mathrm{Ker}(g) = \mathrm{Im}(f)$ and so $(y_1, y_2) \in \mathcal{L}_{\mathrm{Im}(f)}$. Otherwise, $g(y_1) = g(y_2) \neq 0$. Therefore, $y_1 = y_2$, since g is admissible. So, $(y_1, y_2) \in \mathcal{L}_{\mathrm{Im}(f)}$. Now, let $(y_1, y_2) \in \mathcal{L}_{\mathrm{Im}(f)}$. If $y_1, y_2 \in \mathrm{Im}(f)$, then $g(y_1) = g(y_2) = 0$, and so $(y_1, y_2) \in \mathcal{K}_g$. Otherwise, $y_1 = y_2$ which implies that $(y_1, y_2) \in \mathcal{K}_g$. □

Corollary 3.11. *Let* X *be an* S*-act, and let* $L \to M \to N \to 0$ *be an admissible exact sequence of* S*-acts and* S*-homomorphisms. Then* $X \otimes_S L \to X \otimes_S M \to X \otimes_S N \to 0$ *is a Rees exact sequence.*

Proof. This follows from Proposition 3.10 and Remark 3.9. □

Proposition 3.12. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be a Rees short exact sequence of* S*-acts and* S*-homomorphisms. Then it is short exact.*

Proof. It is enough to show that $\mathrm{Ker}(g) = \mathrm{Im}(f)$. Let $y \in \mathrm{Ker}(g)$. Then $g(y) = 0 = g(0)$. Hence, $(y, 0) \in \mathcal{K}_g = \mathcal{L}_{\mathrm{Im}(f)}$. Therefore, $y = 0$ or $(y, 0) \in \mathrm{Im}(f) \times \mathrm{Im}(f)$. Since $0 \in \mathrm{Im}(f)$, we get that $y \in \mathrm{Im}(f)$. Hence $\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$. Now, suppose that $y \in \mathrm{Im}(f)$. Then

$$(y, 0) \in \mathrm{Im}(f) \times \mathrm{Im}(f) \subseteq \mathcal{L}_{\mathrm{Im}(f)} = \mathcal{K}_g.$$

Hence $g(y) = g(0) = 0$. So, $y \in \mathrm{Ker}(g)$ as desired. □

Let R be a commutative unital ring, and let $U(R)$ denote the monoid $(R, .)$. This construction induces a functor $U : R\text{-Mod} \to U(R)\text{-Act}_0$, where R-Mod is the category of R-modules. To every R-module M, the $U(R)$-act $U(M)$ has no addition and retains its R-action. The functor U which is called the *forgetful functor* was introduced in [3].

Example 3.13. Let C_3 and C_4 be the pointed cyclic group of order 3 and 4, respectively, and let $\langle 1, 0\rangle : C_3 \to C_3 \times C_4$ and $\pi_2 : C_3 \times C_4 \to C_4$ be the canonical inclusion and projection. The sequence

$$0 \to U(C_3) \xrightarrow{U(\langle 1,0\rangle)} U(C_3 \times C_4) \xrightarrow{U(\pi_2)} U(C_4) \to 0 \tag{3.2}$$

is short exact, because

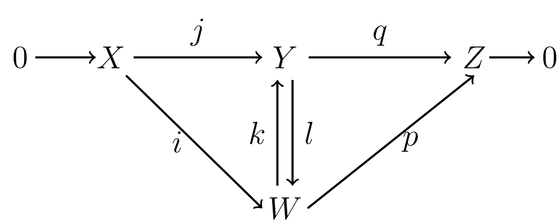
$$\begin{aligned}&\mathrm{Ker}(U(\pi_2))\\&=\mathrm{Ker}(\pi_2)\\&=\{(x,y)\in C_3\times C_4 \mid \pi_2((x,y))=y=0_{C_4}\}\\&=\mathrm{Im}(U(\langle 1,0\rangle))\\&=\mathrm{Im}(\langle 1,0\rangle)\\&=\{(x,y)\in C_3\times C_4 \mid \exists x\in C_3, \langle 1,0\rangle(x)=(x,0_{C_4})=(x,y)\},\end{aligned}$$

