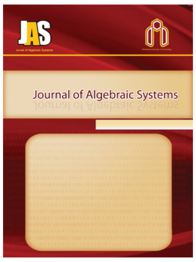
Journal of Algebraic Systems



ISSN:(2345-5128)(2345-511X)
Journal Homepage: www.jas.shahroodut.ac.ir

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M. Zaoui, D. Gretete and B. Fahid*

To cite this article: M. Zaoui, D. Gretete and B. Fahid* (9 June 2024): (f, g)-Derivations in residuated lattices, Journal of Algebraic Systems, DOI: 10.22044/JAS.2023.11989.1615

To link to this article: htpps://doi.org/10.22044/JAS.2023.11989.1615

Published online: 9 June 2024

Journal of Algebraic Systems, vol. xx, no. xx, (202x), pp xx-xx
https://doi.org/10.22044/JAS.2023.11989.1615

(f, g)-DERIVATIONS IN RESIDUATED LATTICES

M. ZAOUI, D. GRETETE AND B. FAHID*

ABSTRACT. In this paper, we present and examine the characteristics of (f, g)-derivations for a residuated lattice. Some relationships between (f, g)-derivation and isotone, contractive and ideal (f, g)-derivations are given. The set of fixed point of an (f, g)-derivation is introduced and its structure is studied. More precisely, we show that the set of fixed points is also a residuated lattice.

1. INTRODUCTION

The classical two-valued logic, also known as Boolean logic, has traditionally been used for information processing and inference. However, with the increasing interest in fuzzy logic, there is a growing need to establish logical systems that can handle uncertain information. Consequently, numerous non-classical logic systems such as BL-algebras [18], MV-algebras [3], and MTV-algebras [7] have been extensively proposed and researched. Residuated lattices [22] are fundamental and crucial algebraic structures since all other logical algebras are considered as special cases of residuated lattices.

The notion of derivation is a very interesting and important area of research, which also allows the study of the structures and properties of algebraic systems. In 1957, Posner [15] introduced the notion of derivation in a prim ring $(R, +, .)$. In 2004, Jun and Xin [11] applied the notion of derivations to BCI-algebras. In 2005, Zhan and Liu [23] introduced the notion of f-derivation of BCI-algebras. In 2008, Xin et al. [19] proposed the concept of a derivation on a lattice $(L, \wedge, \vee)$. In the same year, Çeven and Özturk [2] introduced the notion of an f-derivation on a lattice. In 2016, He et al. [9] introdused the concept of derivation in a residuated lattice, and they characterized some special types of residuated lattices in terms of derivations. In 2018, Rachunek and Salunova [16] have introduced the concept of

Published online: 9 June 2024
MSC(2010): Primary: 06B05; Secondary: 16W25, 11Y50.
Keywords: Residuated Lattice; Derivation; (f, g)-derivation; Ideal (f, g)-derivation.
Received: 7 June 2022, Accepted: 12 May 2023.
*Corresponding author.

derivations and a complete description of all derivations on a non-commutative generalization of MV-algebras. In the same year, Liang et al. [13] have presented the notions of derivations on EQ-algebras and obtained many special types of them. In addition, Wang et al. [20] introduced the notion of derivations of commutative multiplicative semilattices, they investigated the related properties of some special derivations and gave some characterizations. In 2019, Wang et al. [21] gave some representations of MV -algebras in terms of derivations. Rasheed and Majeed [17] studied some results of (α, β)-derivations on prime seeding. Dey et al. [6] considered generalized orthogonal derivations of semiprimary rings. Ciungu [5] studied the properties of implicit derivations in pseudo-BCI-algebras. Chaudhuri [4] discussed (σ, τ)-derivations of group rings. In 2020, Guven [8] proposed the notion of (σ, τ)-derivations generalized on rings and discussed some related aspects. Hosseini and Fosner [10] studied the image of left Jordan derivations on algebras. Ali and Rahaman [1] studied a pair of generalized derivations in rings. Zhu et al. [24] introduced the notion of a generalized derivation and investigated some related properties of them. In 2021, Ling and Zhu [14] proposed a generalization of a derivation in a residuated lattice and some related properties are investigated.

In this paper, as a generalization of a derivation in a residuated lattice, the notion of multiplicative (f, g)-derivation d is introduced, determined by two derivations f and g from L to L. More precisely, for any $x, y \in L$, we propose the following formula:

$$d(x \otimes y) = (f(x) \otimes y) \vee (x \otimes g(y)).$$

Meanwhile, we discuss and investigate some related properties.

