Journal of Algebraic Systems
Vol. 12, No. 2, (2025), pp 367-377

AN IDENTITY RELATED TO θ-CENTRALIZERS IN SEMIPRIME RINGS

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Abstract. Let R be a 2-torsion free semiprime ring and θ be an epimorphism of R. In this paper, under special hypotheses, we prove that if $T(xyx) = \theta(x)T(y)\theta(x)$ holds for all $x, y \in R$, then T is a θ-centralizer.

1. Introduction and Preliminaries

This research has been motivated by the works of Albas [1], Daif and Tammam El-Sayiad [2] and Vukman [4, 5]. Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n > 1$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is semiprime in case $xRx = 0$ implies $x = 0$.

An additive mapping $T : R \longrightarrow R$ is called *left centralizer* (*right centralizer*) if for all $x, y \in R$,

$$T(xy) = T(x)y, \qquad \big(T(xy) = xT(y)\big).$$

We follow Zalar [6] and call T a centralizer in case T is both a left and a right centralizer. In case R has an identity element $T : R \longrightarrow R$ is a left (right) centralizer if and only if T is of the form $T(x) = ax$, $(T(x) = xa)$ for some fixed element $a \in R$.

An additive mapping $T : R \longrightarrow R$ is called a left (right) Jordan centralizer in case $T(x^2) = T(x)x$ $\big(T(x^2) = xT(x)\big)$ holds for $x \in R$.

DOI: 10.22044/JAS.2023.11856.1607.
MSC(2010): Primary: 16A12; Secondary: 16A68, 16A72.
Keywords: Semiprime ring; Centralizer; θ-Centralizer; Epimorphism.
Received: 24 April 2022, Accepted: 5 April 2023.

Clearly, every centralizer is a Jordan centralizer, but the converse is not true in general, as was demonstrated in [3, Example 2.6].

Zalar in [6] has proved that every left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer.

Vukman in [4] prove that in case we have an additive mapping $T : R \longrightarrow R$, where R is a 2-torsion free semiprime ring, satisfying the relation

$$2T(x^2) = T(x)x + xT(x)$$

for all $x \in R$, then T is a centralizer.

Let θ be an endomorphism of R. An additive mapping $T : R \longrightarrow R$ is called *left θ-centralizer* (*right θ-centralizer*) if for all $x, y \in R$,

$$T(xy) = T(x)\theta(y), \quad \big(T(xy) = \theta(x)T(y)\big).$$

If T is both left and right θ-centralizer then it is called a θ-centralizer.

This concept introduced by E. Albas [1], and some interesting generalization of centralizer to the case of θ-centralizer on semiprime rings obtained. Clearly, every centralizer is a special case of a θ-centralizer with $\theta = id$, the identity map on R.

The result of Vukman [4, Theorem 1] generalized for θ-centralizer as follows.

Theorem 1.1. [2, Theorem 1.2] *Let R be a 2-torsion free semiprime ring with identity element, and let $T : R \longrightarrow R$ be an additive mapping such that*

$$2T(x^2) = T(x)\theta(x) + \theta(x)T(x), \qquad x \in R,$$

where θ is a epimorphism of R. Then T is both a left and a right θ-centralizer.

Theorem 1.2. [2, Theorem 1.3] *Let R be a 2-torsion free semiprime ring with center $Z(R)$, and let $T : R \longrightarrow R$ be an additive mapping such that*

$$2T(x^2) = T(x)\theta(x) + \theta(x)T(x), \qquad x \in R,$$

where θ is a epimorphism of R with $\theta(Z(R)) = Z(R)$. Then T is both a left and a right θ-centralizer.

If $T : R \longrightarrow R$ is a θ-centralizer associated with an endomorphism $\theta : R \longrightarrow R$, where R is an arbitrary ring, then T satisfies the relation

$$T(xyx) = \theta(x)T(y)\theta(x), \quad x, y \in R, \tag{1.1}$$

but the converse is false in general (See example 2.1 below). Hence it seems natural to ask whether the converse statement is true. More precisely, we are asking whether an additive mapping T on a ring R

satisfying relation (1.1) is a θ-centralizer.