and $U(\langle 1,0\rangle)=\langle 1,0\rangle$ is a monomorphism and $U(\pi_2)=\pi_2$ is an epimorphism. Note that π_2 is not admissible, because its restriction to $(C_3\times C_4)\setminus\mathrm{Ker}(\pi_2)$ is not an injection. So, the sequence 3.2 is not admissible. Notice that the element

$$(z,t)=((1_{C_3},1_{C_4}),(0_{C_3},1_{C_4}))\in\mathcal{K}_{\pi_2}$$

and $(z,t)\notin\mathcal{L}_{\mathrm{Im}_{\langle 1,0\rangle}}$. Hence, the sequence 3.2 is not Rees short exact.

Proposition 3.14. *Consider the following commutative diagram with Rees short exact sequence in row of S-acts and S-homomorphisms:*



Let k be an isomorphism with inverse l. Then $0\to X\xrightarrow{i} W\xrightarrow{p} Z\to 0$ is a Rees short exact sequence.

Proof. Let $(w,w')\in\mathcal{K}_p$. Then $p(w)=p(w')$ and so, $qk(w)=qk(w')$. Hence, $(k(w),k(w'))\in\mathcal{K}_q=\mathcal{L}_{\mathrm{Im}(j)}$. Therefore, $k(w)=k(w')$ or $(k(w),k(w'))\in\mathrm{Im}(j)\times\mathrm{Im}(j)$. If $k(w)=k(w')$, then $w=w'$. Now, let $(k(w),k(w'))\in\mathrm{Im}(j)\times\mathrm{Im}(j)$. Then there exists $(x,x')\in X\times X$ such that $(j(x),j(x'))=(k(w),k(w'))$. So,

$$(i(x),i(x'))=(lj(x),lj(x'))=(lk(w),lk(w'))=(w,w').$$

Hence $(w,w')\in\mathcal{L}_{\mathrm{Im}(i)}$ which implies that $\mathcal{K}_p\subseteq\mathcal{L}_{\mathrm{Im}(i)}$. Now, suppose that $(w,w')\in\mathcal{L}_{\mathrm{Im}(i)}$. Then $w=w'$ or $(w,w')\in\mathrm{Im}(i)\times\mathrm{Im}(i)$. If $w=w'$, then $p(w)=p(w')$. Now, suppose that

$$(w,w')\in\mathrm{Im}(i)\times\mathrm{Im}(i).$$

Then there exists $(x, x') \in X \times X$ such that $(i(x), i(x')) = (w, w')$, and $(j(x), j(x')) \in \operatorname{Im}(j) \times \operatorname{Im}(j) \subseteq \mathcal{K}_q$. Hence $qj(x) = qj(x')$. Note that

$$\begin{aligned}(p(w), p(w')) &= (qk(w), qk(w')) \\ &= (qk(i(x)), qk(i(x'))) \\ &= (q(ki)(x), q(ki)(x')) \\ &= (qj(x), qj(x')),\end{aligned}$$

and then $p(w) = qj(x) = qj(x') = p(w')$. Therefore, $(w, w') \in \mathcal{K}_p$. Since p is an epimorphism and i is a monomorphism, we get the result. $\square$

Proposition 3.15. *Consider the following commutative diagram with Rees exact sequences in rows of S-acts and S-homomorphisms:*

$$\begin{array}{ccccccc} X & \xrightarrow{i} & Y & \xrightarrow{p} & Z & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & & & \\ X' & \xrightarrow{j} & Y' & \xrightarrow{q} & Z' & \longrightarrow & 0 \end{array}$$