This paper is organized as follows. In section 2, we recall some concepts and results on residuated lattices. In section 3, we propose the notion of multiplicative (f, g)-derivation in residuated lattices and investigate some related properties of isotone, contractive, ideal and good commutative (f, g)-derivation. Also, we define the notion of fixed point. In particular, we obtain that the fixed point set is a residuated lattice.

2. Preliminaries

In this section, we briefly present some basic notions used in the rest of the work.

Definition 1. [22] An algebraic structure $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a bounded commutative residuated lattice (simply called a residuated lattice) if

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
(2) $(L, \otimes, 1)$ is a commutative monoid with unit element 1;
(3) For all $x, y, z \in L$, $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$.

In what follows, we denote by L a residuated lattice

$$(L, \wedge, \vee, \otimes, \rightarrow, 0, 1).$$

For all $x \in L$ and a natural number n, we mean $x^{'} = x \rightarrow 0$, $x^{''} = (x^{'})^{'}$, $x^0 = 1$ and $x^n = x^{n-1} \otimes x$.

Proposition 1. [22] For all $x, y, z, w \in L$, we have

(1) $1 \rightarrow x = x$, $x \rightarrow 1 = 1$;
(2) $x \leq y$ if and only if $x \rightarrow y = 1$;
(3) $x \otimes (x \rightarrow y) \leq y$;
(4) If $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$;
(5) If $x \leq y$ and $z \leq w$, then $x \otimes z \leq y \otimes w$;
(6) $x \otimes y \leq x \wedge y$;
(7) $x \otimes y = 0$ if and only if $x \leq y^{'}$;
(8) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$.

An element $x \in L$ is called *complemented* if there exists an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$. By $B(L)$, we mean the set of all complemented elements of L, i.e.,

$$B(L) = \{x \in L : \ \ \exists y \in L, \ \ x \wedge y = 0 \ \ \textit{and} \ \ x \vee y = 1\}.$$

Proposition 2. [12] For a residuated lattice L we have

(1) If $x \in B(L)$, then $x \wedge y = x \otimes y$ for all $y \in L$;
(2) If $x \in B(L)$, then $x \otimes x = x$.

At the end of this section, we remind the notion of derivation in a residuated lattice L as follows.

Definition 2. [9] A mapping d: $L \longrightarrow L$ is called a derivation on L if it satisfies the following condition, for any $x, y \in L$,

$$d(x \otimes y) = (d(x) \otimes y) \vee (x \otimes d(y)).$$

Proposition 3. [9] Let d be a derivation of L. For all $x, y \in L$,

(1) $d(0) = 0$;
(2) $x \otimes d(1) \leq d(x)$;
(3) $d(x^n) = x^{n-1} \otimes d(x)$, for any $n \geq 1$;
(4) if $x \leq y^{'}$, then $d(y) \leq x^{'}$ and $d(x) \leq y^{'}$;
(5) $d(x^{'}) \leq (d(x))^{'}$.

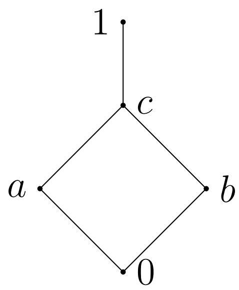
3. (f, g)-derivations in Residuated Lattices

Definition 3. Let f and $g : L \longrightarrow L$ be derivations of L. A mapping d: $L \longrightarrow L$ is called an (f, g)-derivation determined by f and g if

$$d(x \otimes y) = (f(x) \otimes y) \vee (x \otimes g(y)),$$

for any $x, y \in L$.

Example 1. Let $L = \{0, a, b, c, 1\}$ such that the order on L is given by the following Hasse diagram:



The operations $\otimes$ and $\longrightarrow$ on L are defined by the following tables:

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then L is a residuated lattice.
Now, we define a map $d : L \longrightarrow L$ as follows: for all $x \in L$,

$$d(x) = \begin{cases} x, & x = 0, a, b \\ c, & x = c, 1 \end{cases} .$$

A map d is an (f,g)-derivation, determined by two derivations f and g on L as follows: for all $x \in L$,

$$f(x) = \begin{cases} 0, & x = 0, a \\ b, & x = b, c, 1 \end{cases}, \; g(x) = \begin{cases} 0, & x = 0, b \\ a, & x = a, c, 1 \end{cases}.$$

Proposition 4. Let d be a (f,g)-derivation on L, then the following statements hold.

