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FURTHER STUDIES OF THE PERPENDICULAR GRAPHS OF MODULES

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Abstract. In this paper we continue our study of perpendicular graph of modules, that was introduced in [7]. Let R be a ring and M be an R-module. Two modules A and B are called orthogonal, written $A \perp B$, if they do not have non-zero isomorphic submodules. We associate a graph $\Gamma_\perp(M)$ to M with vertices $\mathcal{M}_\perp = \{(0) \neq A \leq M \mid \exists (0) \neq B \leq M \text{ such that } A \perp B\}$, and for distinct $A, B \in \mathcal{M}_\perp$, the vertices A and B are adjacent if and only if $A \perp B$. The main object of this article is to study the interplay of module-theoretic properties of M with graph-theoretic properties of $\Gamma_\perp(M)$. We study the clique number and chromatic number of $\Gamma_\perp(M)$. We prove that if $\omega(\Gamma_\perp(M)) < \infty$ and M has a simple submodule, then $\chi(\Gamma_\perp(M)) < \infty$. Among other results, it is shown that for a semi-simple module M, $\omega(\Gamma_\perp({}_RM)) = \chi(\Gamma_\perp({}_RM))$.

1. Introduction

The present paper is sequal to [7] and so the notations introduced in Introduction of [7] will remain in force. Thus throughout the paper, R is a ring with identity and M is a left R-module,

$$\mathcal{M}_\perp = \{(0) \neq A \lneq M \mid \exists (0) \neq B \lneq M \text{ such that } A \perp B\}$$

is the set of all vertices of the perpendicular graph. As [7], we say that two modules A and B are orthogonal, written $A \perp B$, when they

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do not have non-zero isomorphic submodules. Then the perpendicular graph of M, denoted by $\Gamma_\perp(M)$, is an undirected simple graph with the vertex set $\mathcal{M}_\perp$ in which every two distinct vertices A and B are adjacent if and only if $A \perp B$ (see [7] for more details).
An R-module M is said to be *simple*, if it is not a zero module and it has no non-trivial submodule. The *socle* of an R-module M, written $\mathrm{Soc}(M)$, is the sum of all simple submodules of M. An R-module M is said to be *semi-simple*, if $\mathrm{Soc}(M) = M$. By $N \leq_e M$, we mean N is an essential submodule of M. We say an R-module N is *subisomorphic submodule* of an R-module M, and denoted by $N \lesssim M$, when N is isomorphic to a submodule of M. A module M is called *atomic* if $M \neq 0$ and for any $x, y \in M \setminus \{0\}$, xR and yR have isomorphic non-zero submodules. For more details and some basic facts about atomic modules, the reader is referred to [3]. We should remind the reader that these atomic modules are different from those defined in [4].

Let G be a graph with the vertex set V(G). By *order* of G, we mean the number of vertices of G and we denote it by $|G|$. The *degree* of a vertex v in graph G, denoted by $\deg(v)$, is the number of edges incident with v. A *locally finite graph* is a graph in which every degree of any vertex is finite. A *complete graph* is a graph in which every pair of distinct vertices are adjacent. A *clique* of a graph is a maximal complete subgraph and the number of vertices in the largest clique of a graph G, denoted by $\omega(G)$, is called the *clique number* of G. An *independent set* is a set of vertices in a graph, no two of which are adjacent. A maximal independent set is either an independent set such that adding any other vertex to the set forces the set to contain an edge or the set of all vertices of an empty graph. The size of a maximal independent set of largest possible for a graph G is called the *independence number* of G, denoted by $\alpha(G)$. Let $\chi(G)$, denote the *chromatic number* of G, that is, the minimum number of colors can be assigned to the vertices of G such that every two adjacent vertices have different colors. Obviously $\omega(G) \leq \chi(G)$. In section 2, we look at the coloring of the perpendicular graph of modules. We study conditions under which the chromatic number of $\Gamma_\perp(M)$ is finite. We prove that if $\omega(\Gamma_\perp(M)) < \infty$ and M has a simple submodule, then $\chi(\Gamma_\perp(M)) < \infty$. Also it is shown that for a semi-simple module M, $\chi(\Gamma_\perp({}_RM)) = \omega(\Gamma_\perp({}_RM))$. We give an example, which show that the semi-simple hypothesis is needed for the previous fact. In section 3, we study conditions under which the $\Gamma_\perp(M)$ is finite or infinite. Finally, it is proved that $\Gamma_\perp(M)$ is n-regular, if $\Gamma_\perp(M) \cong K_{n,n}$.