Then there exists a unique S-homomorphism $\gamma : Z \longrightarrow Z'$ which commutes the following diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{i} & Y & \xrightarrow{p} & Z & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ X' & \xrightarrow{j} & Y' & \xrightarrow{q} & Z' & \longrightarrow & 0 \end{array}$$

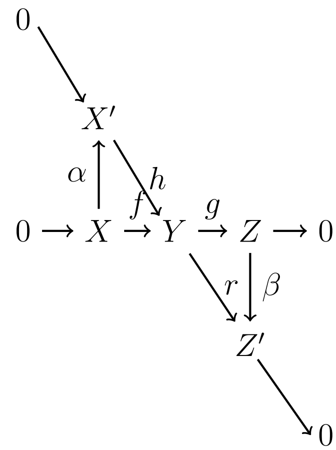
Proof. Let $z \in Z$. Then there exists $y \in Y$ such that $p(y) = z$. Define a map $\gamma : Z \to Z'$ by $\gamma(z) = q\beta(y)$. If there exists $y \neq y' \in Y$ with $p(y') = z$, then $p(y) = p(y')$ and so $(y, y') \in \mathcal{K}_p = \mathcal{L}_{\operatorname{Im}(i)}$. This gives that $(y, y') \in \operatorname{Im}(i) \times \operatorname{Im}(i)$, then there exists $(x, x') \in X \times X$ such that $(i(x), i(x')) = (y, y')$. So,

$$(\beta(y), \beta(y')) = (\beta(i(x)), \beta(i(x'))) = (j(\alpha(x)), j(\alpha(x'))).$$

Then $(\beta(y), \beta(y')) \in \mathcal{L}_{\operatorname{Im}(j)} = \mathcal{K}_q$ and therefore, $q(\beta(y)) = q(\beta(y'))$ which implies that γ is well-defined. It is routine to check that γ is an S-homomorphism and $\gamma \circ p = q \circ \beta$. Now, let $\gamma' : Z \to Z'$ be an S-homomorphism such that $\gamma' \circ p = q \circ \beta$. Since $\gamma' \circ p = \gamma \circ p$ and p is an epimorphism, $\gamma = \gamma'$. $\square$

Proposition 3.16. *Consider the following commutative diagram with Rees exact horizontal and diagonal rows of S-acts and S-*

homomorphisms:



If α is an epimorphism, then β is a monomorphism.

Proof. Let $z_1, z_2 \in Z$ such that $\beta(z_1) = \beta(z_2)$. Then there exists $y_1, y_2 \in Y$ such that $g(y_1) = z_1$ and $g(y_2) = z_2$. Hence

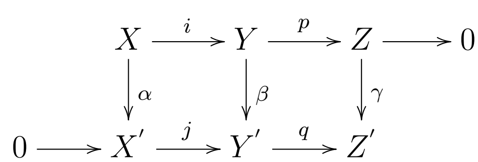
$$r(y_1) = \beta(g(y_1)) = \beta(z_1) = \beta(z_2) = \beta(g(y_2)) = r(y_2).$$

Therefore, $(y_1, y_2) \in \mathcal{K}_r = \mathcal{L}_{\mathrm{Im}(h)}$. If $y_1 = y_2$, then $z_1 = z_2$, as desired. Otherwise, there exist $x_1', x_2' \in X'$ such that $h(x_1') = y_1$ and $h(x_2') = y_2$. Also, there exist $x_1, x_2 \in X$ such that $\alpha(x_1) = x_1'$ and $\alpha(x_2) = x_2'$. Therefore, $f(x_1) = h(\alpha(x_1)) = h(x_1') = y_1$ and

$$f(x_2) = h(\alpha(x_2)) = h(x_2') = y_2.$$

But $(y_1, y_2) \in \mathcal{L}_{\mathrm{Im}(f)} = \mathcal{K}_g$, and so $z_1 = g(y_1) = g(y_2) = z_2$, which shows that β is a monomorphism. □

Proposition 3.17. *Consider the following commutative diagram with Rees exact rows of S-acts and S-homomorphisms:*



Then the sequence

$$\mathrm{Ker}(\alpha) \xrightarrow{\bar{i}} \mathrm{Ker}(\beta) \xrightarrow{\bar{p}} \mathrm{Ker}(\gamma) \tag{3.3}$$

is Rees exact at $\mathrm{Ker}(\beta)$*, where* $\bar{i} = i|_{\mathrm{Ker}(\alpha)}$ *and* $\bar{p} = p|_{\mathrm{Ker}(\beta)}$.