The aim of this paper is to prove that the answer is affirmative under some additional conditions (Theorem 2.8).

2. θ-CENTRALIZERS

We commence with the next example which shows that an additive mapping T on a ring R satisfying relation (1.1) is not a θ-centralizer, in general.

Example 2.1. Let $R = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : \quad a, b, c \in \mathbb{R} \right\}$, and define $\theta, T : R \longrightarrow R$ by

$$\theta\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & c & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, $\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in R$, hence θ is a endomorphism of R. Moreover,

$$T(xyx) = \theta(x)T(y)\theta(x) = 0,$$

for all $x, y \in R$. Therefore T satisfies in (1.1), but it is not left (right) θ-centralizer, because $\theta(x)T(y) = T(x)\theta(y) = 0$ while $T(xy) \neq 0$, in general.

For the proof of the main result we shall need some lemmas.

Lemma 2.2. [5, Lemma 1] *Let R be a semiprime ring and let the identity $atb + btc = 0$ holds for all $t \in R$ and for some $a, b, c \in R$. In this case $(a + c)tb = 0$ for all $t \in R$.*

We shall write $[x, y]$ for $xy - yx$ and use basic commutator identities

$$[xy, z] = [x, z]y + x[y, z] \quad \text{and} \quad [x, yz] = [x, y]z + y[x, z].$$

For simplicity, throughout the paper we put $\phi(x) = [T(x), \theta(x)]$ for all $x \in R$.

Lemma 2.3. *Let R be a 2-torsion free semiprime ring and θ be an epimorphism of R. If $T : R \longrightarrow R$ is an additive mapping such that*

$$T(xyx) = \theta(x)T(y)\theta(x), \tag{2.1}$$

holds for all $x, y \in R$, then $[\phi(x), \theta(x)] = 0$, for all $x \in R$.

Proof. Replacing x by $a+x$ in (2.1) we get

$$T(xya+ayx)=\theta(x)T(y)\theta(a)+\theta(a)T(y)\theta(x), \quad x,y,a\in R. \tag{2.2}$$

Putting $y=x$ and $a=y$ in (2.2) one obtains

$$T(x^2y+yx^2)=\theta(x)T(x)\theta(y)+\theta(y)T(x)\theta(x), \quad x,y\in R. \tag{2.3}$$

Interchanging y by xyx in (2.3) we obtain

$$T(x^3yx+xyx^3)=\theta(x)T(x)\theta(xyx)+\theta(xyx)T(x)\theta(x), \tag{2.4}$$

for all $x,y\in R$. Replacing y by x^2y+yx^2 in (2.1) to get

$$T(x^3yx+xyx^3)=\theta(x)T(x^2y+yx^2)\theta(x), \tag{2.5}$$

Using (2.3) and (2.5),

$$T(x^3yx+xyx^3)=\theta(x)^2T(x)\theta(y)\theta(x)+\theta(x)\theta(y)T(x)\theta(x)^2, \tag{2.6}$$

for all $x,y\in R$. From (2.4) and (2.6) we arrive at

$$-\theta(x)\phi(x)\theta(y)\theta(x)+\theta(x)\theta(y)\phi(x)\theta(x)=0, \quad x,y\in R. \tag{2.7}$$

Since θ is surjective, by (2.7) and Lemma 2.2 $\big($with $a=-\theta(x)\phi(x)$, $t=\theta(y)$, $b=\theta(x)$ and $c=\phi(x)\theta(x)\big)$ we get

$$[\phi(x),\theta(x)]\theta(y)\theta(x)=0, \quad x,y\in R. \tag{2.8}$$

For $z=\theta(y)\phi(x)$, there exist $a\in R$ such that $\theta(a)=z$. If we take $y=a$ in (2.8) we obtain

$$[\phi(x),\theta(x)]\theta(y)\phi(x)\theta(x)=[\phi(x),\theta(x)]z\theta(x)=0. \tag{2.9}$$

