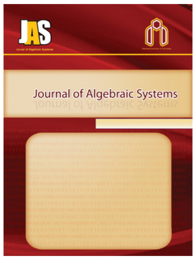
Journal of Algebraic Systems



ISSN:(2345-5128)(2345-511X)
Journal Homepage: www.jas.shahroodut.ac.ir

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To cite this article: A. Fathi* and A. Hajikarimi (9 June 2024): On the finiteness of local homology modules , Journal of Algebraic Systems, DOI: 10.22044/JAS.2023.12118.1631

To link to this article: https://doi.org/10.22044/JAS.2023.12118.1631

Published online: 9 June 2024

Journal of Algebraic Systems, vol. xx, no. xx, (202x), pp xx-xx
https://doi.org/ 10.22044/JAS.2023.12118.1631

ON THE FINITENESS OF LOCAL HOMOLOGY MODULES

A. FATHI* AND A. HAJIKARIMI

Abstract. Let R be a commutative Noetherian ring and $\mathfrak{a}$ be an ideal of R. Suppose M is a finitely generated R-module and N is an Artinian R-module. We define the concept of filter coregular sequence to determine the infimum of integers i such that the generalized local homology $\mathrm{H}_i^{\mathfrak{a}}(M, N)$ is not finitely generated as an $\widehat{R}^{\mathfrak{a}}$-module, where $\widehat{R}^{\mathfrak{a}}$ denotes the $\mathfrak{a}$-adic completion of R. In particular, if R is a complete semi-local ring, then $\mathrm{H}_i^{\mathfrak{a}}(M, N)$ is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module for all non-negative integers i if and only if $(0 :_N \mathfrak{a} + \mathrm{Ann}(M))$ has finite length.

1. Introduction

In this paper, we consider a commutative Noetherian ring R with non-zero identity, and an ideal $\mathfrak{a} \subseteq R$, as well as two R-modules M and N. We denote the $\mathfrak{a}$-adic completion of N by $\Lambda_{\mathfrak{a}}(N)$, and note that the $\mathfrak{a}$-adic completion functor $\Lambda_{\mathfrak{a}}(\cdot)$ is an additive covariant functor on the category of R-modules. We use $L_i^{\mathfrak{a}}(\cdot)$ to denote the i-th left derived functor of $\Lambda_{\mathfrak{a}}(\cdot)$. However, since the tensor functor is not left exact and the inverse limit is not right exact on the category of R-modules, computing the left-derived functors of $\Lambda_{\mathfrak{a}}(\cdot)$ is generally difficult. Moreover, it is important to note that $L_0^{\mathfrak{a}}(\cdot) \ncong \Lambda_{\mathfrak{a}}(\cdot)$.

Matlis studied $L_i^{\mathfrak{a}}(\cdot)$ in the case where $\mathfrak{a}$ is generated by a regular sequence and R is a local ring in [9, 10], and proved some duality between this functor and the local cohomology functor. Recently, Divaani-Aazar et al. in [4] studied the containment of $L_i^{\mathfrak{a}}(\cdot)$ in a Serre class of R-modules up to a given upper bound $s \geq 0$.

Cuong and Nam in [3] defined the i-th *local homology* $\mathrm{H}_i^{\mathfrak{a}}(N)$ of N with respect to $\mathfrak{a}$ as follows:

$$\mathrm{H}_i^{\mathfrak{a}}(N) := \varprojlim_{n\in\mathbb{N}} \mathrm{Tor}_i^R(R/\mathfrak{a}^n, N).$$

They also showed that $L_i^{\mathfrak{a}}(N) \cong \mathrm{H}_i^{\mathfrak{a}}(N)$ when N is Artinian. Similarly, the i-th *generalized local homology* $\mathrm{H}_i^{\mathfrak{a}}(M, N)$ of M and N with respect to $\mathfrak{a}$ is

Published online: 1 April 2024
MSC(2010): Primary: 13D07; Secondary: 13E05, 13C05, 13C15, 13D45.
Keywords: Local homology; Tor functor; Finiteness; Filter coregular sequence.
Received: 21 July 2022, Accepted: 29 May 2023.
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defined by

$$\mathrm{H}_i^{\mathfrak{a}}(M,N) := \varprojlim_{n\in\mathbb{N}} \mathrm{Tor}_i^R(M/\mathfrak{a}^n M, N);$$

see [12, 13] for basic properties and more details.

Matlis in [8] introduced the concept of cosequence (or coregular sequence) as a dual of the concept of regular sequence (see [14] and [16] for more details and basic properties). If N is Artinian and $(0:_N \mathfrak{a}) \neq 0$, then all maximal coregular N-sequences in $\mathfrak{a}$ have the same length, denoted by $\mathrm{width}(\mathfrak{a}, N)$, where $(0:_N \mathfrak{a})$ denotes the set of all elements $x \in N$ such that $rx = 0$ for all $r \in \mathfrak{a}$. Moreover,

$$\mathrm{width}(\mathfrak{a}, N) = \inf\{i \in \mathbb{Z} : \mathrm{H}_i^{\mathfrak{a}}(N) \neq 0\}$$

(see [2, Theorem 4.11]).

The filter regular sequences can be used to study the Artinianess of local cohomology modules of finitely generated R-modules (see [5, Sec. 3]). In this paper as a dual of the concept of filter regular sequence, we introduce the concept of filter coregular sequence to study the finiteness of local homology modules of Artinian R-modules.

Let $\mathrm{Cosupp}(N)$ denote the set of all prime ideals of R containing $\mathrm{Ann}(N)$. A sequence $x_1, \ldots, x_n$ of elements of $\mathfrak{a}$ is called a filter coregular N-sequence (of length n) in $\mathfrak{a}$ if

$$\mathrm{Cosupp}\left((0:_N (x_1,\ldots,x_{i-1})R)/x_i(0:_N (x_1,\ldots,x_{i-1})R)\right) \subseteq \mathrm{Max}(R)$$

for all $1 \leq i \leq n$, where $\mathrm{Max}(R)$ denotes the set of all maximal ideals of R.