2. Clique Number and Chromatic Number of Perpendicular Graph

Let M be an R-module. In this section, we obtain some results on the clique number of $\Gamma_{\perp}(M)$. We study the condition under which the chromatic number of $\Gamma_{\perp}(M)$ is finite. We show that if $\omega(\Gamma_{\perp}(M)) < \infty$ and M has a simple submodule, then $\chi(\Gamma_{\perp}(M)) < \infty$. The next result is a counterpart of [1, Theorem 3.8].

Proposition 2.1. *Let M be an R-module, $\Gamma_{\perp}(M)$ be infinite and S be a simple submodule of M of finite degree.Then the following statements hold :*

(1) *The number of non-isomorphic simple submodules of M is finite.*
(2) *$\chi(\Gamma_{\perp}(M)) < \infty$.*
(3) *$\alpha(\Gamma_{\perp}(M)) = \infty$.*

Proof. (1) Every two non-isomorphic simple submodules are adjacent. Now, if the number of non-isomorphic simple submodules of M is infinite and hence $\deg(S) = \infty$, which is a contradiction.

(2) Let $\mathcal{X} = \{X_i\}_{i\in I}$ be all of the vertices of $\Gamma_{\perp}(M)$ which are not adjacent to S and $\mathcal{Y} = \{Y_j\}_{j\in J}$ be all of vertices of $\Gamma_{\perp}(M)$ which are adjacent to S. It is clear that $|\mathcal{Y}|$ is finite. For each $i \in I$, $X_i \not\perp S$ and for each $j \in J$, $Y_j \perp S$. But for any $X_\alpha, X_\beta \in \mathcal{X}$ we have $X_\alpha \not\perp S$ and $X_\beta \not\perp S$ such that $\alpha, \beta \in I$. Furthermore any two distinct elements X_α and X_β of $\mathcal{X}$ have isomorphic simple submodules, i.e., $X_\alpha \not\perp X_\beta$. This shows that all of vertices in $\mathcal{X}$ can be colored by one color. Inasmuch as $\mathcal{Y}$ is finite, we deduce that of $\chi(\Gamma_{\perp}(M)) < \infty$.

(3) By proof of (2), $\mathcal{X}$ is a maximal independent set of $\Gamma_{\perp}(M)$. Also note that $\mathcal{X}$ is infinite independent set in $\Gamma_{\perp}(M)$ and so $\alpha(\Gamma_{\perp}(M)) = \infty$. □

The next lemma states every element of a clique in $\Gamma_{\perp}(M)$ is an atomic module.

Lemma 2.2. *Let M be an R-module and $\omega(\Gamma_{\perp}(M)) = n$ such that $\{A_1, A_2, ..., A_n\}$ is the complete subgraph in $\Gamma_{\perp}(M)$. Then for any $A, B \leq A_i$ such that $1 \leq i \leq n$, $A \not\perp B$.*

Proof. Suppose that there exists $1 \leq i \leq n$ and $A, B \leq A_i$ such that $A \perp B$. For any j such that $1 \leq i \neq j \leq n$, we have $A \perp A_j$ and $B \perp A_j$, as $A_i \perp A_j$. Hence

$$\mathcal{A} = \{A, B, A_1, A_2, ..., A_{i-1}, A_{i+1}, ..., A_n\}$$

induces a clique in $\Gamma_{\perp}(M)$, which is a contradiction. □

The next result is a counterpart of [1, Theorem 3.10].

Theorem 2.3. *Let M be an R-module which has a simple submodule and $\omega(\Gamma_{\perp}(M)) < \infty$. Then the following hold:*

(1) *The number of non-isomorphic simple submodules of M is finite.*
(2) $\chi(\Gamma_{\perp}(M)) < \infty$.

Proof. (1) On the contrary, assume that the number of non-isomorphic simple submodules of M is infinite, but the set of all non-isomorphic simple submodules of M induces a clique in $\Gamma_{\perp}(M)$, and it is contradicted with $\omega(\Gamma_{\perp}(M)) < \infty$.

