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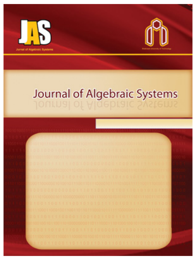

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ON THE COMINIMAXNESS OF LOCAL COHOMOLOGY MODULES

G. GHASEMI* AND J. A'ZAMI

Abstract. Let I be an ideal of a commutative Noetherian ring R. It is shown that the R-modules $H_I^i(M)$ are I-cominimax, for all finitely generated R-modules M and all $i \in \mathbb{N}_0$, if the R-modules $H_I^i(R)$ are I-cominimax with dimension not exceeding 1, for all integers $i \geq 2$. This is an analogue result of Bahmanpour in [6].

1. Introduction

Throughout this paper, R denotes a commutative Noetherian ring (with non-zero identity) and I will denote an ideal of R. The symbol $\mathbb{Z}$ denotes the set of integers; in addition, $\mathbb{N}$ (respectively $\mathbb{N}_0$) will denote the set of positive (respectively non-negative) integers. For each R-module L, the set of minimal elements of $\operatorname{Ass}_R L$ with respect to inclusion is denoted by $\operatorname{mAss}_R L$; also, $\operatorname{Assh}_R L$ denotes the set $\{\mathfrak{p} \in \operatorname{Ass}_R L : \dim R/\mathfrak{p} = \dim L\}$. We denote $\operatorname{Supp} R/I = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq I\}$ by $V(I)$.

For an R-module M, the ith local cohomology module of M with support in $V(I)$ is defined as:

$$H_I^i(M) = \varinjlim_{n \geq 1} \operatorname{Ext}_R^i(R/I^n, M).$$

We refer the reader to [11] or [19] for more details about local cohomology.

Recall that for an R-module M, the notion $\operatorname{cd}(I, M)$, *the cohomological dimension of M with respect to I*, is defined as:

$$\operatorname{cd}(I, M) = \sup\{i \in \mathbb{N}_0 : H_I^i(M) \neq 0\}$$

and the notion $q(I, M)$, which for the first time was introduced by Hartshorne, is defined as:

$$q(I, M) = \sup\{i \in \mathbb{N}_0 : H_I^i(M) \text{ is not Artinian}\},$$

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with the usual convention that the supremum of the empty set of integers is interpreted as $-\infty$. These two notions have been studied by several authors (see [4, 7, 14, 15, 17, 18, 20]).

In the sequel the symbol $\mathscr{C}(R, I)_{com}$ denotes the category of all I-cominimax R-modules and $\mathscr{C}^1(R, I)_{com}$ denotes the category of all R-modules $M \in \mathscr{C}(R, I)_{com}$ such that $\dim M \leq 1$. An R-module M is called $\mathfrak{a}$-cominimax if the support of M is contained in $V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is minimax for all $i \geq 0$. The concept of the $\mathfrak{a}$-cominimax modules is introduced in [2]. Also, throughout this paper, let $\mathscr{I}'(R)$ denote the class of all ideals I of R such that $H^i_I(M) \in \mathscr{C}(R, I)_{com}$, for all finitely generated R-modules M and all $i \in \mathbb{N}_0$.

Recall that the *I-transform functor*, denoted by $D_I(-)$ is defined as:

$$D_I(-) = \varinjlim_{n \geq 1} \operatorname{Hom}_R(I^n, -).$$

In general, the R-module $D_I(R)$ has an R-algebra structure (see [11, Exercise 2.2.3]). In fact, with this structure $D_I(R)$ is a commutative ring with identity. Also, it is well known that if $D_I(-)$ is an exact functor then $D_I(R)$ is a finitely generated R-algebra. But, in general we don't know when the ring $D_I(R)$ is Noetherian.