Assuming that M is finitely generated and N is Artinian, we prove that if there exists a filter coregular N-sequence in $\mathfrak{a}$ of infinite length, then every filter coregular N-sequence in $\mathfrak{a}$ can be extended to a filter coregular N-sequence in $\mathfrak{a}$ of infinite length, and in this case we set $\text{f-width}(\mathfrak{a}, N) = \infty$. Now suppose that all filter coregular N-sequences in $\mathfrak{a}$ have finite length. Then all maximal filter coregular N-sequences in $\mathfrak{a}$ are of the same length, denoted by $\text{f-width}(\mathfrak{a}, N)$. We prove (see Theorem 2.8 and Remark 2.9) that:

$$\begin{aligned}&\text{f-width}(\mathrm{Ann}(M), N)\\ &= \inf\{i \in \mathbb{N}_0 : \mathrm{Tor}_i^R(M,N) \text{ has infinite length as an } R\text{-module}\}\end{aligned}$$

and

$$\begin{aligned}&\text{f-width}(\mathfrak{a} + \mathrm{Ann}(M), N)\\ &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}_i^{\mathfrak{a}}(M,N) \text{ is not a finitely generated } \widehat{R}^{\mathfrak{a}}\text{-module}\}.\end{aligned}$$

In particular,

$$\begin{aligned}&\text{f-width}(\mathfrak{a}, N)\\ &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}_i^{\mathfrak{a}}(N) \text{ is not a finitely generated } \widehat{R}^{\mathfrak{a}}\text{-module}\}.\end{aligned}$$

We also show in Corollary 2.11 that if $\mathrm{H}_i^{\mathfrak{a}}(M, N)$ is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module for all $i \in \mathbb{N}_0$, then $(0 :_N \mathfrak{a} + \mathrm{Ann}(M))$ has finite length. The converse statement is true when R is a semi-local ring that is complete with respect to its Jacobson radical.

2. Main results

We shall use the following notations and terminologies. Let $\mathfrak{a}$ be an ideal of R and N be an R-module. The radical of $\mathfrak{a}$ will be denoted by $\sqrt{\mathfrak{a}}$; also, $\mathrm{Ann}(N)$ will denote the ideal

$$\{r \in R : rx = 0 \text{ for all } x \in N\}$$

of R; and $(0 :_N \mathfrak{a})$ will denote the submodule

$$\{x \in N : rx = 0 \text{ for all } r \in \mathfrak{a}\}$$

of N. We denote by $\mathrm{V}(\mathfrak{a})$ the set of all prime ideals of R containing $\mathfrak{a}$; and we use $\mathrm{Cosupp}(M)$ to denote $\mathrm{V}(\mathrm{Ann}(M))$. The symbol $\mathbb{N}$ (respectively $\mathbb{N}_0$) will denote the set of positive (respectively non-negative) integers. We refer the reader for any unexplained terminology or notation to [1, 11, 15].

Definition 2.1. Let N be an R-module. We say a prime ideal $\mathfrak{p}$ of R is an *attached prime* of N, if there exists a submodule M of N such that $\mathfrak{p} = \mathrm{Ann}(N/M)$. We denote by $\mathrm{Att}(N)$ the set of all attached primes of N.

For an R-module N, it is clear that $\mathrm{Att}(N) \subseteq \mathrm{Cosupp}(N)$ (we refer the reader to [14] for basic properties and more details of these notations). When N has a *secondary representation* in the sense of [7], our definition of $\mathrm{Att}(N)$ coincides with that of Macdonald (see [1, Exercise 7.2.5]). In particular, the set of attached primes of an Artinian module is a finite set.

Definition 2.2. Let N be an R-module. A sequence $x_1, \ldots, x_n$ of elements of R is called a filter coregular N-sequence (of length n) whenever

$$\mathrm{Cosupp}\left((0 :_N (x_1, \ldots, x_{i-1})R)/x_i(0 :_N (x_1, \ldots, x_{i-1})R)\right) \subseteq \mathrm{Max}(R)$$

for all $1 \leq i \leq n$, where $\operatorname{Max}(R)$ denotes the set of all maximal ideals of R. If, in addition, $x_1, \dots, x_n$ belong to an ideal $\mathfrak{a}$, then we say that $x_1, \dots, x_n$ is a filter coregular N-sequence in $\mathfrak{a}$.

Lemma 2.3. *Let N be an Artinian R-module. The following conditions are equivalent:*

(i) *N has finite length;*
(ii) $\operatorname{Cosupp}(N) \subseteq \operatorname{Max}(R)$*; and*
(iii) $\operatorname{Att}(N) \subseteq \operatorname{Max}(R)$.

Proof. Assume that N has finite length. Since N is finitely generated, we have $\operatorname{Cosupp}(N) = \operatorname{Supp}(N)$. Also, the Artinianness of N implies that $\operatorname{Supp}(N) \subseteq \operatorname{Max}(R)$, and so $\operatorname{Cosupp}(N) \subseteq \operatorname{Max}(R)$. This proves the implication (i)⇒(ii). The implication (ii)⇒(iii) is clear. Finally, to prove the implication (iii)⇒(i), suppose that $\operatorname{Att}(N) \subseteq \operatorname{Max}(R)$. Then, by [1, Proposition 7.2.11], we have $\sqrt{\operatorname{Ann}(N)} = \bigcap_{\mathfrak{q} \in \operatorname{Att}(N)} \mathfrak{q}$, and so $\left(\bigcap_{\mathfrak{q} \in \operatorname{Att}(N)} \mathfrak{q}\right)^n N = 0$ for some positive integer n. It follows that N has finite length because $\operatorname{Att}(N)$ consists of finitely many maximal ideals. □

Proposition 2.4. *Let $x_1, \dots, x_n$ be elements of R, and let N be an Artinian R-module. The following conditions are equivalent:*

(i) *$x_1, \dots, x_n$ is a filter coregular N-sequence;*
(ii) $(0 :_N (x_1, \dots, x_{i-1})R)/x_i(0 :_N (x_1, \dots, x_{i-1})R)$ *has finite length for all* $1 \leq i \leq n$*;*
(iii) $\operatorname{Att}\left((0 :_N (x_1, \dots, x_{i-1})R)/x_i(0 :_N (x_1, \dots, x_{i-1})R)\right) \subseteq \operatorname{Max}(R)$ *for all* $1 \leq i \leq n$*; and*
(iv) $x_i \notin \bigcup_{\mathfrak{p} \in \operatorname{Att}(0:_N (x_1, \dots, x_{i-1})R) \setminus \operatorname{Max}(R)} \mathfrak{p}$ *for all* $1 \leq i \leq n$.