(2) Assume that $\omega(\Gamma_{\perp}(M)) < \infty$ and S is a simple submodule of M. Two cases may happen:

(Case 1) If $\Gamma_{\perp}(M)$ is a finite graph, then it is clear that

$$\chi(\Gamma_{\perp}(M)) < \infty.$$

(Case 2) If $\Gamma_{\perp}(M)$ is an infinite graph, then the number of vertices of $\Gamma_{\perp}(M)$ is infinite. Two cases may happen:

(Case a) If $\deg(S) < \infty$, then by Proposition 2.1, $\chi(\Gamma_{\perp}(M)) < \infty$.

(Case b) If $\deg(S) = \infty$, then the number of vertices of $\Gamma_{\perp}(M)$ which are adjacent to S is infinite. Assume that $\mathcal{X} = \{X_i\}_{i\in I}$ is all of the vertices of $\Gamma_{\perp}(M)$ which are adjacent to S and $\mathcal{Y} = \{Y_j\}_{j\in J}$ is all of the vertices of $\Gamma_{\perp}(M)$ which are not adjacent to S. It is clear that no two vertices of $\mathcal{Y}$ are adjacent. Therefore the vertices in $\mathcal{Y}$ are mutually non-adjacent, hence all of vertices in $\mathcal{Y}$ can be colored by one color. Since $\omega(\Gamma_{\perp}(M)) < \infty$, assume that $\omega(\Gamma_{\perp}(M)) = n$. There exist $A_1, A_2, ..., A_n \in \mathcal{M}_{\perp}$ such that $A_i \perp A_j$, for any $1 \leq i \neq j \leq n$. Hence $\mathcal{A} = \{A_1, A_2, ..., A_n\}$ is a clique in $\Gamma_{\perp}(M)$, which all of vertices in $\mathcal{A}$ can be colored by n color. We let $\mathcal{M}_{\perp} = \mathcal{X} \cup \mathcal{Y}$, it is easy to see that at most one of A_i is in $\mathcal{Y}$. Without less of generality, assume that $A_1 \in \mathcal{Y}$ and we give $A_i \in \mathcal{X}$ for $2 \leq i \leq n$. Now we put $\mathcal{T} = \mathcal{X} \setminus \{A_2, ..., A_n\}$. It is clear that for any $X_i \in \mathcal{T}$, X_i is not adjacent to all of the vertices in $\mathcal{A}$. Now, suppose that

$$\begin{aligned}
\mathcal{A}_2 &= \{X_i \in \mathcal{T} | X_i \not\perp A_2\},\\
\mathcal{A}_3 &= \{X_i \in \mathcal{T} | X_i \not\perp A_3\} \setminus \mathcal{A}_2,\\
&\vdots\\
\mathcal{A}_n &= \{X_i \in \mathcal{T} | X_i \not\perp A_n\} \setminus \bigcup_{i=2}^{i=n-1} \mathcal{A}_i.
\end{aligned}$$

We claim that $\mathcal{T} = \bigcup_{i=2}^{i=n} \mathcal{A}_i$. It is clear that $\bigcup_{i=2}^{i=n} \mathcal{A}_i \subset \mathcal{T}$ and we show that $\mathcal{T} \subset \bigcup_{i=2}^{i=n} \mathcal{A}_i$. Suppose that $X \in \mathcal{T}$, so there exists $2 \leq i \leq n$ such that $X \not\perp A_i$. Hence $X \in \mathcal{A}_i$. Finally, we claim that for any $2 \leq i \leq n$, every two vertices of $\mathcal{A}_i$ are not adjacent. Let $Z, Y \in \mathcal{A}_i$ such that $Z \perp Y$. We know that $Z \not\perp A_i$, $Y \not\perp A_i$. So there exist $Z_1 \leq Z$ and $Y_1 \leq Y$ and $A, B \leq A_i$ such that $Z_1 \cong A$, $Y_1 \cong B$. By previous lemma, $A \not\perp B$, i.e., A and B have isomorphic submodules. In this case Z_1 and Y_1 have isomorphic submodules, which is a contradiction. Hence all vertices of $\mathcal{A}_i$ can be colored by A_i color, for some $2 \leq i \leq n$. Thus the graph $\Gamma_{\perp}(M)$ can be colored so that adjacent vertices have different color. In this case, we have $\chi(\Gamma_{\perp}(M)) = \omega(\Gamma_{\perp}(M))$. □

The next result is a counterpart of [2, Theorem 2.5]. Now for any semi-simple R-module M, we find the clique number and chromatic number of $\Gamma_{\perp}(M)$ and observe that they are in fact the same.