Proof. Let $x_1, x_2 \in \mathrm{Ker}(\alpha)$. Then $\bar{p}\bar{i}(x_1) = pi(x_1) = 0$ and

$$\bar{p}\bar{i}(x_2) = pi(x_2) = 0.$$

Therefore, $(\bar{i}(x_1), \bar{i}(x_2)) \in \mathcal{K}_{\overline{p}}$ so that $\mathcal{L}_{\mathrm{Im}(\bar{i})} \subseteq \mathcal{K}_{\overline{p}}$. Now suppose that $(y_1, y_2) \in \mathcal{K}_{\overline{p}}$. Then $\beta(y_1) = \beta(y_2) = 0$, and $\overline{p}(y_1) = \overline{p}(y_2)$. So, $(y_1, y_2) \in \mathcal{K}_p = \mathcal{L}_{\mathrm{Im}(i)}$. Hence, $y_1 = y_2$ or there exists $(x_1, x_2) \in X \times X$ such that $(i(x_1), i(x_2)) = (y_1, y_2)$. Therefore,

$$(\beta i(x_1), \beta i(x_2)) = (j\alpha(x_1), j\alpha(x_2)) = (\beta(y_1), \beta(y_2)).$$

So,

$$j\alpha(x_1) = j\alpha(x_2) = 0$$

and then $\alpha(x_1) = \alpha(x_2) = 0$. Hence, $(y_1, y_2) \in \mathcal{L}_{\mathrm{Im}(\bar{i})}$ which implies that $\mathcal{K}_{\overline{p}} \subseteq \mathcal{L}_{\mathrm{Im}(\bar{i})}$. Therefore, 3.3 is Rees exact at $\mathrm{Ker}(\beta)$. □

4. Characterization of flat acts

Let X be an S-act. As we mentioned in Remark 3.3, the functor $X \otimes_S -$ does not preserve admissible morphisms. In this section, we characterize flat acts via preserving admissible short exact sequences. As a consequence, we prove that the functor $F \otimes_S -$ preserves admissible morphisms, provided that F is a flat S-act.

Theorem 4.1. *The following statements are equivalent for an S-act F:*

(i) *If $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is a Rees short exact sequence of S-acts and S-homomorphisms, then*

$$0 \longrightarrow F \otimes_S X \xrightarrow{1_F \otimes f} F \otimes_S Y \xrightarrow{1_F \otimes g} F \otimes_S Z \longrightarrow 0$$

is also a Rees short exact sequence of S-acts and S-homomorphisms.

(ii) *F is flat.*

(iii) *If $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is an admissible short exact sequence of S-acts and S-homomorphisms, then the sequence $0 \longrightarrow F \otimes_S X \xrightarrow{1_F \otimes f} F \otimes_S Y \xrightarrow{1_F \otimes g} F \otimes_S Z \longrightarrow 0$ is also an admissible short exact sequence of S-acts and S-homomorphisms.*

Proof. (i) ⇒ (ii) Let $f : X \to Y$ be a monomorphism of S-acts. Then the sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Y/X \longrightarrow 0$ is Rees exact (see [1]). Hence, the sequence $0 \longrightarrow F \otimes_S X \longrightarrow F \otimes_S Y \longrightarrow F \otimes_S Y/X \longrightarrow 0$ is Rees exact, by assumption. Therefore, $F \otimes_S X \longrightarrow F \otimes_S Y$ is a monomorphism, as desired.