Proof. The statements (i)–(iii) are equivalent by Lemma 2.3. For each $1 \leq i \leq n$, we set $N_{i-1} := (0 :_N (x_1, \dots, x_{i-1})R)$. Then, in view of [14, Proposition 2.13], we have $x_i \notin \bigcup_{\mathfrak{p} \in \operatorname{Att}(N_{i-1}) \setminus \operatorname{Max}(R)} \mathfrak{p}$ if and only if

$$\operatorname{Att}(N_{i-1}/x_i N_{i-1}) = \operatorname{V}(x_i R) \cap \operatorname{Att}(N_{i-1}) \subseteq \operatorname{Max}(R).$$

Therefore (iii) and (iv) are also equivalent. □

Proposition 2.5. *Let M and N be R-modules, and let $x_1, \ldots, x_n$ be elements of R. For each $i \in \mathbb{N}_0$, there are the following inclusions:*

$$\begin{aligned}
&\operatorname{Cosupp}\left(\operatorname{Tor}_i^R\left(M, (0:_N (x_1,\ldots,x_n)R)\right)\right)\\
&\subseteq \left(\bigcup_{j=i}^{i+n} \operatorname{Cosupp}\left(\operatorname{Tor}_j^R(M,N)\right)\right)\\
&\cup \left(\bigcup_{k=1}^{n} \bigcup_{j=i+2}^{i+2+n-k} \operatorname{Cosupp}\left(\operatorname{Tor}_j^R\left(M, \frac{(0:_N (x_1,\ldots,x_{k-1})R)}{x_k(0:_N (x_1,\ldots,x_{k-1})R)}\right)\right)\right);
\end{aligned} \tag{2.1}$$

and if, in addition, $x_1, \ldots, x_n$ belong to $\operatorname{Ann}(M)$, *then*

$$\begin{aligned}
&\operatorname{Cosupp}\left(\operatorname{Tor}_i^R(M,N)\right)\\
&\subseteq \operatorname{Cosupp}\left(\operatorname{Tor}_{i-n}^R\left(M, (0:_N (x_1,\ldots,x_n)R)\right)\right)\\
&\cup \left(\bigcup_{k=1}^{n} \bigcup_{j=i+1-k}^{i+2-k} \operatorname{Cosupp}\left(\operatorname{Tor}_j^R\left(M, \frac{(0:_N (x_1,\ldots,x_{k-1})R)}{x_k(0:_N (x_1,\ldots,x_{k-1})R)}\right)\right)\right).
\end{aligned} \tag{2.2}$$

Proof. We prove the claimed inclusions by induction on n. The following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to (0:_N x_1R) & \to & N & \xrightarrow{x_1} & x_1N & \to & 0 & & \\
 & & \downarrow{\scriptstyle x_1} & \searrow^{x_1} & & & & & \\
 & 0 \to & x_1N & \xrightarrow{\subseteq} & N & \to & N/x_1N & \to & 0
\end{array}$$

induces the commutative diagram

$$\begin{array}{ccccccc}
\cdots \to T_i(0:_N x_1R) & \to & T_i(N) & \xrightarrow{x_1^{(i)}} & T_i(x_1N) & \to & T_{i-1}(0:_N x_1R) \to \cdots\\
 & & \downarrow{\scriptstyle x_1^{(i)}} & \searrow^{x_1} & & & \\
\cdots \to T_{i+1}(N/x_1N) & \to & T_i(x_1N) & \xrightarrow{f_i} & T_i(N) & \to & T_i(N/x_1N) \to \cdots
\end{array} \tag{2.3}$$

with exact rows, where $T_i(\cdot) := \operatorname{Tor}_i^R(M,\cdot)$ and $x_1^{(i)} := \operatorname{Tor}_i^R(\mathrm{id}_M, x_1)$. Therefore [14, Proposition 2.9(4)] implies that

$$\begin{aligned}
&\operatorname{Cosupp}\left(T_i(0:_N x_1R)\right)\\
&\subseteq \operatorname{Cosupp}\left(T_i(N)\right) \cup \operatorname{Cosupp}\left(T_{i+1}(x_1N)\right)\\
&\subseteq \operatorname{Cosupp}\left(T_i(N)\right) \cup \operatorname{Cosupp}\left(T_{i+1}(N)\right) \cup \operatorname{Cosupp}\left(T_{i+2}(N/x_1N)\right)
\end{aligned} \tag{2.4}$$

for all $i \in \mathbb{N}_0$ (we note that if $L \to M \to N$ is an exact sequence of R-modules, then we can deduce from [14, Proposition 2.9(4)] that

$$\operatorname{Cosupp}(M) \subseteq \operatorname{Cosupp}(L) \cup \operatorname{Cosupp}(N).$$

This proves the inclusion (2.1) in the case when $n = 1$. Now assume, inductively, that $n > 1$ and the inclusion (2.1) holds for smaller values of n. If we replace N by $(0 :_N x_1R)$, then, by the inductive hypothesis for elements $x_2, \ldots, x_n$, we have

$$\begin{aligned} &\operatorname{Cosupp}\left(T_i(0 :_N (x_1, \ldots, x_n)R)\right) \\ &\subseteq \left(\bigcup_{j=i}^{i+n-1} \operatorname{Cosupp}\left(T_j(0 :_N x_1R)\right)\right) \\ &\cup \left(\bigcup_{k=2}^{n} \bigcup_{j=i+2}^{i+2+n-k} \operatorname{Cosupp}\left(T_j\left(\frac{(0 :_N (x_1, \ldots, x_{k-1})R)}{x_k(0 :_N (x_1, \ldots, x_{k-1})R)}\right)\right)\right) \end{aligned} \tag{2.5}$$

(note that if we set $y_1 := x_2, \ldots, y_{n-1} := x_n$ and $l := k-1$, then $1 \leq l \leq n-1$ and $i+2 \leq j \leq i+2+n-1-l$ yield $2 \leq k \leq n$ and $i+2 \leq j \leq i+2+n-k$). Now combining the inclusion (2.4) with the inclusion (2.5) yields the inclusion (2.1) and the inductive step is complete.