Proposition 2.4. *Let M be semi-simple R-module such that $\Gamma_{\perp}(M) \neq \emptyset$. Then the following statements hold:*

(1) *The clique number and the chromatic number of $\Gamma_{\perp}(M)$ are equal to the cardinal number of the set of non-isomorphic simple submodules of M.*
(2) *The girth of $\Gamma_{\perp}(M)$ is 3 except when M has exactly two non-isomorphic simple submodules.*

Proof. (1) Let $\{N_i\}_{i \in I}$ be a clique in $\Gamma_{\perp}(M)$. So for any $i \neq j \in I$, $N_i \perp N_j$. Also for any $i \in I$, N_i contains a simple submodule of M such as S_i. Thus for any $i, j \in I$, $S_i \perp S_j$, i.e., $S_i \ncong S_j$. Assume that $\mathcal{S}$ is all of non-isomorphic simple submodules of M, hence $|I| \leq |\mathcal{S}|$ such that $\mathcal{S}$ is the largest clique in $\Gamma_{\perp}(M)$. Therefore the clique number of $\Gamma_{\perp}(M)$ is the cardinal number of the set of non-isomorphic simple submodules of M. But $|\mathcal{S}| = \omega(\Gamma_{\perp}(M)) \leq \chi(\Gamma_{\perp}(M))$ and so $\chi(\Gamma_{\perp}(M)) \geq |\mathcal{S}|$. We show that $\chi(\Gamma_{\perp}(M)) = |\mathcal{S}|$. Let $N \in \mathcal{M}_{\perp}$, so N contains a simple submodule of M such as S and we put:

$$N_S = \{T \in \mathcal{S} | T \cong N_1 \leq N\}.$$

By the Axiom of choice, for any $N \in \mathcal{M}_{\perp}$, we choose $T \in N_S$. Hence N, T can be colored by one color. We claim that by these conditions, $\Gamma_{\perp}(M)$ is colored. In fact we find the least of the number of color in $\Gamma_{\perp}(M)$ which adjacent vertices have different color. Suppose that N, K are two adjacent vertices in $\Gamma_{\perp}(M)$ such that have the same color. So N and $T_1 \in \mathcal{S}$ have the same color, also K and $T_2 \in \mathcal{S}$ have the same color. Therefore T_1 and T_2 have the same color, but $T_1, T_2 \in \mathcal{S}$ and so $T_1 = T_2$. Also $T_2 \cong K_1 \leq K$ and $T_1 \cong N_1 \leq N$, that is $N_1 \cong K_1$, i.e., $N \not\perp K$, which is a contradiction. Thus $\chi(\Gamma_{\perp}(M)) \leq |\mathcal{S}|$, so $\chi(\Gamma_{\perp}(M)) = |\mathcal{S}|$.

(2) Assume that the number of non-isomorphic simple submodules of M is $|I|$. If $|I| \geq 3$, then it is clear that $\mathrm{gr}(\Gamma_\perp(M)) = 3$. But if M has two non-isomorphic simple submodules S_1 and S_2, then $\Gamma_\perp(M)$ is a bipartite graph. Thus $\mathrm{gr}(\Gamma_\perp(M)) = 4$ or $\mathrm{gr}(\Gamma_\perp(M)) = \infty$. $\square$

The following example illustrate that condition of semi-simple is required on Proposition 2.4.

Example 2.5. Let $R = \mathbb{Z}$ and consider the R-module

$$M = \mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_q$$

such that p, q are distinct prime number. Note that M is not semi-simple module and $\omega(\Gamma_\perp(M)) = 3$ such that the number of non-isomorphic simple submodules of M is 2. Also $\mathrm{gr}(\Gamma_\perp(M)) = 3$.

The following figure shows that $\omega(\Gamma_\perp(\mathbb{Z}_{210})) = \chi(\Gamma_\perp(\mathbb{Z}_{210})) = 4$ and the number of non-isomorphic simple submodules is 4.