(1) $d(0) = 0$;
(2) $f(1) \otimes x \leq d(x)$ and $g(x) \leq d(x)$, for any $x \in L$;
(3) $x^{n-1} \otimes f(x) \leq d(x^n)$ and $x^{n-1} \otimes g(x) \leq d(x^n)$, for any $x \in L$ and $n > 1$;
(4) if $x \otimes y = 0$, then $y \otimes f(x) = x \otimes g(y) = d(x \otimes y) = 0$, for any $x, y \in L$;
(5) $d(x^{'}) \leq (f(x))^{'} \vee (g(x))^{'}$, for any $x \in L$.

Proof. (1) It follows from definition that

$$d(0) = d(0 \otimes 0) = (f(0) \otimes 0) \vee (0 \otimes g(0)) = (0 \otimes 0) \vee (0 \otimes 0) = (0 \otimes 0) = 0,$$

i.e., $d(0) = 0$.

(2) Let $x \in L$. Then $d(x) = d(1 \otimes x) = (f(1) \otimes x) \vee (1 \otimes g(x))$. Hence, $f(1) \otimes x \leq d(x)$ and $1 \otimes g(x) = g(x) \leq d(x)$.

(3) Let $x \in L$. Then $d(x^2) = d(x \otimes x) = (f(x) \otimes x) \vee (x \otimes g(x))$. Thus, $f(x) \otimes x = x \otimes f(x) \leq d(x^2)$ and $x \otimes g(x) \leq d(x^2)$. By induction, we have
$x^{n-1} \otimes f(x) \leq d(x^n)$ and $x^{n-1} \otimes g(x) \leq d(x^n)$, for any $n > 1$.

(4) Let $x, y \in L$ and $x \otimes y = 0$. Then it follows from (1) that

$$d(x \otimes y) = y \otimes f(x) = x \otimes g(y) = 0.$$

(5) Let $x \in L$, Then

$$\begin{aligned} d(x^{'}) &= d(x^{'} \otimes 1) \\ &= (f(x^{'}) \otimes 1)) \vee (x^{'} \otimes g(1)) \\ &= f(x^{'}) \vee (x^{'} \otimes g(1)) \\ &\leq f(x^{'}) \vee g(x^{'}) \\ &\leq (f(x))^{'} \vee (g(x))^{'}. \end{aligned}$$

□

Definition 4. Let d be an (f,g)-derivation on L,

(1) d is called an isotone (f,g)-derivation provided that $x \leq y$ implies
$d(x) \leq d(y)$, for any $x, y \in L$;
(2) d is called a contractive (f,g)-derivation provided that $d(x) \leq x$, for any $x \in L$.

In particular, if d is both isotone and contractive, we call d an ideal (f,g)-derivation.

Example 2. Let $L = \{0, a, b, c, 1\}$ be a chain. Define the operations $\otimes$ and $\rightarrow$ as follows:

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then L is a residuated lattice, where $x \wedge y = min\{x, y\}$ and $x \vee y = max\{x, y\}$ for all $x, y \in L$.

Now, we define a map $d : L \longrightarrow L$ as follows: for all $x \in L$,

$$d(x) = \begin{cases} x, & x = 0, a, b, c \\ c, & x = 1 \end{cases}.$$

A map d is an (f,g)-derivation, determined by two derivations f and g on L as follows: for all $x \in L$,

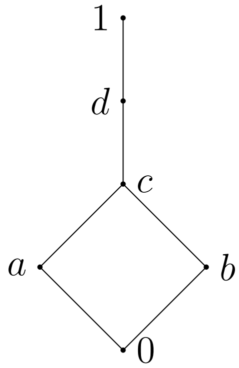
$$f(x) = \begin{cases} x, & x = 0, a, b \\ c, & x = c, 1 \end{cases}, \; g(x) = \begin{cases} 0, & x = 0, a \\ b, & x = b, c, 1 \end{cases}.$$

d is an ideal (f,g)-derivation on L.

Example 3. Let $t \in L$, Define a mapping f: $L \longrightarrow L$ by $f(x) = x \otimes t$ and g: $L \longrightarrow L$ by $g(x) = 0$, for any $x \in L$, then f and g are derivations on L. Now, we define a mapping d: $L \longrightarrow L$ by $d(x) = x \otimes t$, for any $x \in L$. It is easy to verify that d is an ideal (f,g)-derivation on L.

In the following example, we will provide an illustration of an (f,g)-derivation on L, which is both contractive and not isotone.