Now assume that $x_jM = 0$ for all $1 \leq j \leq n$ and we prove, by induction on n, that the inclusion (2.2) holds. Since the functor $T_i(\cdot)$ is R-linear, the endomorphism of $T_i(N)$ given by multiplication by x_j is the zero map for all $i \in \mathbb{N}_0$ and all $1 \leq j \leq n$. The triangle in the diagram (2.3) commutes, and so $\operatorname{Im} x_1^{(i)} \subseteq \operatorname{Ker} f_i$ for all $i \in \mathbb{N}_0$. Therefore

$$\operatorname{Cosupp}\left(\operatorname{Im} x_1^{(i)}\right) \subseteq \operatorname{Cosupp}\left(\operatorname{Ker} f_i\right) \subseteq \operatorname{Cosupp}\left(T_{i+1}(N/x_1N)\right). \tag{2.6}$$

Also, the exactness of rows in the diagram (2.3) implies that

$$\operatorname{Cosupp}\left(T_i(x_1N)\right) \subseteq \operatorname{Cosupp}\left(\operatorname{Im} x_1^{(i)}\right) \cup \operatorname{Cosupp}\left(T_{i-1}(0 :_N x_1)\right) \tag{2.7}$$

and

$$\operatorname{Cosupp}\left(T_i(N)\right) \subseteq \operatorname{Cosupp}\left(T_i(x_1N)\right) \cup \operatorname{Cosupp}\left(T_i(N/x_1N)\right). \tag{2.8}$$

The inclusions (2.6)–(2.8) yield

$$\begin{aligned} \operatorname{Cosupp}\left(T_i(N)\right) \subseteq\ & \operatorname{Cosupp}\left(T_{i-1}(0 :_N x_1R)\right) \\ & \cup \operatorname{Cosupp}\left(T_i\left(N/x_1N\right)\right) \\ & \cup \operatorname{Cosupp}\left(T_{i+1}\left(N/x_1N\right)\right). \end{aligned} \tag{2.9}$$

Hence, the inclusion (2.2) is true in the case when $n = 1$. Next suppose, inductively, that $n > 1$ and that the inclusion (2.2) has been proved for smaller values of n. If we use $(0 :_N x_1R)$ and $i-1$ instead of N and i respectively, then the inductive hypothesis for elements $x_2, \ldots, x_n$ yields

$$\begin{aligned} &\operatorname{Cosupp}\left(T_{i-1}(0 :_N x_1R)\right) \\ &\subseteq \operatorname{Cosupp}\left(T_{i-n}(0 :_N (x_1, \ldots, x_n)R)\right) \\ &\cup \left(\bigcup_{k=2}^{n} \bigcup_{j=i+1-k}^{i+2-k} \operatorname{Cosupp}\left(T_j \left(\frac{(0 :_N (x_1, \ldots, x_{k-1})R)}{x_k(0 :_N (x_1, \ldots, x_{k-1})R)} \right)\right)\right). \end{aligned} \tag{2.10}$$

By combining the inclusions (2.9) and (2.10), we obtain the inclusion (2.2). This completes the inductive step. □

Corollary 2.6. *Let M and N be R-modules, and let $x_1, \ldots, x_n$ be a filter coregular N-sequence in* $\operatorname{Ann}(M)$. *Then*

$$\operatorname{Cosupp}\left(\operatorname{Tor}_i^R(M,N)\right) \subseteq \operatorname{Max}(R) \tag{2.11}$$

for all $i < n$, and

$$\begin{aligned} &\operatorname{Cosupp}\left(\operatorname{Tor}_n^R(M,N)\right) \cup \operatorname{Max}(R) \\ &= \operatorname{Cosupp}\left(M \otimes_R (0 :_N (x_1, \ldots, x_n)R)\right) \cup \operatorname{Max}(R). \end{aligned} \tag{2.12}$$

Proof. For each $1 \leq k \leq n$, since

$$\operatorname{Cosupp}\left(\frac{(0 :_N (x_1, \ldots, x_{k-1})R)}{x_k(0 :_N (x_1, \ldots, x_{k-1})R)}\right) \subseteq \operatorname{Max}(R),$$

we have

$$\operatorname{Cosupp}\left(\operatorname{Tor}_i^R\left(M, \frac{(0 :_N (x_1, \ldots, x_{k-1})R)}{x_k(0 :_N (x_1, \ldots, x_{k-1})R)}\right)\right) \subseteq \operatorname{Max}(R) \tag{2.13}$$

for all $i \in \mathbb{N}_0$. Hence the inclusion (2.11) is an immediate consequence of the inclusion (2.2). Now we prove the equation (2.12). If we set $i = 0$ in the inclusion (2.1), then it follows from the inclusions (2.11) and (2.13) that

$$\begin{aligned} &\operatorname{Cosupp}\left(M \otimes_R (0 :_N (x_1, \ldots, x_n)R)\right) \\ &\subseteq \operatorname{Cosupp}\left(\operatorname{Tor}_n^R(M,N)\right) \cup \operatorname{Max}(R). \end{aligned} \tag{2.14}$$

Conversely, if we set $i = n$ in the inclusion (2.2), then the inclusion (2.13) implies that

$$\begin{aligned} &\operatorname{Cosupp}\left(\operatorname{Tor}_n^R(M,N)\right) \\ &\subseteq \operatorname{Cosupp}\left(M \otimes_R (0 :_N (x_1, \ldots, x_n)R)\right) \cup \operatorname{Max}(R). \end{aligned} \tag{2.15}$$

Now the equation (2.12) follows from the inclusions (2.14) and (2.15). □

Lemma 2.7. *Let M, N and L be R-modules such that M and L are finitely generated, and let $n \in \mathbb{N}$. If* $\mathrm{Cosupp}\left(\mathrm{Tor}_i^R(M,N)\right) \subseteq \mathrm{Max}(R)$ *for all $i < n$ and* $\mathrm{Supp}(L) \subseteq \mathrm{Supp}(M)$*, then*

$$\mathrm{Cosupp}\left(\mathrm{Tor}_i^R(L,N)\right) \subseteq \mathrm{Max}(R)$$

for all $i < n$. In particular, $\mathrm{Cosupp}\left(\mathrm{Tor}_i^R(L,N)\right) \subseteq \mathrm{Max}(R)$ *for all $i < n$ if and only if* $\mathrm{Cosupp}\left(\mathrm{Tor}_i^R(M,N)\right) \subseteq \mathrm{Max}(R)$ *for all $i < n$ whenever* $\mathrm{Supp}(L) = \mathrm{Supp}(M)$*.*

Proof. Assume that $\mathrm{Cosupp}\left(\mathrm{Tor}_i^R(M,N)\right) \subseteq \mathrm{Max}(R)$ for all $i < n$ and we prove by induction on n that for every finitely generated R-module L with $\mathrm{Supp}(L) \subseteq \mathrm{Supp}(M)$, $\mathrm{Cosupp}\left(\mathrm{Tor}_i^R(L,N)\right) \subseteq \mathrm{Max}(R)$ for all $i < n$. Assume that L is a finitely generated R-module such that $\mathrm{Supp}(L) \subseteq \mathrm{Supp}(M)$. By Gruson's theorem [17, Theorem 4.1] there exists a chain