Multiplying (2.8) from the right by $\phi(x)$, we have

$$[\phi(x),\theta(x)]\theta(y)\theta(x)\phi(x)=0. \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$[\phi(x),\theta(x)]\theta(y)[\phi(x),\theta(x)]=0, \quad x,y\in R.$$

Since θ is surjective and R is semiprime, we obtain $[\phi(x),\theta(x)]=0$ for all $x\in R$. □

Lemma 2.4. *With the same hypotheses of Lemma 2.3, we have*

$$\theta(x)\phi(x)\theta(x)=0, \qquad x\in R.$$

Proof. By Lemma 2.3, $[\phi(x),\theta(x)]=0$, for all $x\in R$. Replacing x by $x+y$, we get

$$[\phi(x),\theta(y)]+[[T(x),\theta(y)],\theta(x)]+[\phi(y),\theta(x)]$$
$$+[[T(y),\theta(x)],\theta(y)]+[[T(x),\theta(y)],\theta(y)]+[[T(y),\theta(x)],\theta(x)]=0.$$

Putting $-x$ for x in the above relation and comparing the relation so obtained with the above relation we arrive at

$$[\phi(x),\theta(y)]+[[T(x),\theta(y)],\theta(x)]+[[T(y),\theta(x)],\theta(x)]=0. \tag{2.11}$$

Putting xyx for y in (2.11) and using (2.1), Lemma 2.3 and (2.11) we obtain

$$\begin{aligned}0=&[\phi(x),\theta(xyx)]+[[T(x),\theta(xyx)],\theta(x)]+[[T(xyx),\theta(x)],\theta(x)]\\=&\theta(x)[\phi(x),\theta(y)]\theta(x)+[\theta(x)[T(y),\theta(x)]\theta(x),\theta(x)]\\&+\big[\phi(x)\theta(y)\theta(y)+\theta(x)[T(x),\theta(y)]\theta(x)+\theta(x)\theta(y)\phi(x),\theta(x)\big]\\=&\theta(x)[\phi(x),\theta(y)]\theta(x)+\theta(x)[[T(y),\theta(x)],\theta(x)]\theta(x)\\&+\theta(x)[\theta(y),\theta(x)]\phi(x)+\phi(x)[\theta(y),\theta(x)]\theta(x)\\&+\theta(x)[[T(x),\theta(y)],\theta(x)]\theta(x)\\=&\phi(x)[\theta(y),\theta(x)]\theta(x)+\theta(x)[\theta(y),\theta(x)]\phi(x)\\=&\phi(x)\theta(y)\theta(x)^2-\theta(x)^2\theta(y)\phi(x)\\&+\theta(x)\theta(y)\theta(x)\phi(x)-\phi(x)\theta(x)\theta(y)\theta(x).\end{aligned}$$

Thus,

$$\begin{aligned}&\phi(x)\theta(y)\theta(x)^2-\theta(x)^2\theta(y)\phi(x)\\&+\theta(x)\theta(y)\theta(x)\phi(x)-\phi(x)\theta(x)\theta(y)\theta(x)=0.\end{aligned} \tag{2.12}$$

By Lemma 2.3, $\phi(x)\theta(x)=\theta(x)\phi(x)$, so from (2.7) and (2.12) we get

$$\phi(x)\theta(y)\theta(x)^2-\theta(x)^2\theta(y)\phi(x)=0, \quad x,y\in R. \tag{2.13}$$

Multiplying (2.13) from the left by $\theta(x)$, gives

$$\theta(x)\phi(x)\theta(y)\theta(x)^2-\theta(x)^3\theta(y)\phi(x)=0. \tag{2.14}$$

Using (2.7) the equality (2.14) reduces to

$$\theta(x)\theta(y)\phi(x)\theta(x)^2-\theta(x)^3\theta(y)\phi(x)=0, \quad x,y\in R. \tag{2.15}$$

Left multiplication of the above relation by $T(x)$ gives

$$T(x)\theta(x)\theta(y)\phi(x)\theta(x)^2-T(x)\theta(x)^3\theta(y)\phi(x)=0. \tag{2.16}$$