Example 4. Let $L = \{0, a, b, c, d, 1\}$ such that the order on L is given by the following Hasse diagram:



The operations $\otimes$ and $\longrightarrow$ on L are defined by the following tables:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	b	b	b	b
c	0	a	b	c	c	c
d	0	a	b	c	c	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	a	a	1	1	1	1
c	0	a	b	1	1	1
d	0	a	b	d	1	1
1	0	a	b	c	d	1

Then L is a residuated lattice.

Now, we define a map $d : L \longrightarrow L$ as follows: for all $x \in L$,

$$d(x) = \begin{cases} x, & x = 0, a, b, c, d \\ c, & x = 1 \end{cases}.$$

A map d is an (f, g)-derivation, determined by two derivations f and g on L as follows: for all $x \in L$,

$$f(x) = \begin{cases} x, & x = 0, a, b, c, d \\ a, & x = 1 \end{cases}, \qquad g(x) = \begin{cases} x, & x = 0, a, b, c, d \\ b, & x = 1 \end{cases}.$$

d is an (f, g)-derivation on L contractive. However, d is not isotone.

Now, some properties are investigated.

Proposition 5. Let d be an (f, g)-derivation on L. If d is contractive, then f and g are also contractive.

Proof. We know that $f(x) \leq d(x)$ and $g(x) \leq d(x)$, for any $x \in L$. Since d is contractive (i.e., $d(x) \leq x$, for any $x \in L$), it holds that $f(x) \leq x$ and $g(x) \leq x$, for any $x \in L$. Thus, f and g are contractive. $\square$

Proposition 6. Let d be an (f,g)-derivation on L. If f or g is contractive, then d is contractive.

Proof. Assume that f is contractive and let $x \in L$. We have $d(x) = f(x) \vee (x \otimes g(1))$. Then the fact that $f(x) \leq x$ and $x \otimes g(1) \leq x$ imply that $f(x) \vee (x \otimes g(1)) \leq x$. Therefore, d is contractive. $\square$

Proposition 7. Let d be an (f,g)-derivation on L. If f or g is isotone, then d is isotone.

Proof. Assume that f is isotone and let $x, y \in L$ such that $x \leq y$. Then $f(x) \leq f(y)$. Thus, $d(x) = f(x) \vee (x \otimes g(1)) \leq f(y) \vee (y \otimes g(1)) = d(y)$. Consequently, d is isotone. $\square$

Proposition 8. Let d be an isotone (f,g)-derivation on L. Then the following statements hold.

(1) If $z \leq x \to y$, then $z \leq f(x) \to d(y)$ and $x \leq g(z) \to d(y)$, for any $x, y, z \in L$;
(2) $x \to y \leq f(x) \to d(y)$ and $g(x \to y) \leq x \to d(y)$, for any $x, y \in L$;
(3) $y \leq f(x) \to d(y)$ and $x \leq g(y) \to d(x)$, for any $x, y \in L$.

Proof. (1) Let $x, y, z \in L$ and $z \leq x \to y$. Then $x \otimes z \leq y$. Since d is an isotone (f,g)-derivation on L, we have $(f(x) \otimes z) \vee (x \otimes g(z)) \leq d(y)$, then $f(x) \otimes z \leq d(y)$ and $x \otimes g(z) \leq d(y)$. Therefore $z \leq f(x) \to d(y)$ and $x \leq g(z) \to d(y)$.

(2) Since $x \otimes (x \to y) \leq y$, we have $d(x \otimes (x \to y)) \leq d(y)$. It follows that $(f(x) \otimes x \to y) \vee (x \otimes g(x \to y)) \leq d(y)$, which implies $f(x) \otimes (x \to y) \leq d(y)$ and $x \otimes g(x \to y) \leq d(y)$, Therefore $x \to y \leq f(x) \to d(y)$ and $g(x \to y) \leq x \to d(y)$ for all $x, y \in L$.

(3) Since $x \otimes y \leq x$ for all $x, y \in L$, we have $d(x \otimes y) \leq d(x)$. It follows that $(f(x) \otimes y) \vee (x \otimes g(y)) \leq d(x)$. Thus $f(x) \otimes y \leq d(x)$ and $x \otimes g(y) \leq d(x)$. Therefore, $y \leq f(x) \to d(y)$ and $x \leq g(y) \to d(x)$ for all $x, y \in L$. $\square$

Proposition 9. Let d be a contractive (f,g)-derivation on L. Then the following statements hold.