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_m = L$$

of submodules of L such that, for each $1 \leq j \leq m$, L_j/L_{j-1} is a homomorphic image of a direct sum of finitely many copies of M. For each $1 \leq j \leq m$, the exact sequence

$$0 \to L_{j-1} \to L_j \to L_j/L_{j-1} \to 0$$

induces the following long exact sequence

$$\cdots \to \mathrm{Tor}_i^R(L_{j-1},N) \to \mathrm{Tor}_i^R(L_j,N) \to \mathrm{Tor}_i^R(L_j/L_{j-1},N) \to \cdots.$$

Hence

$$\begin{aligned}&\mathrm{Cosupp}(\mathrm{Tor}_i^R(L_j,N))\\ &\subseteq \mathrm{Cosupp}(\mathrm{Tor}_i^R(L_{j-1},N)) \cup \mathrm{Cosupp}(\mathrm{Tor}_i^R(L_j/L_{j-1},N))\end{aligned}$$

for all $1 \leq j \leq m$ and all i. It follows that

$$\begin{aligned}
&\operatorname{Cosupp}(\operatorname{Tor}_i^R(L,N)) \\
&= \operatorname{Cosupp}(\operatorname{Tor}_i^R(L_m,N)) \\
&\subseteq \operatorname{Cosupp}(\operatorname{Tor}_i^R(L_{m-1},N)) \cup \operatorname{Cosupp}(\operatorname{Tor}_i^R(L_m/L_{m-1},N)) \\
&\vdots \\
&\subseteq \operatorname{Cosupp}(\operatorname{Tor}_i^R(L_0,N)) \cup \left(\bigcup_{j=1}^{m} \operatorname{Cosupp}(\operatorname{Tor}_i^R(L_j/L_{j-1},N)\right) \\
&= \bigcup_{j=1}^{m} \operatorname{Cosupp}(\operatorname{Tor}_i^R(L_j/L_{j-1},N)
\end{aligned}$$

for all i. Thus to prove the assertion it is sufficient for us to prove that $\operatorname{Cosupp}(\operatorname{Tor}_i^R(L_j/L_{j-1},N) \subseteq \operatorname{Max}(R)$ for all $1 \leq j \leq m$ and all $i < n$. Hence the situation can be reduced to the case $m = 1$. Thus there exists an exact sequence

$$0 \to K \to M^t \to L \to 0$$

for some $t \in \mathbb{N}$ and some finitely generated R-module K. This exact sequence induces the following long exact sequence

$$\cdots \to \operatorname{Tor}_i^R(M,N)^t \to \operatorname{Tor}_i^R(L,N) \to \operatorname{Tor}_{i-1}^R(K,N) \to \cdots . \tag{2.16}$$

For $n = 1$, it follows from the exact sequence

$$(M \otimes_R N)^t \to L \otimes_R N \to 0$$

that

$$\operatorname{Cosupp}(L \otimes_R N) \subseteq \operatorname{Cosupp}(M \otimes_R N) \subseteq \operatorname{Max}(R).$$

Therefore the result holds for $n = 1$. Now assume, inductively, that $n > 1$ and the result has been proved for smaller values of n. It follows from the exact sequence (2.16) that

$$\begin{aligned}
&\operatorname{Cosupp}(\operatorname{Tor}_i^R(L,N)) \\
&\subseteq \operatorname{Cosupp}(\operatorname{Tor}_i^R(M,N)) \cup \operatorname{Cosupp}(\operatorname{Tor}_{i-1}^R(K,N))
\end{aligned} \tag{2.17}$$

for all i. Since $\operatorname{Supp}(K) \subseteq \operatorname{Supp}(M)$, the induction hypothesis implies that

$$\operatorname{Cosupp}(\operatorname{Tor}_i^R(K,N)) \subseteq \operatorname{Max}(R)$$

for all $i < n-1$. Thus, by the hypothesis and the inclusion (2.17), we have

$$\operatorname{Cosupp}(\operatorname{Tor}_i^R(L,N)) \subseteq \operatorname{Max}(R)$$

for all $i < n$. This completes the inductive step. □

Now, we are ready to state and prove the main result of this paper. Let $\mathfrak{a}$ be an ideal of R and let N be an Artinian R-module. Among the other things, the following theorem shows that the infimum of integers i with the property that the local homology module $\mathrm{H}_i^{\mathfrak{a}}(N)$ is not finitely generated as an $\widehat{R}^{\mathfrak{a}}$-module and the common length of all maximal filter coregular N-sequences in $\mathfrak{a}$ are same.

Theorem 2.8. *Let $\mathfrak{a}$ be an ideal of R, and let M and N be R-modules such that M is finitely generated and N is Artinian. For each $n \in \mathbb{N}$, the following conditions are equivalent:*

(i) *there is a filter coregular N-sequence in $\mathfrak{a}$ of length n;*
(ii) *any filter coregular N-sequence in $\mathfrak{a}$ of length less than n can be extended to a filter coregular N-sequence in $\mathfrak{a}$ of length n;*
(iii) $\mathrm{Cosupp}\left(\mathrm{Tor}_i^R(R/\mathfrak{a}, N)\right) \subseteq \mathrm{Max}(R)$ *(or equivalently* $\mathrm{Tor}_i^R(R/\mathfrak{a}, N)$ *has finite length) for all $i < n$;*
(iv) *if* $\mathrm{Supp}(M) = \mathrm{V}(\mathfrak{a})$, *then* $\mathrm{Cosupp}\left(\mathrm{Tor}_i^R(M, N)\right) \subseteq \mathrm{Max}(R)$ *(or equivalently* $\mathrm{Tor}_i^R(M, N)$ *has finite length) for all $i < n$; and*
(v) *if* $\mathrm{Ann}(M) \subseteq \mathfrak{a}$, *then* $\mathrm{H}_i^{\mathfrak{a}}(M, N)$ *is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module for all $i < n$.*

Proof. The statements (iii) and (iv) are equivalent by Lemma 2.7. The implication (ii)⇒(i) is clear. Also, (i)⇒(iii) is an immediate consequence of the inclusion (2.11) in Corollary 2.6.