Throughout this paper, for each pair of the sets X and Y, the expression $X \subseteq Y$ means that X is a subset of Y and the expression $X \subset Y$ means that $X \subseteq Y$ and $X \neq Y$. For an Artinian R-module A, the set of attached prime ideals of A is denoted by $\operatorname{Att}_R A$. Also, for any non-nilpotent element x of R and any R-module M, the localization of M at the multiplicatively closed subset $S = \{1_R, x, x^2, x^3, \dots\}$ of R is shown by M_x. For each ideal I of a Noetherian ring R and each R-module M, we denote the submodule $\bigcup_{n=1}^{\infty}(0 :_M I^n)$ of M by $\Gamma_I(M)$. Furthermore, for any ideal I of a commutative ring T, we denote the set of minimal prime ideals over I by $\operatorname{Min} I$. Also, we show set of all maximal ideals of a ring T by $\operatorname{Max}(T)$. Finally, for any ideal J of T, *the radical of* J, denoted by $\operatorname{Rad}(J)$, is defined to be the set $\{x \in T : x^n \in J \text{ for some } n \in \mathbb{N}\}$. For any unexplained notation and terminology we refer the reader to [11, 12, 21].

2. PRELIMINARIES

In this section we establish some technical results which will be used later. We start this section with some auxiliary lemmas.

Lemma 2.1. For an ideal I of a ring R, the following statements hold:

(1) $\mathscr{C}^1(R,I)_{cof}$ is an Abelian category.
(2) Suppose that M is an R-module with $\operatorname{Supp} M \subseteq \operatorname{Max}(R) \cap V(I)$. If the R-module $(0 :_M I)$ is finitely generated then the R-module M is Artinian and I-cofinite.

Proof. See [10, Theorem 2.7] and [22, Lemma 2.1]. □

Let R be a Noetherian ring and I be an ideal of R. Recall that a subcategory $\mathscr{M}$ of the category of all R-modules is said to be a *Serre category* if in any short exact sequence of R-modules and R-homomorphisms, the middle module is in $\mathscr{M}$ if and only if the two other modules are in $\mathscr{M}$. Let $\mathscr{C}^1(R,I)$ be the Serre category of all I-torsion R-modules M with $\dim M \leq 1$. We want to emphasize at the outset, that two categories $\mathscr{C}^1(R,I)$ and $\mathscr{C}^1(R,I)_{cof}$ are different. In fact always $\mathscr{C}^1(R,I)_{com}$ is a proper subcategory of $\mathscr{C}^1(R,I)$. Now, for any R-module N, we define the notation $\mathrm{c}^1(I,N)$ as the greatest integer i such that $H^i_I(N)$ is not in $\mathscr{C}^1(R,I)$ if there exist such i's and $-\infty$ otherwise. Finally, we recall that in [4] the notion $\widetilde{q}(I,N)$ is defined as the greatest integer i such that $H^i_I(N)$ is not an Artinian I-cofinite module if there exist such i's and $-\infty$ otherwise.

Lemma 2.2. Let I be an ideal of a ring R. Assume that M and N are two finitely generated R-modules such that $\operatorname{Supp} M \subseteq \operatorname{Supp} N$. Then the following statements hold:

(1) $\mathrm{c}^1(I,M) \leq \mathrm{c}^1(I,N)$.
(2) $q(I,M) \leq q(I,N)$.
(3) $\widetilde{q}(I,M) \leq \widetilde{q}(I,N)$.
(4) $\operatorname{cd}(I,M) \leq \operatorname{cd}(I,N)$.

Proof. (1) Considering the fact that $\mathscr{C}^1(R,I)$ is a Serre category, the assertion follows immediately from [4, Theorem 2.3].
(2) See [14, Theorem 3.2].
(3) See [4, Theorem 2.6].
(4) See [15, Theorem 2.2].

□

The following result is needed in the proof of Theorem 3.10.

Lemma 2.3. Let I, J be two ideals of a ring R and M be an R-module with $JM = 0$ and $\operatorname{Supp} M \subseteq V(I)$. Then M is I-cominimax (as an R-module) if and only if M is $(I+J)/J$-cominimax (as an R/J-module).

Proof. The assertion follows by applying a method similar to the proof of [13, Proposition 2]. □

Lemma 2.4. (See [4, Theorem 4.10]) Let I be an ideal of a ring R with $q(I,R) \leq 1$. Then, $I \in \mathscr{I}(R)$.

3. RESULTS

The main goal of this section is to prove Theorem 3.10. But, first we need some useful lemmas.