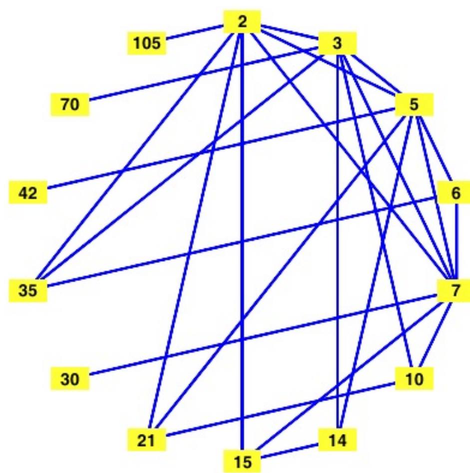

FIGURE 1. $\mathbb{Z}_{210}$

3. Perpendicular Graph and Some Finiteness Conditions

In this section, we find results with hypothesis which all vertices are finite degree. Also we show that $\Gamma_\perp(M)$ is finite if and only if M contains a simple submodule and every simple submodule of M has a finite degree. Finally, it is proved that $\Gamma_\perp(M)$ is n-regular, if $\Gamma_\perp(M) \cong K_{n,n}$.

Let $A \in \mathcal{M}_\perp$. By $\mathrm{N}(A)$, we mean the set of all vertices which are adjacent to A. It is called the neighbor of A in $\Gamma_\perp(M)$. Also $\mathrm{S}(A)$ means that all of submodules of A.

Lemma 3.1. *Let M be an R-module such that every vertex of $\Gamma_\perp(M)$ has a finite degree. Then the number of submodules of any vertex of $\Gamma_\perp(M)$ is finite.*

Proof. Assume that $N \in \mathcal{M}_\perp$ so there exists $K \leq M$ such that $K \perp N$. Since all submodules of N are adjacent to K and the degree of every vertex of $\Gamma_\perp(M)$ is finite, the number of submodules of N is finite. □

Now, we reduced the conditions of Lemma 3.1, we infer:

Lemma 3.2. *Let M be an R-module. If M contains a simple submodule and every simple submodule of M has a finite degree, then the number of submodules of any vertex of $\Gamma_\perp(M)$ is finite.*

Proof. Assume that S is a simple submodule of M such that $S \in \mathcal{M}_\perp$. We know that the number of vertices of $\Gamma_\perp(M)$ which adjacent to S is finite, so suppose that $\mathrm{N}(S) = \{N_1, N_2, ..., N_n\}$. Since $S \perp N_i$, S is adjacent to all submodules of N_i. That is the number of submodules of N_i is finite. Now, suppose that $\mathcal{B} = \{M_j\}_{j \in J}$ is the family all vertices of $\Gamma_\perp(M)$ which are not adjacent to S. We note that the vertices of $\mathcal{B}$ are mutually non-adjacent (If $M_1, M_2 \in \mathcal{B}$ and $M_1 \perp M_2$, inasmuch as $S \not\perp M_1$ and $S \not\perp M_2$ so there exist $M_1' \leq M_1$ and $M_2' \leq M_2$ such that $M_1' \cong S \cong M_2'$, which is a contradiction). For any $K \in \mathrm{S}(M_j)$, since $M_j \in \mathcal{M}_\perp$, there exists $N_i \in \mathrm{N}(S)$ such that $M_j \perp N_i$. Hence $K \perp N_i$. On the other hand, for every $N_i \in \mathrm{N}(S)$, N_i contains a simple submodule, hence assume that $S_i \subset N_i$, for every $1 \leq i \leq n$. Thus $K \perp S_i$ for any $K \in \mathrm{S}(M_j)$. Since $\deg(S_i) < \infty$, $|\mathrm{S}(M_j)| < \infty$. Hence the number of submodules of M_j is finite. □

The next example shows that the converse Lemmas 3.1 and 3.2 are not true.

Example 3.3. Let $R = \mathbb{Z}$ and consider the R-module

$$M = \mathbb{Z}_q \oplus \mathbb{Z}_{p^\infty}$$

where p, q are distinct prime number. By [6, Theorem 2.4], every submodule of M is $A \oplus B$ such that $A \leq \mathbb{Z}_{p^\infty}$ and $B \leq \mathbb{Z}_q$. Hence $\Gamma_\perp(M)$ is an infinite star graph such that for any $N \in \mathcal{M}_\perp$, $|\mathrm{S}(N)| < \infty$ but $\deg(\mathbb{Z}_\mathrm{q}) = \infty$.