(iii)⇒(ii). Assume that $\mathrm{Cosupp}\left(\mathrm{Tor}_i^R(R/\mathfrak{a}, N)\right) \subseteq \mathrm{Max}(R)$ for all $i < n$, and suppose, for the sake of contradiction, that $x_1, \ldots, x_m$ is a maximal filter coregular N-sequence in $\mathfrak{a}$ of length $0 \leq m < n$. The maximality of $x_1, \ldots, x_m$ yields

$$\mathfrak{a} \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Att}(0:_N (x_1,\ldots,x_m)R) \setminus \mathrm{Max}(R)} \mathfrak{p}.$$

Since $\mathrm{Att}\,(0 :_N (x_1, \ldots, x_m)R)$ is a finite set, it follows from the Prime Avoidance Theorem that $\mathfrak{a} \subseteq \mathfrak{p}$ for some

$$\mathfrak{p} \in \mathrm{Att}\,(0 :_N (x_1, \ldots, x_m)R) \setminus \mathrm{Max}(R).$$

Hence, by the equation (2.12) in Corollary 2.6 and the hypothesis, we have

$$\begin{aligned}\mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \cap \mathrm{Att}(0 :_N (x_1, \ldots, x_m)R) &= \mathrm{Att}(R/\mathfrak{a} \otimes_R (0 :_N (x_1, \ldots, x_m)R)) \\ &\subseteq \mathrm{Cosupp}(R/\mathfrak{a} \otimes_R (0 :_N (x_1, \ldots, x_m)R)) \\ &\subseteq \mathrm{Cosupp}(\mathrm{Tor}_m^R(R/\mathfrak{a}, N)) \cup \mathrm{Max}(R) \\ &\subseteq \mathrm{Max}(R),\end{aligned}$$

which is a contradiction. Hence the statements (i)–(iv) are equivalent.

(i)⇔(v). We prove, by induction on n, that (i) and (v) are equivalent. Assume that M is a finitely generated R-module such that $\mathrm{Ann}(M) \subseteq \mathfrak{a}$. We first assume that $n = 1$. Since $M \otimes_R N$ is Artinian, we have

$$\begin{aligned}\mathrm{H}_0^{\mathfrak{a}}(M, N) &\cong \Lambda_{\mathfrak{a}}(M \otimes_R N) \\ &\cong (M \otimes_R N)/\mathfrak{a}^s(M \otimes_R N) \\ &\cong \mathrm{Tor}_0^R(M/\mathfrak{a}^s M, N)\end{aligned}$$

for all sufficiently large integers s. Also, since $\mathrm{Supp}(M/\mathfrak{a}^s M) = \mathrm{V}(\mathfrak{a})$, the equivalence of (i) and (iv) implies that $\mathrm{H}_0^{\mathfrak{a}}(M, N)$ is a finitely generated R-module or equivalently it is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module if and only if $\mathfrak{a}$ contains a filter coregular element on N (note that since $\mathrm{H}_0^{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-torsion, its submodules as an R-module and as an $\widehat{R}^{\mathfrak{a}}$-module are same; see [6, Lemma 1.3]). Thus the result holds in the case $n = 1$.

Now assume, inductively, that $n > 1$ and the result has been proved for smaller values of n. Since N is Artinian, there exists $t \in \mathbb{N}$ such that $\mathfrak{a}^s N = \mathfrak{a}^t N$ for all $s \geq t$ and so $\Lambda_{\mathfrak{a}}(N) \cong N/\mathfrak{a}^t N$. Assume that either (i) or (v) holds. Since $n > 1$ and

$$\mathrm{H}_0^{\mathfrak{a}}(M, N) \cong \mathrm{Tor}_0^R(M/\mathfrak{a}^s M, N)$$

for sufficiently large integers s, if (v) holds, then $\mathrm{Tor}_0^R(M/\mathfrak{a}^s M, N)$ has finite length by the hypothesis of (v). Since

$$\mathrm{Supp}(R/\mathfrak{a}^t) = \mathrm{Supp}(M/\mathfrak{a}^s M),$$

by Lemma 2.7,

$$\Lambda_{\mathfrak{a}}(N) \cong \mathrm{Tor}_0^R(R/\mathfrak{a}^t, N)$$

has finite length in this case. Also, if (i) holds, then, by the equivalence of (i) and (iv), $\Lambda_{\mathfrak{a}}(N) \cong \mathrm{Tor}_0^R(R/\mathfrak{a}^t, N)$ has finite length. Therefore in both cases

$\Lambda_{\mathfrak{a}}(N)$ has finite length. Now, the exact sequence

$$0 \to \mathfrak{a}^t N \to N \to \Lambda_{\mathfrak{a}}(N) \to 0$$

of Artinian R-modules induces the following long exact sequences

$$\begin{aligned} \cdots \to \mathrm{H}^{\mathfrak{a}}_{i+1}(M, \Lambda_{\mathfrak{a}}(N)) \to \mathrm{H}^{\mathfrak{a}}_{i}(M, \mathfrak{a}^t N) \to \mathrm{H}^{\mathfrak{a}}_{i}(M, N) \\ \to \mathrm{H}^{\mathfrak{a}}_{i}(M, \Lambda_{\mathfrak{a}}(N)) \to \cdots \end{aligned} \tag{2.18}$$

(see [12, Proposition 2.4]), and

$$\begin{aligned} \cdots \to \mathrm{Tor}^R_{i+1}\left(R/\mathfrak{a}, \Lambda_{\mathfrak{a}}(N)\right) \to \mathrm{Tor}^R_{i}\left(R/\mathfrak{a}, \mathfrak{a}^t N\right) \to \mathrm{Tor}^R_{i}\left(R/\mathfrak{a}, N\right) \\ \to \mathrm{Tor}^R_{i}\left(R/\mathfrak{a}, \Lambda_{\mathfrak{a}}(N)\right) \to \cdots . \end{aligned} \tag{2.19}$$

Since $\Lambda_{\mathfrak{a}}(N)$ is Artinian, by [13, Theorems 2.3(i) and 3.2], we have

$$\mathrm{H}^{\mathfrak{a}}_{i}(M, \Lambda_{\mathfrak{a}}(N)) \cong H_i\left(\Lambda_{\mathfrak{a}}(M \otimes_R F_\bullet)\right),$$

where $F_\bullet$ is a free resolution of $\Lambda_{\mathfrak{a}}(N)$. Now $\Lambda_{\mathfrak{a}}(N)$ is finitely generated and so we can assume that every component of $F_\bullet$ is finitely generated. On the other hand, $\Lambda_{\mathfrak{a}}(\cdot)$ is an additive exact functor on the category of finitely generated R-modules, and hence it commutes with the homological functor in this category. Therefore

$$H_i\left(\Lambda_{\mathfrak{a}}(M \otimes_R F_\bullet)\right) \cong \Lambda_{\mathfrak{a}}\left(H_i\left(M \otimes_R F_\bullet\right)\right) \cong \Lambda_{\mathfrak{a}}\left(\mathrm{Tor}^R_i\left(M, \Lambda_{\mathfrak{a}}(N)\right)\right).$$