Lemma 3.1. Let I be an ideal of a ring R such that $\mathscr{C}(R,I)_{com}$ is Abelian. Then the following statements hold:

(1) Suppose that

$$X^\bullet : \cdots \longrightarrow X^i \xrightarrow{f^i} X^{i+1} \xrightarrow{f^{i+1}} X^{i+2} \longrightarrow \cdots ,$$

is a complex such that $X^i \in \mathscr{C}(R,I)_{com}$ for all $i \in \mathbb{Z}$. Then for each $i \in \mathbb{Z}$ the i^{th} cohomology module $H^i(X^\bullet)$ is in $\mathscr{C}(R,I)_{com}$.

(2) Assume that $M \in \mathscr{C}(R,I)_{com}$ and N is a finitely generated R-module. Then for each $i \in \mathbb{N}_0$, the R-modules $\operatorname{Tor}_i^R(N,M)$ and $\operatorname{Ext}_R^i(N,M)$ are in $\mathscr{C}(R,I)_{com}$.

Proof. (1) The assertion follows easily from the definition.

(2) Since N is finitely generated it follows that N has a free resolution with finitely generated free R-modules. Now the assertion follows from applying part (i) and computing the R-modules $\operatorname{Tor}_i^R(N,M)$ and $\operatorname{Ext}_R^i(N,M)$ by this free resolution. □

Lemma 3.2. (See [1, Lemma 2.3]) Let I be an ideal of a ring R and $\mathscr{M}$ be a Serre subcategory of the category of R-modules. Let $n \in \mathbb{N}_0$ and M be an R-module such that $\operatorname{Ext}_R^j(R/I, H_I^i(M)) \in \mathscr{M}$, for all $0 \leq i < n$ and all $j \in \mathbb{N}_0$. If the R-modules $\operatorname{Ext}_R^n(R/I,M)$ and $\operatorname{Ext}_R^{n+1}(R/I,M)$ are in $\mathscr{M}$, then the R-modules $\operatorname{Hom}_R(R/I, H_I^n(M))$ and $\operatorname{Ext}_R^1(R/I, H_I^n(M))$ are in $\mathscr{M}$.

Lemma 3.3. (See [10, Proposition 2.6]) Let I be an ideal of a Noetherian ring R and M be an R-module such that $\dim M \leq 1$ and $\operatorname{Supp} M \subseteq V(I)$. Then the following statements are equivalent:

(1) M is I-cominimax.
(2) The R-modules $\operatorname{Hom}_R(R/I,M)$ and $\operatorname{Ext}_R^1(R/I,M)$ are cominimax.

Lemma 3.4. (See [3, Corollary 2.10]) Let I be an ideal of a ring R with $q(I,R) \leq 1$. Then $\mathscr{C}(R,I)_{cof}$ is Abelian.

Lemma 3.5. (See [5, Theorem 3.11]) Let I be an ideal of a ring R such that the I-transform functor $D_I(-)$ is exact. Then $D_I(R)$ is a flat R-algebra.

The following lemma is needed in the proof of Proposition 3.9.

Lemma 3.6. (See [6, Lemma 4.6]) Suppose that I is an ideal of a ring R such that $\Gamma_I(R) = 0$ and $q(I,R) \leq 1$. Let N be a finitely generated R-module. Then the R-modules $\mathrm{Tor}_i^R(N, D_I(R))$ are Artinian and I-cofinite, for all $i \in \mathbb{N}$, and the R-modules $\mathrm{Ext}_R^j(R/I, N \otimes_R D_I(R))$ are finitely generated, for all $j \in \mathbb{N}_0$.

Lemma 3.7. (See [8, Lemma 2.4]) Let $(R, \mathfrak{m})$ be a local ring and A be an Artinian R-module. Suppose that x is an element in $\mathfrak{m}$ such that $V(xR) \cap \mathrm{Att}_R A \subseteq \{\mathfrak{m}\}$. Then the R-module A/xA has finite length.

Lemma 3.8. (See [8, Lemma 2.5]) Let $(R, \mathfrak{m})$ be a local ring and A be an Artinian R-module. Suppose that I is an ideal of R such that the R-module $\mathrm{Hom}_R(R/I, A)$ is finitely generated. Then $V(I) \cap \mathrm{Att}_R A \subseteq V(\mathfrak{m})$.