(ii) ⇒ (i) Follows from [6, Corollary 3.1.1].

(i) ⇒ (iii) Let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \qquad (4.1)$$

be an admissible exact sequence of S-acts and S-homomorphisms. Then the sequence 4.1 is a Rees exact sequence by Proposition 3.10. Hence,

$$0 \longrightarrow F \otimes_S X \stackrel{1_F \otimes f}{\longrightarrow} F \otimes_S Y \stackrel{1_F \otimes g}{\longrightarrow} F \otimes_S Z \longrightarrow 0$$

is a Rees exact sequence by assumption. Therefore, the sequence 4.1 is exact by Proposition 3.12. Hence, $F \otimes_S X \stackrel{1_F \otimes f}{\longrightarrow} F \otimes_S Y$ is an admissible homomorphism by Proposition 3.2. Then it is sufficient to show that $1_F \otimes g$ is an admissible S-homomorphism. Note that the epimorphism $g : Y \longrightarrow g(Y) = Z$ is a cokernel, since g is an admissible. Therefore, the epimorphism $1_F \otimes g : F \otimes_S Y \longrightarrow F \otimes_S g(Y) = F \otimes_S Z$ is a cokernel by [3, Proposition 2.3.3]. So, it is an admissible S-homomorphism by [3, Remark 3.2.3]. Hence, the sequence

$$0 \longrightarrow F \otimes_S X \stackrel{1_F \otimes f}{\longrightarrow} F \otimes_S Y \stackrel{1_F \otimes g}{\longrightarrow} F \otimes_S Z \longrightarrow 0$$

is admissible exact. So, we get the assertion.

(iii) $\Rightarrow$ (ii) This is proved as the same line as (i) $\Rightarrow$ (ii), since for S-monomorphism $X \to Y$, the sequence $0 \to X \to Y \to Y/X \to 0$ is admissible exact by Proposition 3.2. □

Corollary 4.2. *Let F be a flat S-act, and let $f : X \to Y$ be a monomorphism of S-acts. Then* $\dfrac{F \otimes_S Y}{F \otimes_S X} \cong F \otimes_S (Y/X)$.

Proof. Consider admissible exact sequence $0 \to X \to Y \to Y/X \to 0$ of S-acts and S-homomorphisms. By Theorem 4.1, the sequence

$$0 \to F \otimes_S X \to F \otimes_S Y \to F \otimes_S (Y/X) \to 0$$

is admissible exact. Hence, $\dfrac{F \otimes_S Y}{F \otimes_S X} \cong F \otimes_S (Y/X)$. □

Corollary 4.3. *Let F be a flat S-act. Then the functor $F \otimes_S -$ preserves admissible morphism.*

Proof. Let F be a flat S-act, and let $f : X \to Y$ be an admissible S-homomorphism. By Proposition 3.2, the sequences

$$0 \to \mathrm{Ker}(f) \to X \to \mathrm{Im}(f) \to 0,$$

and

$$0 \to \mathrm{Im}(f) \to Y \to Y/\mathrm{Im}(f) \to 0$$

are admissible exact. Therefore, the sequences

$$0 \to F \otimes_S \mathrm{Ker}(f) \to F \otimes_S X \to F \otimes_S \mathrm{Im}(f) \to 0,$$

and

$$0 \to F \otimes_S \mathrm{Im}(f) \to F \otimes_S Y \to F \otimes_S (Y/\mathrm{Im}(f)) \to 0$$

are also admissible exact, by Theorem 4.1. Hence, the morphism $F \otimes_S X \to F \otimes_S Y$ is admissible, by Proposition 3.2. □

Let $\{X_i \mid i \in I\}$ be a family of S-acts. The coproduct of this family is $\bigvee_{i \in I} X_i = (\dot{\bigcup}(X_i \setminus \{0_{X_i}\}))\dot{\bigcup}\{0\}$ with $x_i s = 0$, if $x_i s = 0_{X_i}$ in X_i and $0s = 0$ for all $s \in S$.