Since $\mathrm{Tor}^R_i\left(M, \Lambda_{\mathfrak{a}}(N)\right)$ is Artinian, we obtain

$$\Lambda_{\mathfrak{a}}\left(\mathrm{Tor}^R_i\left(M, \Lambda_{\mathfrak{a}}(N)\right)\right) \cong \mathrm{Tor}^R_i\left(M, \Lambda_{\mathfrak{a}}(N)\right)/\mathfrak{a}^r\, \mathrm{Tor}^R_i\left(M, \Lambda_{\mathfrak{a}}(N)\right)$$

for all sufficiently large integers r. Since $\mathfrak{a}^r\,\mathrm{Tor}^R_i\left(M, \Lambda_{\mathfrak{a}}(N)\right) = 0$ for $r \geq t$, the above isomorphisms yield

$$\mathrm{H}^{\mathfrak{a}}_{i}(M, \Lambda_{\mathfrak{a}}(N)) \cong \mathrm{Tor}^R_i\left(M, \Lambda_{\mathfrak{a}}(N)\right) \cong \mathrm{Tor}^R_i\left(M, N/\mathfrak{a}^t N\right).$$

Hence $\mathrm{H}^{\mathfrak{a}}_{i}(M, \Lambda_{\mathfrak{a}}(N))$ is a finitely generated R-module for all $i \in \mathbb{N}_0$. Also, the above isomorphism shows that $\mathrm{H}^{\mathfrak{a}}_{i}(M, \Lambda_{\mathfrak{a}}(N))$ is $\mathfrak{a}$-torsion, and so $\mathrm{H}^{\mathfrak{a}}_{i}(M, \Lambda_{\mathfrak{a}}(N))$ is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module for all $i \in \mathbb{N}_0$ by [6, Lemma 1.3]. Now, for each $i \in \mathbb{N}_0$, it follows from the long exact sequence (2.18) that $\mathrm{H}^{\mathfrak{a}}_{i}(M, N)$ is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module if and only if $\mathrm{H}^{\mathfrak{a}}_{i}(M, \mathfrak{a}^t N)$ is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module. Also, for each $i \in \mathbb{N}_0$, it follows from the long exact sequence (2.19) that $\mathrm{Tor}^R_i\left(R/\mathfrak{a}, N\right)$ has finite length if and only if $\mathrm{Tor}^R_i\left(R/\mathfrak{a}, \mathfrak{a}^t N\right)$ has finite length because $\mathrm{Tor}^R_i\left(M, \Lambda_{\mathfrak{a}}(N)\right)$ has finite length for all i. Thus to prove the equivalence of (i) and (v), in view of the equivalence of (i) and (iii), we can replace N by $\mathfrak{a}^t N$ and assume, in addition, that $\mathfrak{a}N = N$. Therefore $\mathrm{V}(\mathfrak{a}) \cap \mathrm{Att}(N) = \emptyset$, and so $\mathfrak{a} \nsubseteq \bigcup_{\mathfrak{p} \in \mathrm{Att}(N)} \mathfrak{p}$. Let

$x_1 \in \mathfrak{a} \setminus \bigcup_{\mathfrak{p}\in \mathrm{Att}(N)} \mathfrak{p}$. Then $\mathrm{V}(x_1R) \cap \mathrm{Att}(N) = \emptyset$, and so $N = x_1N$. The exact sequence

$$0 \to (0 :_N x_1R) \to N \xrightarrow{x_1} N \to 0$$

induces the long exact sequence

$$\begin{aligned} \cdots \to \mathrm{H}^{\mathfrak{a}}_{i+1}(M,N) \xrightarrow{x_1} \mathrm{H}^{\mathfrak{a}}_{i+1}(M,N) \to \mathrm{H}^{\mathfrak{a}}_{i}(M,(0 :_N x_1R)) \\ \to \mathrm{H}^{\mathfrak{a}}_{i}(M,N) \xrightarrow{x_1} \cdots . \end{aligned} \tag{2.20}$$

We first assume that (i) holds. By the equivalence of (i) and (ii), we can extend x_1 to a filter coregular N-sequence of length n in $\mathfrak{a}$, say $x_1, x_2, \ldots, x_n$. Hence $x_2, \ldots, x_n$ is a filter coregular $(0 :_N x_1R)$-sequence in $\mathfrak{a}$, and so, by the inductive hypothesis, $\mathrm{H}^{\mathfrak{a}}_{i}(M,(0 :_N x_1R))$ is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module for all $i < n-1$. It follows from long exact sequence (2.20) that $\mathrm{H}^{\mathfrak{a}}_{i}(M,N)/x_1 \mathrm{H}^{\mathfrak{a}}_{i}(M,N)$ and consequently its homomorphic image $\mathrm{H}^{\mathfrak{a}}_{i}(M,N)/(\mathfrak{a}\widehat{R}^{\mathfrak{a}}) \mathrm{H}^{\mathfrak{a}}_{i}(M,N)$ are finitely generated $\widehat{R}^{\mathfrak{a}}$-modules for all $i < n$. Also, by [12, Proposition 2.3(i)], we have

$$\bigcap_{t\in\mathbb{N}} (\mathfrak{a}\widehat{R}^{\mathfrak{a}})^t \mathrm{H}^{\mathfrak{a}}_{i}(M,N) = \bigcap_{t\in\mathbb{N}} (\mathfrak{a}^t\widehat{R}^{\mathfrak{a}}) \mathrm{H}^{\mathfrak{a}}_{i}(M,N) = \bigcap_{t\in\mathbb{N}} \mathfrak{a}^t \mathrm{H}^{\mathfrak{a}}_{i}(M,N) = 0.$$

Hence, by [11, Theorem 8.4], $\mathrm{H}^{\mathfrak{a}}_{i}(M,N)$ is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module for all $i < n$. Conversely, assume that $\mathrm{H}^{\mathfrak{a}}_{i}(M,N)$ is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module for all $i < n$. It follows from the long exact sequence (2.20) that $\mathrm{H}^{\mathfrak{a}}_{i}(M,(0 :_N x_1R))$ is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module for all $i < n-1$, and so, by the inductive hypothesis, there is a filter coregular $(0 :_N x_1R)$-sequence in $\mathfrak{a}$ of length $n-1$, say $x_2, \ldots, x_n$. Therefore $x_1, x_2, \ldots, x_n$ is a filter coregular N-sequence in $\mathfrak{a}$. This completes the inductive step. □