(1) If $f(1) = 1$, then d is an identity derivation, i.e., $d(x) = x$, for any $x \in L$;

(2) If $g(x) = x$, for any $x \in L$, then d is an identity derivation.

Proof. (1) From proposition 4, it follows that $f(1) \otimes x \leq d(x)$, for any $x \in L$. Assume that $f(1) = 1$, for any $x \in L$ we get

$$x = x \otimes 1 = x \otimes f(1) \leq d(x) \leq x,$$

which shows that $d(x) = x$ for all $x \in L$. Therefore d is an identity derivation.

(2) From proposition 4, it follows that $g(x) \leq d(x)$ for any $x \in L$. Assume that $g(x) = x$, we get $x = g(x) \leq d(x) \leq x$, which shows that $d(x) = x$ for all $x \in L$. Therefore d is an identity derivation. □

Corollary 1. Let d be a contractive (f, g)-derivation on L. If f or g is an identity derivation, then d is also an identity derivation.

Theorem 1. Let d be an (f, g)-derivation on L. Then the following statements hold.

(1) If $f(x) \to d(y) = f(x) \to y$, for any $x, y \in L$, then d is an ideal (f, g)-derivation on L;
(2) The converse of (1) holds if we assume that $f(f(x)) = f(x)$, for any $x \in L$.

Proof. (1) Assume that $f(x) \to d(y) = f(x) \to y$, for any $x, y \in L$. Since $f(x) \otimes 1 \leq d(x)$, we have $1 \leq f(x) \to d(x) = f(x) \to x$, then $f(x) \otimes 1 \leq x$. On the other hand $g(1) \otimes x \leq g(1) \wedge x \leq x$, which implies $d(x) = (f(x) \otimes 1) \vee (x \otimes g(1)) \leq x$, for any $x \in L$. Thus, d is contractive. Moreover, for any $x, y \in L$, let $x \leq y$, we have $f(x) \otimes 1 = f(x) \leq x \leq y$. It follows that $1 \leq f(x) \to y = f(x) \to d(y)$, then $f(x) \leq d(y)$. Thus,

$$d(x) = (f(x) \otimes 1) \vee (x \otimes g(1)) \leq d(y) \vee (y \otimes g(1)) \leq d(y) \vee g(y) \leq d(y),$$

then, $d(x) \leq d(y)$. Hence d is isotone. Therefore d is an ideal (f, g)-derivation on L.

(2) Let d be an ideal (f, g)-derivation on L and $f(f(x)) = f(x)$, for any $x \in L$. Since $d(y) \leq y$, for any $y \in L$, it holds that by Proposition 1 (4) that $f(x) \to d(y) \leq f(x) \to y$. On the other hand, let $t \leq f(x) \to y$, for any $t \in L$. We obtain $f(x) \otimes t \leq y$. Since d is isotone, it follows that $d(f(x) \otimes t) \leq d(y)$, for any $x, y, t \in L$. The fact that $d(x \otimes y) = (f(x) \otimes y) \vee (x \otimes g(y))$, implies that $f(x) \otimes y \leq d(x \otimes y)$,

for any $x, y \in L$. Thus, $f(f(x)) \otimes t \leq d(f(x) \otimes t) \leq d(y)$, for any $x, y \in L$. Hence $t \leq f(f(x)) \to d(y)$, for any $t \in L$, which implies that

$$f(x) \to y \leq f(x) \to d(y)$$

for all $x, y \in L$. Therefore, we obtain $f(x) \to d(y) = f(x) \to y$, for any $x, y \in L$. □

Theorem 2. Let d be a contractive (f, g)-derivation on L. If $d(1) \in B(L)$, then the following statements are equivalent.

(1) d is an ideal (f, g)-derivation on L;
(2) $d(x) \leq d(1)$, for any $x \in L$;
(3) $d(x) = d(1) \otimes x$, for any $x \in L$;
(4) $d(x \wedge y) = d(x) \wedge d(y)$, for any $x, y \in L$;
(5) $d(x \vee y) = d(x) \vee d(y)$, for any $x, y \in L$;
(6) $d(x \otimes y) = d(x) \otimes d(y)$, for any $x, y \in L$.

Proof. (1) $\Rightarrow$ (2) Let $x \in L$. Since $x \leq 1$, and d is isotone, then $d(x) \leq d(1)$.