The next result is a counterpart of [1, Corollary 3.9].

Theorem 3.4. *Let M be an R-module such that $\Gamma_{\perp}(M) \neq \emptyset$. Then the following statements hold:*

(1) *If M has no simple submodule, then $\Gamma_{\perp}(M)$ is infinite.*
(2) *If M contains a simple submodule and every simple submodule of M has finite degree, then $\Gamma_{\perp}(M)$ is finite.*

Proof. (1) It is clear.

(2) Suppose that M contains a simple submodule and every simple submodule of M has finite degree. First, the number of non-isomorphic simple submodules of M is finite. Second, assume that $\{S_i\}_{i \in I}$ is the family of all non-isomorphic simple submodules of M such that $|I| > 1$. If there exists $i \in I$ such that the number of simple submodules isomorphic to S_i is infinite, then for any $i \neq j$, $\deg(S_j)$ is infinite which is a contradiction. Hence the number of simple submodules which are isomorphic to S_i is finite, for any $i \in I$. That is the number of simple submodules of M if finite. Assume that $\{S_1, S_2, ..., S_n\}$ is the set of all simple submodules of M and we show that the number of vertices of $\Gamma_{\perp}(M)$ is finite. On the contrary the number of vertices of $\Gamma_{\perp}(M)$ is infinite. If every vertex of $\Gamma_{\perp}(M)$ is adjacent to S_i for some $i \in I$, then as each S_i is simple and has a finite degree, $|\Gamma_{\perp}(M)| \leq \sum_{i=1}^{n} \deg(S_i) < \infty$. Therefore there exists $N \in \mathcal{M}_{\perp}$ such that $N \not\perp S_i$ for any $i \in I$. Hence N has a submodule isomorphic to S_i and so for any $N \neq K \in \mathcal{M}_{\perp}$, $K \not\perp N$ which is the contradiction with $N \in \mathcal{M}_{\perp}$. □

The following corollary is an immediate consequence of Theorem 3.4.

Corollary 3.5. *Let M be an R-module and $\Gamma_{\perp}(M) \neq \emptyset$. The degree of any vertex of $\Gamma_{\perp}(M)$ is finite if and only if $\Gamma_{\perp}(M)$ is finite graph.*

A ray in an infinite graph is an infinite sequence of vertices $v_0, v_1, v_2, ...$ in which each vertex appears at most once in the sequence and each two consecutive vertices in the sequence are the two endpoints of an edge in the graph.

Lemma 3.6. [5, Lemma 1] *A locally finite graph has infinite diameter if and only if it contains a ray.*

Using Konigs Lemma and Lemma 3.6, we are able to give a completely graph theoretic proof for Corollary 3.5. The reader is reminded that Konigs Lemma say " If a graph is connected and locally finite, then if our graph is infinite, it has a ray (an infinite path)". Now we are ready to give the second proof.

Second proof for Corollary 3.5: Let $\Gamma_\perp(M)$ be a locally finite graph. On the contrary assume that $\Gamma_\perp(M)$ is an infinite graph. By Konigs Lemma, it has a ray and by Lemma 3.6, its diameter must be infinite which is a contradiction with $\mathrm{diam}(\Gamma_\perp(M)) \leq 3$ in [7].

Proposition 3.7. *Let M be an R-module and $\Gamma_\perp(M) \neq \emptyset$ such that $\Gamma_\perp(M)$ is finite. Then the following statements hold:*

(1) *Every simple submodule of M is a vertex of $\Gamma_\perp(M)$.*
(2) *The number of simple submodules of M is finite.*

Proof. (1) Since $\Gamma_\perp(M) \neq \emptyset$, there exist $N, K \leq M$ such that $N \perp K$. But the degree of any vertex in $\Gamma_\perp(M)$ is finite, by Lemma 3.1, the number of submodules of any vertex is finite, i.e., every vertex contains a simple submodule. There exist non-isomorphic simple submodules S_1 and S_2 of M such that $S_1 \subset N$ and $S_2 \subset K$. Assume that S is a simple submodule of M such that $S \notin \mathcal{M}_\perp$. Thus $S \not\perp S_1$ and $S \not\perp S_2$, i.e., $S_1 \cong S \cong S_2$ which is a contradiction.