Proposition 4.4. *Let $\{F_i \mid i \in I\}$ be a family of flat S-acts. Then $\bigvee_{i \in I} F_i$ is flat.*

Proof. Let $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ be an admissible short exact sequence of S-acts and S-homomorphisms. Then

$$0 \longrightarrow F_i \otimes_S X \xrightarrow{1_{F_i} \otimes f} F_i \otimes_S Y \xrightarrow{1_{F_i} \otimes g} F_i \otimes_S Z \longrightarrow 0$$

is an admissible exact sequence for every $i \in I$. We show that

$$0 \longrightarrow \bigvee_{i \in I}(F_i \otimes_S X) \xrightarrow{\overline{f}} \bigvee_{i \in I}(F_i \otimes_S Y) \xrightarrow{\overline{g}} \bigvee_{i \in I}(F_i \otimes_S Z) \longrightarrow 0$$

is an admissible exact sequence, where

$$\overline{f}((f_i \otimes x)_{i \in I}) = ((1_{F_i} \otimes f)(f_i \otimes x))_{i \in I}$$

and

$$\overline{g}((f_i \otimes y)_{i \in I}) = ((1_{F_i} \otimes g)(f_i \otimes y))_{i \in I}$$

for $f_i \otimes x \in F_i \otimes_S X$ and $f_i \otimes y \in F_i \otimes_S Y$. It is clear that $\overline{f}$ and $\overline{g}$ are admissible S-homomorphisms. It is evident to see that $\overline{g}$ is an epimorphism and $\overline{f}$ is a monomorphism. Note that

$$(1_{F_i} \otimes g) \circ (1_{F_i} \otimes f) = 0.$$

So, $\overline{g} \circ \overline{f} = ((1_{F_i} \otimes g) \circ (1_{F_i} \otimes f))_{i \in I} = (0)_{i \in I} = 0$. Hence, $\mathrm{Im}(\overline{f}) \subseteq \mathrm{Ker}(\overline{g})$. Let $(f_i \otimes y)_{i \in I} \in \mathrm{Ker}(\overline{g})$. Then $\overline{g}((f_i \otimes y)_{i \in I}) = ((1_{F_i} \otimes g)(f_i \otimes y))_{i \in I} = 0$. Therefore, $f_i \otimes y \in \mathrm{Ker}(1_{F_i} \otimes g) = \mathrm{Im}(1_{F_i} \otimes f)$ for all $i \in I$. So, there exists $f_i \otimes x \in F_i \otimes_S X$ such that $(1_{F_i} \otimes f)(f_i \otimes x) = f_i \otimes y$ and we have $\mathrm{Ker}(\overline{g}) \subseteq \mathrm{Im}(\overline{f})$. Now, [7, Proposition 2.5.14] implies the result. □

Proposition 4.5. *If F and F' are flat S-acts, then so is $F \otimes_S F'$.*

Proof. Let $0 \to X \to Y \to Z \to 0$ be an admissible short exact sequence of S-acts. By assumption,

$$0 \to F' \otimes_S X \to F' \otimes_S Y \to F' \otimes_S Z \to 0,$$

and so

$$0 \to F \otimes_S (F' \otimes_S X) \to F \otimes_S (F' \otimes_S Y) \to F \otimes_S (F' \otimes_S Z) \to 0,$$

are admissible exact sequences. Hence

$$0 \to (F \otimes_S F') \otimes_S X \to (F \otimes_S F') \otimes_S Y \to (F \otimes_S F') \otimes_S Z \to 0,$$

is an admissible short exact sequence, by [3, Proposition 2.3.3]. Now, in view of Theorem 4.1 the result follows. □

Corollary 4.6. *Let T be a multiplicatively closed subset of S, and let F be a flat S-act. Then $T^{-1}F$ is a flat S-act.*