(2) $\Rightarrow$ (3) Suppose that $d(x) \leq d(1)$, for any $x \in L$. Notice that $d(1) \in B(L)$, we obtain $d(x) = d(1) \wedge d(x) = d(1) \otimes d(x) \leq d(1) \otimes x$. On the other hand, since, $f(1) \otimes x \leq d(x)$ and $g(1) \otimes x \leq d(x)$ we have $(f(1) \otimes x) \vee (g(1) \otimes x) \leq d(x)$. Then, $(f(1) \vee g(1)) \otimes x \leq d(x)$. So $d(1) \otimes x \leq d(x)$. Thus, $d(x) = d(1) \otimes x$, for any $x \in L$.

(3) $\Rightarrow$ (4) Let $d(x) = d(1) \otimes x$, for any $x \in L$. Then, for any $x, y \in L$, we have

$$\begin{aligned} d(x \wedge y) &= d(1) \otimes (x \wedge y) \\ &= d(1) \wedge (x \wedge y) \\ &= (d(1) \wedge x) \wedge (d(1) \wedge y) \\ &= (d(1) \otimes x) \wedge (d(1) \otimes y) \\ &= d(x) \wedge d(y). \end{aligned}$$

(4) $\Rightarrow$ (1) Assume that $x \leq y$, then, $x \wedge y = x$. An consequence of (4) is that $d(x) = d(x \wedge y) = d(x) \wedge d(y)$, which implies $d(x) \leq d(y)$ for all $x, y \in L$. Thus, d is an ideal (f, g)-derivation on L.

(3) $\Rightarrow$ (5) For any $x, y \in L$, it follows from (3) and proposition 1 that

$$d(x \vee y) = d(1) \otimes (x \vee y) = (d(1) \otimes x) \vee (d(1) \otimes y) = d(x) \vee d(y).$$

(5) $\Rightarrow$ (1) Let $x, y \in L$ such that $x \leq y$. It follows from (5) that

$$d(y) = d(x \vee y) = d(x) \vee d(y),$$

for any $x, y \in L$. Then we conclude that $d(x) \leq d(y)$, for any $x, y \in L$. Therefore, d is an ideal (f,g)-derivation on L.

(3) $\Rightarrow$ (6) For any $x, y \in L$, it follows from (3) that

$$d(x \otimes y) = d(1) \otimes (x \otimes y) = (d(1) \otimes x) \otimes (d(1) \otimes y) = d(x) \otimes d(y).$$

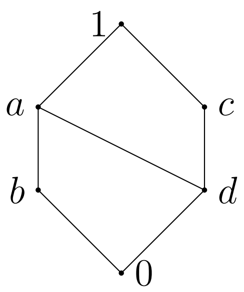
(6) $\Rightarrow$ (2) Let $x \in L$. Then it holds that

$$d(x) = d(x \otimes 1) = d(x) \otimes d(1) = d(x) \wedge d(1).$$

Thus, $d(x) \leq d(1)$ for any $x, y \in L$. □

An ideal (f,g)-derivation is said to be *good* if $d(1) \in B(L)$.

Example 5. Let $L = \{0, a, b, c, d, 1\}$ such that the order on L is given by the following Hasse diagram:



The operations $\otimes$ and $\longrightarrow$ on L are defined by the following tables:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	b	b	d	0	a
b	0	b	b	0	0	b
c	0	d	0	c	d	c
d	0	0	0	d	0	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Then L is a residuated lattice and $B(L) = \{0, b, c, 1\}$.

Now, we define a map $d : L \longrightarrow L$ by $d(x) = x$ for all $x \in L$.

A map d is an (f,g)-derivation, determined by two derivations f and g on L as follows: for all $x \in L$,

$$f(x)=\begin{cases}0, & x=0\\ b, & x=a,b,1\\ c, & x=c\\ d, & x=d\end{cases}, \quad g(x)=\begin{cases}0, & x=0,b\\ c, & x=c,1\\ d, & x=a,d\end{cases}.$$

One can check that d is a good ideal (f,g)-derivation on L.

In the following, we discuss the properties of good ideal (f,g)-derivation.

Proposition 10. Let d be a good ideal (f,g)-derivation on L. Then it holds that

(1) for any $x \in L$, $d^2(x)=d(x)$;
(2) for any $x,y \in L$, $d(x) \leq y$ if and only if $d(x) \leq d(y)$;
(3) for any $x,y \in L$,
$$d(d(x)\wedge d(y))=d(d(x)\wedge y)=d(x\wedge d(y))=d(x\wedge y).$$

Proof. (1) Since d is a good ideal (f,g)-derivation on L and $d(1)\in B(L)$, we have $d(x)=d(1)\otimes x$. It follows that

$$d(d(x))=d(1)\otimes d(x)=d(1)\otimes(d(1)\otimes x)=d(1)\otimes x=d(x),$$

that is, $d^2(x)=d(x)$, for any $x\in L$.