In [25] H. Zöschinger introduced the interesting class of minimax modules, and in [25, 26] he has given many equivalent conditions for a module to be minimax. The R-module N is said to be a *minimax module*, if there is a finitely generated submodule L of N, such that N/L is Artinian. Hence, the class of minimax modules includes all finitely generated and all Artinian modules. Also, from [9, Lemma 2.1] we know that the category of minimax modules is a Serre category. It was shown by T. Zink [24] and by E. Enochs [16] that a module over a complete local ring is minimax if and only if it is Matlis reflexive. Finally, we recall that the *arithmetic rank* of an ideal J in a commutative Noetherian ring R, denoted by $\mathrm{ara}(J)$, is the least number of elements of J required to generate an ideal which has the same radical as J, i.e.,

$$\begin{aligned}\mathrm{ara}(J) = \min\{n \in \mathbb{N}_0 : \exists\, x_1, \ldots, x_n \in J \text{ with}\\ \mathrm{Rad}((x_1, \cdots, x_n)R) = \mathrm{Rad}(J)\}.\end{aligned}$$

The following proposition plays an important role in the proof of Theorem 3.10.

Proposition 3.9. Let I be an ideal of a ring R such that $\Gamma_I(R) = 0$ and $H_I^i(R) \in \mathscr{C}^1(R,I)_{com}$, for all integers $i \geq 2$. Then, for each finitely generated R-module N and each integer $i \in \mathbb{N}$, the R-module $\mathrm{Tor}_i^R(N, D_I(R))$ is I-cominimax and the R-modules $\mathrm{Ext}_R^j(R/I, N \otimes_R D_I(R))$ are minimax, for all $j \in \mathbb{N}_0$.

Proof. If $q(I,R) \leq 1$ then the assertion follows from Lemma 3.6. So, we may assume that $q(I,R) \geq 2$. Then by applying Lemma 2.1, from the hypothesis $H_I^{q(I,R)}(R) \in \mathscr{C}^1(R,I)_{com}$ we can deduce that $\dim H_I^{q(I,R)}(R) = 1$.(If $\dim H_I^{q(I,R)}(R) = 0$, then $\operatorname{Supp} H_I^{q(I,R)}(R) \subsetneq \operatorname{Max}(R)$ and since $0 :_{H_I^{q(I,R)}(R)} I$ is minimax, it follows that $0 :_{H_I^{q(I,R)}(R)} I$ is Artinian and so $H_I^{q(I,R)}(R)$ is Artinian which is a contradiction).

Suppose that N is a finitely generated R-module. In order to prove the assertion, we use induction on $t = \operatorname{ara}(I + \operatorname{Ann}_R N / \operatorname{Ann}_R N)$. If $t = 0$, then it follows from the definition that $\operatorname{Supp} N \subseteq V(I)$. By the hypothesis the R-module $H_I^i(R)$ is I-cominimax, for all integers $i \geq 2$. Also, by the assumption we have $H_I^0(R) \simeq \Gamma_I(R) = 0$. Therefore, for each $i \neq 1$ the R-module $H_I^i(R)$ is I-cominimax. Hence, by [23, Proposition 3.11] the R-module $H_I^1(R)$ is I-cominimax too. Therefore, by [23, Corollary 2.5], for each $i \in \mathbb{N}_0$ the R-module $\operatorname{Tor}_i^R(N, H_I^1(R))$ is minimax. On the other hand, the short exact sequence

$$0 \longrightarrow R \longrightarrow D_I(R) \longrightarrow H_I^1(R) \longrightarrow 0,$$

induces the long exact sequence

$$\begin{aligned} \ldots \longrightarrow \operatorname{Tor}_1^R(N,R) &\longrightarrow \operatorname{Tor}_1^R(N, D_I(R)) \longrightarrow \operatorname{Tor}_1^R(N, H_I^1(R)) \\ &\longrightarrow N \otimes_R R \longrightarrow N \otimes_R D_I(R) \longrightarrow N \otimes_R H_I^1(R) \longrightarrow 0. \end{aligned}$$