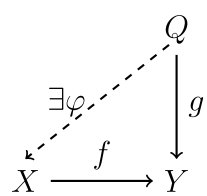
Proof. By [3, Proposition 2.4.4(ii)], $T^{-1}F \cong F \otimes_S T^{-1}S$. Also, $T^{-1}S$ is flat by [5, Theorem 2.2]. Now, the assertion follows from Proposition 4.5. □

Corollary 4.7. *Let I be an ideal of the monoid S such that S/I is a flat S-act. Then F/IF is flat, provided that F is a flat S-act.*

Proof. By [6, Theorem 3.2], $S/I \otimes_S F \cong F/IF$. Hence, we get the assertion by Proposition 4.5. □

The notion of admissibly projective S-act was defined in [3] as follows:

Definition 4.8. An S-act Q is called *admissibly projective* if it satisfies the lifting property:



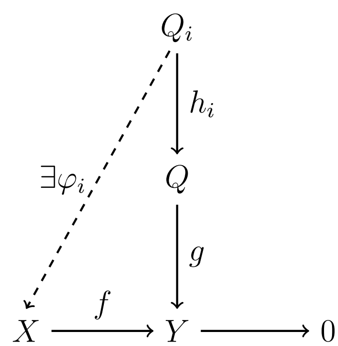
whenever $f : X \to Y$ is an admissible surjection, that is, for any S-homomorphism $g : Q \to Y$ there exists an S-homomorphism $\varphi : Q \to X$ such that $g = f \circ \varphi$.

Recall that a non-zero element $s \in S$ is called *zero-divisor*, if there exists a non-zero element $t \in S$ such that $st = 0$. An element $s \in S$ is called *cancelable*, if $rs = ts$ implies that $r = t$ where $r, t \in S$. An S-act X is called *torsion free*, if for any $x, x' \in X$ and for any cancelable element $s \in S$ the equality $xs = x's$ implies that $x = x'$. Note that any torsion free S-act is admissibly projective, by [3, Proposition 3.3.10]. Also, every projective S-act is admissibly projective, but not vice versa, see [3, Example 3.3.9].

In the following, we show that the coproduct of a family of admissibly projective S-acts are also admissibly projective.

Proposition 4.9. *Let* $\{Q_i \mid i \in I\}$ *be a family of admissibly projective* S*-acts. Then* $\bigvee_{i\in I} Q_i$ *is admissibly projective.*

Proof. Let $Q = \bigvee_{i\in I} Q_i$. Since Q_i is admissibly projective for each $i \in I$, we have the following commutative diagram of S-acts and S-homomorphisms:



where f is an admissible epimorphism and h_i is the canonical injection map. Define $\varphi : Q \to X$ by

$$\varphi(q) = \begin{cases} \varphi_i(q) & \text{if } \ 0_Q \neq q \in Q_i \text{ for some } i \in I, \\ 0_X & \text{if } \ q = 0_Q. \end{cases}$$

It is easily seen that $f \circ \varphi = g$, which means that Q is admissibly projective. □

In the following, we show that the class of flat S-acts is a subclass of admissibly projective S-acts.

Proposition 4.10. *Every flat* S*-act is admissibly projective.*

Proof. Let X be a flat S-act. By [7, Lemma 3.9.2 and Proposition 3.10.3], X is torsion free. Now, the assertion follows from [3, Proposition 3.3.10]. □

It is natural to ask whether the tensor products of admissibly projective S-acts is admissibly projective and whether the localization of admissibly projective S-act is admissibly projective. In the following, we answer to these questions in the special cases.

Corollary 4.11. *Let* T *be a multiplicatively closed subset of* S*. Then the following statements hold.*

(i) $T^{-1}S$ *is admissibly projective* S*-act.*
(ii) *Let* F *be a flat* S*-act. Then* $T^{-1}F$ *is an admissibly projective* S*-act.*

(iii) *Let F and F' be flat S-acts. Then $F \otimes_S F'$ is an admissibly projective S-act.*

(iv) *Let $\{F_i \mid i \in I\}$ be a family of flat S-acts. Then $\bigvee_{i \in I} F_i$ is admissibly projective.*

Acknowledgments

The authors would like to thank the referees for helpful comments and suggestions.