(2) Assume that $d(x)\leq y$, we have $d(d(x))\leq d(y)$. By statement (1), we get $d(x)\leq d(y)$. Conversely, suppose that $d(x)\leq d(y)$, we have $d(x)\leq d(y)\leq y$.

(3) It follows from statement (1) that $d(d(x))=d(x)$, for any $x\in L$. From $d(y)\leq y$, we have $x\wedge d(y)\leq x\wedge y$. It follows that $d(x\wedge d(y))\leq d(x\wedge y)$, for any $x,y\in L$. On the other hand, combining $d(x\wedge y)\leq d(y)$ and $d(x\wedge y)\leq d(x)\leq x$, we obtain $d(x\wedge y)\leq x\wedge d(y)$. Then $d(x\wedge y)=d(d(x\wedge y))\leq d(x\wedge d(y))$. Thus, $d(x\wedge y)=d(x\wedge d(y))$, for any $x,y\in L$. It follows that $d(d(x)\wedge y)=d(d(x)\wedge d(y))$. In a similar way, we can prove that $d(x\wedge y)=d(d(x)\wedge y)$. Therefore, we conclude that $d(d(x)\wedge d(y))=d(d(x)\wedge y)=d(x\wedge d(y))=d(x\wedge y)$, for any $x,y\in L$. ☐

Next, we discuss the structure and properties of the fixed point set of an ideal (f,g)-derivation. Firstly, we give the concept of the fixed set of (f,g)-derivation in residuated lattices as follows.

Definition 5. Let d be an (f,g)-derivation on L. We denote by set

$$Fix_d(L)=\{x\in L \mid d(x)=x\},$$

the set of fixed elements of L with respect to an (f,g)-derivation d.

Now, we investigate some operations on $Fix_d(L)$.

Theorem 3. Let d be an ideal (f,g)-derivation on L. Then it holds that

(1) for any $x, y \in Fix_d(L)$, $x \otimes y, x \vee y \in Fix_d(L)$.
(2) If d_1 and d_2 are good, then $d_1 = d_2$ if and only if $Fix_{d_1}(L) = Fix_{d_2}(L)$.

Proof. (1) Let $x, y \in Fix_d(L)$,

$$\begin{aligned} x \otimes y &= d(x) \otimes y \\ &= ((f(1) \otimes x) \vee g(x)) \otimes y \\ &= (f(1) \otimes x \otimes y) \vee (y \otimes g(x)) \\ &\leq (f(y) \otimes x) \vee (y \otimes g(x)) \\ &= d(x \otimes y). \end{aligned}$$

On the other hand, since d is an ideal (f,g)-derivation on L, we have $d(x \otimes y) \leq x \otimes y$, which implies $d(x \otimes y) = x \otimes y$. Therefore, $x \otimes y \in Fix_d(L)$. Moreover, it is easy to know that

$$x \vee y = d(x) \vee d(y) \leq d(x \vee y) \leq x \vee y,$$

then $d(x \vee y) = x \vee y$. Hence, $x \vee y \in Fix_d(L)$.

(2) Let $d_1 = d_2$ it is clear that $Fix_{d_1}(L) = Fix_{d_2}(L)$. Conversely, suppose that $Fix_{d_1}(L) = Fix_{d_2}(L)$. Then $d_1(d_1(x)) = d_1(x)$, for any $x \in L$, which implies $d_1(x) \in Fix_{d_1}(L) = Fix_{d_2}(L)$. Thus, $d_2(d_1(x)) = d_1(x)$ for any $x \in L$. Similarly, we have $d_1(d_2(x)) = d_2(x)$, for any $x \in L$. On the other hand, since d_1 and d_2 are ideal (f,g)-derivation on L, we have

$$d_1(d_2(x)) \leq d_1(x) = d_2(d_1(x)),$$

that is $d_1(d_2(x)) \leq d_2(d_1(x))$, for any $x \in L$. In a similar way, we conclude that $d_2(d_1(x)) \leq d_1(d_2(x))$. Combining them, we can obtain $d_1(d_2(x)) = d_2(d_1(x))$. Thus, $d_1(x) = d_1(d_2(x)) = d_2(d_1(x)) = d_2(x)$, for any $x \in L$. finally, $d_1 = d_2$. $\square$

Remark 1. It follows from Theorem 3 that a good ideal (f,g)-derivation d is determined by its fixed point set $Fix_d(L)$.