For $z=T(x)\theta(y)$, there exist $a\in R$ such that $\theta(a)=z$. Putting a for y in (2.15) we obtain

$$\theta(x)T(x)\theta(y)\phi(x)\theta(x)^2-\theta(x)^3T(x)\theta(y)\phi(x)=0. \tag{2.17}$$

Subtracting (2.17) from (2.16) we obtain

$$\phi(x)\theta(y)\phi(x)\theta(x)^2-[T(x),\theta(x)^3]\theta(y)\phi(x)=0, \quad x,y\in R. \tag{2.18}$$

By (2.18) and Lemma 2.2 we have

$$\big([T(x),\theta(x)^3]-\phi(x)\theta(x)^2\big)\theta(y)\phi(x)=0, \tag{2.19}$$

which by Lemma 2.3 reduces to

$$\big(\theta(x)\phi(x)\theta(x) + \theta(x)^2\phi(x)\big)\theta(y)\phi(x) = 0, \quad x, y \in R. \tag{2.20}$$

By Lemma 2.3,

$$\theta(x)^2\phi(x) = \theta(x)\phi(x)\theta(x), \tag{2.21}$$

hence it follows from (2.20) and (2.21) that

$$\theta(x)\phi(x)\theta(x)\theta(y)\phi(x) = 0, \tag{2.22}$$

for all $x, y \in R$. Multiplying (2.22) from the right by $\theta(x)$, we get

$$\theta(x)\phi(x)\theta(x)\theta(y)\phi(x)\theta(x) = 0. \tag{2.23}$$

Replacing y by yx in (2.23), we obtain

$$\theta(x)\phi(x)\theta(x)\theta(y)\theta(x)\phi(x)\theta(x) = 0, \quad x, y \in R. \tag{2.24}$$

Since θ is surjective and R is semiprime, $\theta(x)\phi(x)\theta(x) = 0$ for all $x \in R$. This finishes the proof. □

Lemma 2.5. *With the same hypotheses of Lemma 2.3, $\theta(x)\phi(x) = 0$ for all $x \in R$.*

Proof. By (2.7) we have

$$-\theta(x)\phi(x)\theta(y)\theta(x) + \theta(x)\theta(y)\phi(x)\theta(x) = 0, \qquad x, y \in R.$$

Replacing y by yx in the above equality and using Lemma 2.4, we get

$$\theta(x)\phi(x)\theta(y)\theta(x)^2 = 0. \tag{2.25}$$

for all $x, y \in R$. For $z = \theta(y)T(x)$, there exist $a \in R$ such that $\theta(a) = z$. Taking $y = a$ in (2.25), we arrive at

$$\theta(x)\phi(x)\theta(y)T(x)\theta(x)^2 = 0, \qquad x, y \in R. \tag{2.26}$$

Right multiplication of (2.25) by $T(x)$ gives

$$\theta(x)\phi(x)\theta(y)\theta(x)^2T(x) = 0. \tag{2.27}$$

Subtracting (2.27) from (2.26) we obtain

$$\theta(x)\phi(x)\theta(y)[T(x), \theta(x)^2] = 0, \tag{2.28}$$

which can be written as

$$\theta(x)\phi(x)\theta(y)\big(\phi(x)\theta(x) + \theta(x)\phi(x)\big) = 0, \qquad x, y \in R. \tag{2.29}$$

Applying Lemma 2.3, we have

$$\theta(x)\phi(x)\theta(y)\theta(x)\phi(x) = 0. \tag{2.30}$$

Since θ is surjective, the semiprimeness of R implies that $\theta(x)\phi(x) = 0$ for all $x \in R$, as required. □

Lemma 2.6. *With the same hypotheses of Lemma 2.3, $\phi(x) = 0$ for all $x \in R$.*

Proof. From Lemma 2.3 and Lemma 2.5, $\phi(x)\theta(x) = 0$, for all $x \in R$. The linearization gives

$$\phi(x)\theta(y) + [T(x), \theta(y)]\theta(x) + [T(y), \theta(x)]\theta(x) \\ + \phi(y)\theta(x) + [T(x), \theta(y)]\theta(y) + [T(y), \theta(x)]\theta(y) = 0.$$