(2) Since $\Gamma_\perp(M)$ is finite, the number of non-isomorphic simple submodules of M is finite. We let $\{S_1, S_2, ..., S_n\}$ be the set of all non-isomorphic simple submodules of M. We note, if M have just one simple submodule S_1, then by Lemma 3.1, every vertex contains S_1 and so $\Gamma_\perp(M) = \emptyset$, which is a contradiction. If the number of simple submodules isomorphic to S_i is infinite, for some $1 \leq i \leq n$ then for any $i \neq j$, $\deg(S_j) = \infty$, which is a contradiction. □

Proposition 3.8. *Let M be an R-module and $\Gamma_\perp(M) \neq \emptyset$ be a finite graph. Then the following statements hold:*

(1) $\mathrm{Soc}(M) \leq_e M$.
(2) *M is finitely cogenerated.*

Proof. (1) Assume that N is a non-trivial submodule of M. Two cases may happen:

(Case 1) If $N \in \mathcal{M}_\perp$, then by Corollary 3.5, $\deg(N) < \infty$ and by Lemma 3.1, N contains a simple submodule S. Hence $\mathrm{Soc}(M) \cap N \neq 0$.

(Case 2) If $N \notin \mathcal{M}_\perp$, since $\Gamma_\perp(M) \neq \emptyset$, there exist $K, L \in \mathcal{M}_\perp$ such that $K \perp L$. But $N \notin \mathcal{M}_\perp$, that is $K \not\perp N$ and $L \not\perp N$. Since $K \not\perp N$, there exist $N_1 \leq N$ and $K_1 \leq K$ such that $N_1 \cong K_1$. Hence $N_1 \in \mathcal{M}_\perp$, i.e., $\deg(N_1) < \infty$ and by Lemma 3.1, N contains a simple submodule S_1. Hence $\mathrm{Soc}(M) \cap N \neq 0$, i.e., $\mathrm{Soc}(M) \leq_e M$.

(2) By (1), $\mathrm{Soc}(M) \leq_e M$ and by Proposition 3.7 the number of simple submodules of M is finite. Hence $\mathrm{Soc}(M)$ is finitely generated and by [8, Proposition 21.3], M is finitely cogenerated. □

The next example shows that the converse Proposition 3.8 is not true.

Example 3.9. Let $R = \mathbb{Z}$ and consider the R-module $M = \mathbb{Z}_q^\infty \oplus \mathbb{Z}_p^\infty$ where p, q are distinct prime number. It is clear that $\mathrm{Soc}(M) \leq_e M$ and M is finitely cogenereted. By [6, Theorem 2.4], we can see that all of submodules of M are $A \oplus B$ which $A \leq \mathbb{Z}_p^\infty$, $B \leq \mathbb{Z}_q^\infty$. Since $\mathrm{S}(\mathbb{Z}_p^\infty)$ and $\mathrm{S}(\mathbb{Z}_q^\infty)$ are infinite and $\mathbb{Z}_p^\infty \perp \mathbb{Z}_q^\infty$, $\deg(\mathbb{Z}_\mathrm{p}^\infty) = \deg(\mathbb{Z}_\mathrm{q}^\infty) = \infty$. Thus $\Gamma_\perp(M)$ is an infinite graph.

Theorem 3.10. *Let n be a positive integer number and M an R-module such that $\Gamma_\perp(M)$ is an n-regular, then the following statements hold:*

1. *Every non-simple vertex of $\Gamma_\perp(M)$ only contains a simple submodule (up to isomorphism).*
2. *M has exactly two non-isomorphic simple submodules.*
3. *The intersection of all non-zero submodules of M is zero.*
4. *$\Gamma_\perp(M)$ is a complete bipartite graph $K_{n,n}$.*

Proof. (1) On the contrary, K be a non-simple vertex of $\Gamma_\perp(M)$ and $S_1, S_2 \subset K$, such that S_1 and S_2 are non-isomorphic simple submodules of M. Since $\Gamma_\perp(M)$ is n-regular, K is adjacent to n vertices of $\Gamma_\perp(M)$. Let $\mathrm{N}(K) = \{A_1, A_2, ..., A_n\}$. Since $S_1 \ncong S_2$, $S_1 \perp S_2$. But S_1 and S_2 are adjacent to all of vertices in $\mathrm{N}(K)$. Thus $\deg(S_1) = \deg(S_2) \geq n+1$, which is a contradiction with n-regular graph.