References

1. Y. Chen and K. P. Shum, Rees short exact sequence of S-systems, *Semigroup Forum*, **65** (2002), 141–148.
2. C. Chu, O. Lorscheid and R. Santhanam, Sheaves and K-theory for $\mathbb{F}_1$-schemes, *Adv. Math.*, **229**(4) (2012), 2239–2286.
3. J. Flores, *Homological Algebra for Commutative Monoids,* PhD thesis, Rutgers University, New Jersey, 2015.
4. P. A. Grillet, Irreducible actions, *Period. Math. Hungar.*, **54**(1) (2007), 51–76.
5. S. Irannezhad and A. Madanshekaf, Act of fractions and flatness properties, *Southeast Asian Bull. Math.*, **44**(2) (2020), 229–243.
6. M. Jafari, A. Golchin and H. Mohammadzadeh Sanny, Preservation of Rees exact sequences, *Math. Slovaca*, **69**(5) (2019), 1–10.
7. M. Kilp, U. Knauer and A. V. Mikhalev, *Monoids, Acts and Categories,* Walter de Gruyter, Berlin, 2000.

Elahe Nafarieh Talkhooncheh
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
Email: elahe.nafariyeh@srbiau.ac.ir

Maryam Salimi
Department of Mathematics, East Tehran Branch, Islamic Azad University, Tehran, Iran.
Email: maryamsalimi@ipm.ir

Hamid Rasouli
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
Email: hrasouli@srbiau.ac.ir

Elham Tavasoli
Department of Mathematics, East Tehran Branch, Islamic Azad University, Tehran, Iran.
Email: elhamtavasoli@ipm.ir

Abolfazl Tehranian
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
Email: tehranian@srbiau.ac.ir

Journal of Algebraic Systems

ADMISSIBLE (REES) EXACT SEQUENCES AND FLAT ACTS

E. NAFARIEH TALKHOONCHEH, M. SALIMI, H. RASOULI,
E. TAVASOLI AND A. TEHRANIAN

رشته‌های دقیق قابل‌قبول (ریس) و اکت‌های یکدست

الهه نفریه طالخونچه[۱]، مریم سلیمی[۲]، حمید رسولی[۳]، الهام توسلی[۴] و ابوالفضل تهرانیان[۵]

[۱,۳,۵]گروه ریاضی، واحد علوم و تحقیقات، دانشگاه آزاد اسلامی، تهران، ایران

[۲,۴]گروه ریاضی، واحد تهران شرق، دانشگاه آزاد اسلامی، تهران، ایران

فرض کنیم S یک مونوئید جابجایی و صفردار باشد. در این مقاله، برخی خواص رشته‌های دقیق کوتاه قابل‌قبول (ریس) از S-اکت‌ها مورد بررسی قرار گرفته است. به‌طور ویژه، نشان داده شده است که هر رشته‌ی دقیق کوتاه قابل‌قبول از S-اکت‌ها، یک رشته‌ی دقیق کوتاه ریس است. همچنین، توصیفی از اکت‌های یکدست با حفظ رشته‌های دقیق کوتاه بیان شده است. به عنوان نتیجه، برای S-اکت یکدست F نشان می‌دهیم که تابعگون $F \otimes_S -$ همریختی‌های قابل‌قبول را حفظ می‌کند. درآخر، ثابت شده است که کلاس S-اکت‌های یکدست یک زیر کلاس از کلاس S-اکت‌های پروژکتیو قابل‌قبول است.

کلمات کلیدی: S-اکت، رشته‌ی دقیق ریس، رشته‌ی دقیق قابل‌قبول، اکت‌های پروژکتیو قابل‌قبول.