As $\operatorname{Tor}_i^R(N,R) = 0$, for each $i \in \mathbb{N}$, it follows the R-module $\operatorname{Tor}_i^R(N, D_I(R))$ is minimax, for all $i \in \mathbb{N}_0$ and

$$\operatorname{Supp} \operatorname{Tor}_i^R(N, D_I(R)) \subseteq \operatorname{Supp} \operatorname{Tor}_i^R(N, H_I^1(R)) \subseteq V(I), \text{ for all } i \in \mathbb{N}.$$

Since $N \otimes_R D_I(R)$ is minimax, it follows that $\operatorname{Ext}_R^j(R/I, N \otimes_R D_I(R))$ is minimax for all $j \in \mathbb{N}_0$.Thus the assertion holds for $t = 0$.

Suppose, inductively, that $t > 0$ and the result has been proved for all smaller values of t. Since $\operatorname{Ann}_R N \subseteq \operatorname{Ann}_R N/\Gamma_I(N)$, it follows that

$$\operatorname{ara}(I + \operatorname{Ann}_R N/\Gamma_I(N) / \operatorname{Ann}_R N/\Gamma_I(N)) \leq \operatorname{ara}(I + \operatorname{Ann}_R N / \operatorname{Ann}_R N).$$

On the other hand, the short exact sequence

$$0 \longrightarrow \Gamma_I(N) \longrightarrow N \longrightarrow N/\Gamma_I(N) \longrightarrow 0,$$

induces the long exact sequence

$$\begin{aligned} \cdots \longrightarrow &\operatorname{Tor}_1^R(\Gamma_I(N), D_I(R)) \longrightarrow \operatorname{Tor}_1^R(N, D_I(R)) \\ \longrightarrow &\operatorname{Tor}_1^R(N/\Gamma_I(N), D_I(R)) \longrightarrow \Gamma_I(N) \otimes_R D_I(R) \longrightarrow N \otimes_R D_I(R) \\ \longrightarrow &(N/\Gamma_I(N)) \otimes_R D_I(R) \longrightarrow 0. \end{aligned}$$

Consequently, applying the inductive assumption for the I-torsion finitely generated R-module $\Gamma_I(N)$ and replacing N by $N/\Gamma_I(N)$, we can make the additional assumption that $\Gamma_I(N)=0$. Then, by [11, Lemma 2.1.1] we have $I \nsubseteq \bigcup_{\mathfrak{p}\in \mathrm{Ass}_R N} \mathfrak{p}$. Next, let $v\in\mathbb{N}$ and

$$\Omega_v := \textstyle\bigcup_{i=1}^{v+1} \mathrm{Supp}\, \mathrm{Tor}_i^R(N, D_I(R)).$$

We claim that $\Omega_v \subseteq \mathrm{Supp} \bigoplus_{i=2}^{\infty} H_I^i(R)$. Assume that the opposite holds. Then there is an integer $1\leq l\leq v+1$ such that

$$\mathrm{Supp}\, \mathrm{Tor}_l^R(N, D_I(R)) \nsubseteq \mathrm{Supp} \textstyle\bigoplus_{i=2}^{\infty} H_I^i(R).$$

Choose an element $\mathfrak{p}\in \mathrm{Supp}\,\mathrm{Tor}_l^R(N, D_I(R))$ such that $\mathfrak{p}\notin \mathrm{Supp}\bigoplus_{i=2}^{\infty} H_I^i(R)$. Then, $H^i_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) \simeq (H_I^i(R))_{\mathfrak{p}} = 0$, for all integers $i\geq 2$. Thus, by [11, Lemma 6.3.1] the $IR_{\mathfrak{p}}$-transform functor $D_{IR_{\mathfrak{p}}}(-)$ is exact and by Lemma 3.5, $D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is a flat $R_{\mathfrak{p}}$-algebra. Hence

$$(\mathrm{Tor}_l^R(N, D_I(R)))_{\mathfrak{p}} \simeq \mathrm{Tor}_l^{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}))=0,$$

which is a contradiction.