Further, as an application of the above propositions and theorems, we leads the following result.

Theorem 4. Let d be a good ideal (f,g)-derivation on L. Then
$$(Fix_d(L), \sqcap, \vee, \otimes, \longmapsto, 0, \overline{1})$$
is a residuated lattice, where $x \sqcap y = d(x \wedge y)$, $x \longmapsto y = d(x \longrightarrow y)$ and $\overline{1} = d(1)$, for any $x, y \in L$.

Proof. We complete the proof by three steps.

(1) First, we show that $(Fix_d(L), \sqcap, \vee, \otimes, \longmapsto, 0, \overline{1})$ is a bounded lattice with 0 as the smallest element and $\overline{1}$ as the greatest element. From theorem 3, it holds that $Fix_d(L)$ is closed under $\vee$. Since d is an ideal (f,g)-derivation on L, then $d(x \wedge y) \leq d(x) = x$ and $d(x \wedge y) \leq d(y) = y$, for any $x, y \in Fix_d(L)$. Thus, $d(x \wedge y)$ is a lower bound of x and y in $Fix_d(L)$. Now, suppose that $m = d(m) \in Fix_d(L)$ is an other lower bound of x and y in $Fix_d(L)$, then we have $m \leq x \wedge y$. Thus, $m = d(m) \leq d(x \wedge y)$, which implies that the greatest lower bound (g.l.b) of x and y exists in $Fix_d(L)$ and is $d(x \wedge y)$, and will be denoted by $x \sqcap y = d(x \wedge y)$. Therefore, $(Fix_d(L), \sqcap, \vee)$ is a lattice. Let $x \in Fix_d(L)$, by theorem 2, it follows that $x \sqcap 0 = d(x \wedge 0) = d(0) = 0$ and $x \vee d(1) = d(x) \vee d(1) = d(x \vee 1) = d(1)$. Therefore, 0 is the smallest element and $\overline{1} = d(1)$ is the greatest element in $Fix_d(L)$.

(2) Next, we prove that $(Fix_d(L), \otimes, \overline{1})$ is a commutative monoid with $\overline{1} = d(1)$ as a neutral element. We have $(Fix_d(L), \otimes)$ is closed under $\otimes$, and it is easy to show that it satisfies associative laws. Thus, $(Fix_d(L), \otimes)$ is a commutative semigroup. Let $x \in Fix_d(L)$, then $x \otimes \overline{1} = x \otimes d(1)$ and from property (3) of theorem 2, we have $x \otimes d(1) = d(x)$. Then $x \otimes \overline{1} = x \otimes d(1) = x$. Thus, $\overline{1} = d(1)$ is a neutral element.

(3) Finally, we show that $x \otimes y \leq z$ if and only if $y \leq x \mapsto z$, for any $x, y, z \in Fix_d(L)$. We define $x \mapsto y = d(x \to y)$. By proposition 10 (2), we have $d(x) \leq y$ if and only if $d(x) \leq d(y)$ for all $x, y \in L$. Then, for any $x, y, z \in Fix_d(L)$ it follows that

$$\begin{aligned} x \otimes y \leq z &\Leftrightarrow y \leq x \to z \\ &\Leftrightarrow d(y) \leq x \to z \\ &\Leftrightarrow d(y) \leq d(x \to z) \\ &\Leftrightarrow y \leq x \mapsto z. \end{aligned}$$

Therefore, $(Fix_d(L), \sqcap, \vee, \otimes, \longmapsto, 0, \overline{1})$ is a residuated lattice. $\square$

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Mbarek Zaoui
National school of Applied Sciences ENSAK, Ibn Tofail University, BP 241, 14000 Kenitra-Morocco.
Email: zaouimbarek@yahoo.fr, mbarek.zaoui@uit.ac.ma

Driss Gretete
National school of Applied Sciences ENSAK, Ibn Tofail University, BP 241, 14000 Kenitra-Morocco.
Email: driss.gretete@uit.ac.ma

Brahim Fahid
Superior School of Technology, Ibn Tofail University, BP 241, 14000 Kenitra-Morocco.
Email: brahim.fahid@uit.ac.ma, fahid.brahim@yahoo.fr