Remark 2.9. Let $\mathfrak{a}$ be an ideal of R, and let N be an Artinian R-module. When there exists a filter coregular N-sequence in $\mathfrak{a}$ of infinite length, then, by the equivalence of (i) and (ii) in Theorem 2.8, any filter coregular N-sequence in $\mathfrak{a}$ can be extended to a filter coregular N-sequence in $\mathfrak{a}$ of arbitrary length, and in this case we set

$$\text{f-width}(\mathfrak{a}, N) = \infty.$$

Now assume that all filter coregular N-sequences in $\mathfrak{a}$ have finite length. Again, by the equivalence of (i) and (ii) in Theorem 2.8, we can extend any filter coregular N-sequence in $\mathfrak{a}$ to a maximal one, and all maximal filter coregular N-sequences in $\mathfrak{a}$ are of the same length which we denote this common length by $\operatorname{f-width}(\mathfrak{a}, N)$. Moreover, if M is a finitely generated R-module such that $\operatorname{Supp}(M) = \mathrm{V}(\mathfrak{a})$, then, by Theorem 2.8, we have

$$\begin{aligned} &\operatorname{f-width}(\mathfrak{a}, N) \\ &= \inf\{i \in \mathbb{N}_0 : \operatorname{Cosupp}\left(\operatorname{Tor}_i^R(M, N)\right) \nsubseteq \operatorname{Max}(R)\} \\ &= \inf\{i \in \mathbb{N}_0 : \operatorname{Tor}_i^R(M, N) \text{ has infinite length as an } R\text{-module}\} \\ &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}_i^{\mathfrak{a}}(N) \text{ is not a finitely generated } \widehat{R}^{\mathfrak{a}}\text{-module}\} \end{aligned} \tag{2.21}$$

(we note that $\mathrm{H}_i^{\mathfrak{a}}(R, N) = \mathrm{H}_i^{\mathfrak{a}}(N)$). Also, for an arbitrary finitely generated R-module L, since $\mathrm{H}_i^{\mathfrak{a}+\operatorname{Ann}(L)}(L, N) \cong \mathrm{H}_i^{\mathfrak{a}}(L, N)$, if we replace $\mathfrak{a}$ by $\mathfrak{a}+\operatorname{Ann}(L)$ in Theorem 2.8, then the equivalence of (ii) and (v) in Theorem 2.8 yields

$$\begin{aligned} &\operatorname{f-width}(\mathfrak{a} + \operatorname{Ann}(L), N) \\ &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}_i^{\mathfrak{a}}(L, N) \text{ is not a finitely generated } \widehat{R}^{\mathfrak{a}}\text{-module}\}. \end{aligned} \tag{2.22}$$

Finally, since $\mathrm{V}(\mathfrak{a}) = \mathrm{V}\left(\sqrt{\mathfrak{a}}\right)$, it follows from the first equality in the equation (2.21) that $\operatorname{f-width}(\mathfrak{a}, N) = \operatorname{f-width}(\sqrt{\mathfrak{a}}, N)$.

Proposition 2.10. *Let $\mathfrak{a}$ be an ideal of R, and let N be an Artinian R-module. If* $\operatorname{f-width}(\mathfrak{a}, N) = \infty$*, then* $(0 :_N \mathfrak{a})$ *has finite length. The converse statement holds whenever R is a semi-local ring which is complete with respect to its Jacobson radical.*

Proof. Assume that $\operatorname{f-width}(\mathfrak{a}, N) = \infty$, and $x_1, x_2, x_3, \ldots$ is a filter coregular N-sequence of infinite length in $\mathfrak{a}$. There is the following descending chain of submodules of N

$$(0 :_N x_1R) \supseteq (0 :_N (x_1, x_2)R) \supseteq (0 :_N (x_1, x_2, x_3)R) \supseteq \cdots .$$

Hence $(0 :_N (x_1, \ldots, x_{n-1})R) = (0 :_N (x_1, \ldots, x_n)R)$ for some $n \in \mathbb{N}$, and so $x_n(0 :_N (x_1, \ldots, x_{n-1})R) = 0$. Thus $(0 :_N (x_1, \ldots, x_{n-1})R)$ has finite length because $(0 :_N (x_1, \ldots, x_{n-1})R)/x_n(0 :_N (x_1, \ldots, x_{n-1})R)$ has finite length by definition. Hence $(0 :_N \mathfrak{a}) \subseteq (0 :_N (x_1, \ldots, x_{n-1})R)$ has finite length. To prove the converse statement, assume that R is a complete semi-local ring and that $(0 :_N \mathfrak{a})$ has finite length. Hence

$$\operatorname{Cosupp}(0 :_N \mathfrak{a}) = \operatorname{Supp}(0 :_N \mathfrak{a}) \subseteq \operatorname{Max}(R).$$

On the other hand, for each $i \in \mathbb{N}_0$, $\mathfrak{a} + \operatorname{Ann}(N) \subseteq \operatorname{Ann}(\operatorname{Tor}_i^R(R/\mathfrak{a}, N))$. Therefore, in view of [14, Proposition 2.12], we have

$$\begin{aligned}\operatorname{Cosupp}(\operatorname{Tor}_i^R(R/\mathfrak{a}, N)) &\subseteq \operatorname{V}(\mathfrak{a} + \operatorname{Ann}(N)) \\ &= \operatorname{V}(\mathfrak{a}) \cap \operatorname{Cosupp}(N) \\ &= \operatorname{Cosupp}(0 :_N \mathfrak{a}) \\ &\subseteq \operatorname{Max}(R)\end{aligned}$$

for all $i \in \mathbb{N}_0$. Hence Theorem 2.8 implies that $\operatorname{f-width}(\mathfrak{a}, N) = \infty$. □

Corollary 2.11. *Let $\mathfrak{a}$ be an ideal of R, and let M and N be R-modules such that M is finitely generated and N is Artinian.*

(i) *If $\operatorname{Tor}_i^R(M, N)$ has finite length for all $i \in \mathbb{N}_0$, then $(0 :_N \operatorname{Ann}(M))$ has finite length.*

(ii) *If $\operatorname{H}_i^{\mathfrak{a}}(M, N)$ is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module for all $i \in \mathbb{N}_0$, then $(0 :_N \mathfrak{a} + \operatorname{Ann}(M))$ has finite length. In particular, $(0 :_N \mathfrak{a})$ has finite length whenever $\operatorname{H}_i^{\mathfrak{a}}(N)$ is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module for all $i \in \mathbb{N}_0$.*

Moreover, the converse statements hold when R is a complete semi-local ring.

Proof. It follows by the equations (2.21), (2.22) and Proposition 2.10. □

Acknowledgments

The authors are deeply grateful to the referee for his/her useful comments and suggestions. Also, we would like to express our gratitude to Prof. Kamran Divaani-Aazar for careful reading of the manuscript and for the helpful comments and suggestions.

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