(2) On the contrary, S_1, S_2, S_3 are three non-isomorphic simple submodules of M so $S_1 \oplus S_2, S_1 \oplus S_3, S_2 \oplus S_3 \in \mathcal{M}_\perp$, which is a contradiction with (1).

(3) The intersection of different simple submodules is zero, thus the result follows from part (2).

(4) By (2), M has exactly two non-isomorphic simple submodules S_1, S_2. Thus for any $N \in \mathcal{M}_\perp$, $S_1 \lesssim N$ or $S_2 \lesssim N$. Set

$$V_1 = \{N \in \mathcal{M}_\perp | S_1 \lesssim N\}$$

and $V_2 = \{N \in \mathcal{M}_\perp | S_2 \lesssim N\}$. Clearly, $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = \mathcal{M}_\perp$ and the elements of V_i are not adjacent, for $i = 1, 2$. Now suppose that $A \in V_1$ so there exists $0 \neq B \lneq M$ such that $A \perp B$. But $S_1 \lesssim A$ and B contains a simple submodule, so $S_2 \lesssim B$. This implies that $\Gamma_\perp(M)$ is a bipartite graph and since $\Gamma_\perp(M)$ is an n-regular graph, $|V_1| = |V_2| = n$, i.e., $\Gamma_\perp(M)$ is $K_{n,n}$. □

Corollary 3.11. *Let M be an R-module. $\Gamma_\perp(M)$ is n-regular graph if and only if $\Gamma_\perp(M) \cong K_{n,n}$.*

The following figure, $\Gamma_{\perp}(\mathbb{Z}_{216})$ is a 3-regular graph.

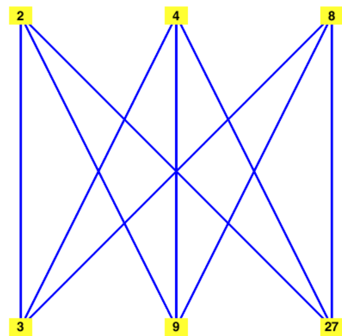

FIGURE 2. $\mathbb{Z}_{216}$

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FURTHER STUDIES OF THE PERPENDICULAR GRAPHS OF MODULES

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مطالعه بیشتر گراف‌های متعامد مدول‌ها

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ما در این مقاله به بررسی گراف‌های متعامد مدول‌ها که در [۷] معرفی شده‌اند، می‌پردازیم. فرض می‌کنیم M یک R-مدول باشد، دراین صورت دو مدول A و B را عمود بر هم نامیده و با نماد $A \perp B$ نشان می‌دهیم، هرگاه A و B دارای زیرمدول‌های غیرصفر یکریخت با هم نباشند. گراف $\Gamma_{\perp}(M)$ را با رأس‌های $\mathcal{M}_{\perp} = \{(0) \neq A \leq M \mid \exists (0) \neq B \leq M,\ \ A \perp B\}$ مرتبط می‌کنیم. هدف اصلی این مقاله بررسی تاثیر متقابل ویژگی‌های مدول M با گراف $\Gamma_{\perp}(M)$ است. در این مقاله عدد خوشه‌ای و عدد رنگی گراف $\Gamma_{\perp}(M)$ را مطالعه کرده و ثابت می‌کنیم اگر $\omega(\Gamma_{\perp}(M)) < \infty$ و M شامل زیرمدول ساده باشد، آن‌گاه $\chi(\Gamma_{\perp}(M)) < \infty$. از جمله نتایج بدست آمده که می‌توان به آن اشاره کرد، این است که برای یک مدول نیم‌ساده $\omega(\Gamma_{\perp}(M)) = \chi(\Gamma_{\perp}(M))$.

کلمات کلیدی: عدد رنگی، عدد خوشه‌ای، گراف متناهی، مدول اتمیک، مدول نیم‌ساده.