From the above relation one obtains

$$\phi(x)\theta(y) + [T(x), \theta(y)]\theta(x) + [T(y), \theta(x)]\theta(x) = 0. \tag{2.31}$$

Multiplying (2.31) from the right by $\phi(x)$ and using Lemma 2.5 we get $\phi(x)\theta(y)\phi(x) = 0$, for every $x, y \in R$. Since θ is surjective, it follows from the semiprimeness of R that $\phi(x) = 0$ for all $x \in R$. □

Lemma 2.7. *With the same hypotheses of Lemma 2.3,*

$$T(xy + yx) = T(y)\theta(x) + \theta(x)T(y),$$

for all $x, y \in R$.

Proof. We first prove that for all $x, y \in R$,

$$\theta(x)h(x, y)\theta(x) = 0, \qquad [h(x, y), \theta(x)] = 0, \tag{2.32}$$

where $h(x, y) = T(xy + yx) - T(y)\theta(x) - \theta(x)T(y)$.

Replacing y by $xy + yx$ in (2.1) we get

$$T(x^2yx + xyx^2) = \theta(x)T(xy + yx)\theta(x), \quad x, y \in R. \tag{2.33}$$

Putting x^2 for a in (2.2),

$$T(xyx^2 + x^2yx) = \theta(x)T(y)\theta(x^2) + \theta(x^2)T(y)\theta(x), \quad x, y \in R. \tag{2.34}$$

Subtracting (2.33) from (2.34), we obtain

$$\theta(x)h(x, y)\theta(x) = 0, \quad x, y \in R, \tag{2.35}$$

and hence the left equation in (2.32) is satisfied. The linearization of (2.35) gives

$$\theta(x)h(x, y)\theta(z) + \theta(x)h(z, y)\theta(x) + \theta(z)h(x, y)\theta(x) = 0, \tag{2.36}$$

for all $x, y, z \in R$. Multiplying (2.36) from the right by $h(x, y)\theta(x)$, to get

$$\theta(x)h(x, y)\theta(z)h(x, y)\theta(x) = 0, \quad x, y, z \in R. \tag{2.37}$$

By Lemma 2.6, $\phi(x) = 0$ for all $x \in R$. Replacing x by $x + y$ we arrive at

$$[T(x), \theta(y)] + [T(y), \theta(x)] = 0, \quad x, y \in R. \tag{2.38}$$

Putting $xy+yx$ for y in (2.38) and using Lemma 2.6,

$$\begin{aligned}0 =&[T(x),\theta(xy+yx)]+[T(xy+yx),\theta(x)]\\ =&\theta(x)[T(x),\theta(y)]+[T(x),\theta(y)]\theta(x)+[T(xy+yx),\theta(x)].\end{aligned}$$

According to (2.38) one can replace $[T(x),\theta(y)]$ by $-[T(y),\theta(x)]$ in the above equality, so we get

$$[T(xy+yx),\theta(x)]-\theta(x)[T(y),\theta(x)]-[T(y),\theta(x)]\theta(x)=0. \quad (2.39)$$

Rewritten (2.38) as

$$[T(xy+yx)-T(y)\theta(x)-\theta(x)T(y),\theta(x)]=0,$$

hence $[h(x,y),\theta(x)]=0$ for all $x,y\in R$. Thus, the right equation in (2.32) is proved.

It follows from $[h(x,y),\theta(x)]=0$ that $h(x,y)\theta(x)=\theta(x)h(x,y)$ and hence by (2.37),

$$h(x,y)\theta(x)\theta(z)h(x,y)\theta(x)=0, \quad x,y,z\in R. \quad (2.40)$$

Since θ is surjective and R is semiprime,

$$h(x,y)\theta(x)=0, \quad x,y\in R, \quad (2.41)$$

and hence $\theta(x)h(x,y)=0$. Replacing x by $x+z$ in (2.41), gives

$$h(x,y)\theta(z)+h(z,y)\theta(x)=0, \quad x,y,z\in R. \quad (2.42)$$

Multiplying (2.42) from the right by $h(x,y)$, we get

$$h(x,y)\theta(z)h(x,y)=0,$$

and consequently, $h(x,y)=0$. This completes the proof. □

Theorem 2.8. *Let R be a 2-torsion free semiprime ring and θ be an epimorphism of R. If $T:R\longrightarrow R$ is an additive mapping such that*

$$T(xyx)=\theta(x)T(y)\theta(x), \quad (2.43)$$

holds for all $x,y\in R$, then T is a θ-centralizer with each of the following conditions.

(i) *R is unital,*
(ii) *$\theta(Z(R))=Z(R)$.*

Proof. By preceding lemma $T(xy+yx)=T(y)\theta(x)+\theta(x)T(y)$, for all $x,y\in R$. Replacing y by x, we get

$$2T(x^2)=T(x)\theta(x)+\theta(x)T(x),$$

for all $x\in R$. Now the result follows from Theorem 1.1 and Theorem 1.2 if (i) and (ii) holds, respectively. □

Corollary 2.9. [5, Theorem 1] *Let R be a 2-torsion free semiprime ring. If $T : R \longrightarrow R$ is an additive mapping such that*

$$T(xyx) = xT(y)x, \qquad x, y \in R,$$

then T is a centralizer.

Let R, θ and T be as in Example 2.1. Then θ is a endomorphism of R and T satisfies in (2.43) for all $x, y \in R$, but it is not left (right) θ-centralizer. Therefore the mentioned condition in the hypotheses of Theorem 2.8 is crucial.

Remark that Theorem 2.8 with condition (ii) can be obtained from [1, Theorem 2]. Indeed, by Lemma 2.7 we have

$$T(xy + yx) = T(y)\theta(x) + \theta(x)T(y), \qquad x, y \in R.$$

Putting $y = x$, we get

$$2T(x^2) = T(x)\theta(x) + \theta(x)T(x), \qquad x \in R.$$

On the other hand, by Lemma 2.6, $\phi(x) = 0$ for all $x \in R$. This means that

$$T(x)\theta(x) = \theta(x)T(x),$$

and hence $T(x^2) = T(x)\theta(x) = \theta(x)T(x)$ for all $x \in R$. Thus, T is a θ-Jordan centralizer of R and by [1, Theorem 2] it is a θ-centralizer.

In the following result we show that in Theorem 2.8, if the equality (2.43) is replaced by the weaker condition

$$T(x^3) = \theta(x)T(x)\theta(x), \qquad x \in R,$$

then the result will be established with condition (i).

Theorem 2.10. *Let R be a 2-torsion free semiprime ring with an identity element 1, and let $\theta : R \longrightarrow R$ be an surjective homomorphism. If $T : R \longrightarrow R$ is an additive mapping such that*

$$T(x^3) = \theta(x)T(x)\theta(x), \tag{2.44}$$

for all $x \in R$, then T is a θ-centralizer.

Proof. Since θ is surjective homomorphism, $\theta(1) = 1$. Replacing x by $x + 1$ in (2.44), we get

$$3T(x^2) + 2T(x) = \theta(x)a\theta(x) + T(x)\theta(x) + a\theta(x) + \theta(x)T(x) + \theta(x)a, \tag{2.45}$$

where $a = T(1)$. Interchanging x by $-x$ in (2.45) and simplify the result we obtain

$$6T(x^2) = 2\theta(x)a\theta(x) + 2T(x)\theta(x) + 2\theta(x)T(x), \tag{2.46}$$

From (2.45) and (2.46),

$$2T(x) = a\theta(x) + \theta(x)a, \qquad x \in R. \tag{2.47}$$

By (2.46) and (2.47),

$$3a\theta(x)^2 + 3\theta(x)^2 a = 2\theta(x)a\theta(x) + 2T(x)\theta(x) + 2\theta(x)T(x), \tag{2.48}$$

for all $x \in R$. Combining (2.47) and (2.48), to get

$$a\theta(x)^2 + \theta(x)^2 a - 2\theta(x)a\theta(x) = 0. \tag{2.49}$$

The relation (2.49) can be written in the form

$$[[a, \theta(x)], \theta(x)] = 0, \qquad x \in R. \tag{2.50}$$

The linearization of (2.50), gives

$$[[a, \theta(x)], \theta(y)] + [[a, \theta(y)], \theta(x)] = 0, \qquad x, y \in R. \tag{2.51}$$

It follow from (2.50) that

$$\theta(x)[a, \theta(x)] = [a, \theta(x)]\theta(x). \tag{2.52}$$

We substitute xy for y in (2.51) and using (2.50), (2.51) and (2.52), we arrive at

$$\begin{aligned} 0 =& [[a, \theta(x)], \theta(xy)] + [[a, \theta(xy)], \theta(x)] \\ =& [[a, \theta(x)], \theta(x)]\theta(y) + \theta(x)[[a, \theta(x)], \theta(y)] \\ &+ [a, \theta(x)][\theta(y), \theta(x)] + \theta(x)[[a, \theta(y)], \theta(x)] \\ =& [a, \theta(x)][\theta(y), \theta(x)]. \end{aligned}$$

Hence

$$[a, \theta(x)][\theta(y), \theta(x)] = 0, \qquad x, y \in R. \tag{2.53}$$

For $z = \theta(y)a$, there exist $t \in R$ such that $\theta(t) = z$. If we take $y = t$ in (2.53), we get

$$[a, \theta(x)][\theta(y)a, \theta(x)] = 0, \qquad x, y \in R. \tag{2.54}$$

It follows from (2.54) that $[a, \theta(x)]\theta(y)[a, \theta(x)] = 0$ for all $x, y \in R$. Thus, $[a, \theta(x)] = 0$ and hence by (2.47), $T(x) = a\theta(x) = \theta(x)a$ for all $x \in R$. This completes the proof. □

Corollary 2.11. *Let R be a 2-torsion free semiprime ring with an identity element 1, and let $\theta : R \longrightarrow R$ be a surjective homomorphism. Suppose that $T : R \longrightarrow R$ is an additive mapping. Then the following conditions are equivalents.*

(i) $2T(x^2) = T(x)\theta(x) + \theta(x)T(x)$,
(ii) $T(x) = T(1)\theta(x) = \theta(x)T(1)$, *i.e. T is a θ-centralizer,*
(iii) $T(x^3) = \theta(x)T(x)\theta(x)$.

Proof. The statement $(i) \implies (ii)$ is Theorem 1.1. $(ii) \implies (i)$ and $(ii) \implies (iii)$ is clear. $(iii) \implies (ii)$ is Theorem 2.10. □

From Theorem 2.8 and Theorem 2.10 the following question can be raised.

Question 2.12. *Let R be a 2-torsion free semiprime ring with center $Z(R)$, and suppose that $T : R \longrightarrow R$ is an additive mapping such that*

$$T(x^3) = \theta(x)T(x)\theta(x), \qquad x \in R,$$

where θ is a epimorphism with $\theta(Z(R)) = Z(R)$. Is T a θ-centralizer?

Acknowledgments

The author gratefully acknowledges the helpful comments of the anonymous referees. This research was partially supported by the Ayatollah Borujerdi University with Grant No. 15664-214257.

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Journal of Algebraic Systems

AN IDENTITY RELATED TO θ–CENTRALIZERS IN SEMIPRIME RINGS

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یک تساوی در رابطه با θ–ضربگرها در حلقه‌های نیم‌اول

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فرض کنید R یک حلقه بدون تاب و θ یک بروریختی روی حلقه R باشد. در این مقاله، تحت شرایط خاص، نشان می‌دهیم که اگر برای هر $x, y \in R$، تساوی $T(xyx) = \theta(x)T(y)\theta(x)$ برقرار باشد، آنگاه T یک θ–ضربگر روی R است.

کلمات کلیدی: حلقه نیم‌اول، ضربگر، θ–ضربگر، بروریختی.