By [11, Corollary 3.3.3], we know that $\mathrm{cd}(I,R)\leq \mathrm{ara}(I)<\infty$. Since by the assumption $H_I^i(R)\in \mathscr{C}^1(R,I)_{com}$, for each integer $i\geq 2$, it follows that the set

$$\Psi := \bigcup_{i=2}^{\infty} \mathrm{Ass}_R\, H_I^i(R),$$

is finite, because for all $i>\mathrm{ara}(I)$, $H_I^i(R)=0$ and also for all $i\geq 2$, $H_I^i(R)$ is I-cominimax and therefore $\mathrm{Ass}_R\, H_I^i(R)$ is finite. Set

$$\Delta := \{\mathfrak{p}\in\Omega_v \ : \ \dim R/\mathfrak{p}=1\}.$$

Then it is clear that $\Delta\subseteq \mathrm{Assh}_R \bigoplus_{i=2}^{\infty} H_I^i(R) \subseteq \Psi$ and Δ is a finite set. Furthermore, by using the assumption $H_I^i(R)\in\mathscr{C}^1(R,I)_{com}$, for each integer $i\geq 2$, and applying Lemma 2.1, it is easy to see that $q(IR_{\mathfrak{p}}, R_{\mathfrak{p}})\leq 1$, for all $\mathfrak{p}\in\Delta$ (for this we show that for all $i\geq 2$, and $\mathfrak{p}\in\Delta$ the $R_{\mathfrak{p}}-$module $(H_I^i(R))_{\mathfrak{p}}$ is Artinian. Let $H:=H_I^i(R)$ for all $i\geq 2$ and $E:=\mathrm{Hom}_R(R/I,H)$. Since H is I-cominimax, it follows that there exists a finitely generated submodule T of E such that E/T is Artinian. Now from the fact that $\dim R/\mathfrak{p}=1$, we conclude $T_{\mathfrak{p}}\simeq E_{\mathfrak{p}}$ and so the $R_{\mathfrak{p}}$-module $E_{\mathfrak{p}}$ is of finite length and $H_{\mathfrak{p}}$ is Artinian for all $i\geq 2$. This shows that $q(IR_{\mathfrak{p}}, R_{\mathfrak{p}})\leq 1$). Thus by Lemma 3.6, the $R_{\mathfrak{p}}$-module $\mathrm{Tor}_i^{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})) = (\mathrm{Tor}_i^R(N, D_I(R)))_{\mathfrak{p}}$, is Artinian and $IR_{\mathfrak{p}}$-cofinite, for all $\mathfrak{p}\in\Delta$ and all $i\in\mathbb{N}$. Assume that $\Delta=\{\mathfrak{p}_1,\ldots,\mathfrak{p}_n\}$ and set

$$\Lambda := \bigcup_{i=1}^{v+1}\bigcup_{j=1}^{n}\{\mathfrak{q}\in \operatorname{Spec} R \mid \mathfrak{q}R_{\mathfrak{p}_j}\in \operatorname{Att}_{R_{\mathfrak{p}_j}}(\operatorname{Tor}_i^R(N,D_I(R)))_{\mathfrak{p}_j}\}.$$

By Lemma 3.8 we have $V(IR_{\mathfrak{p}_j})\cap \operatorname{Att}_{R_{\mathfrak{p}_j}}(\operatorname{Tor}_i^R(N,D_I(R)))_{\mathfrak{p}_j}\subseteq V(\mathfrak{p}_jR_{\mathfrak{p}_j})$, for all $1\leq i\leq v+1$ and all $1\leq j\leq n$. Hence, $\Lambda\cap V(I)\subseteq\Delta$. Also, since for each $\mathfrak{q}\in\Lambda$ we have $\mathfrak{q}R_{\mathfrak{p}_j}\in \operatorname{Att}_{R_{\mathfrak{p}_j}}(\operatorname{Tor}_i^R(N,D_I(R)))_{\mathfrak{p}_j}$, for some integers $1\leq i\leq v+1$ and $1\leq j\leq n$, it follows that

$$(\operatorname{Ann}_R N)R_{\mathfrak{p}_j}\subseteq \operatorname{Ann}_{R_{\mathfrak{p}_j}}(\operatorname{Tor}_i^R(N,D_I(R)))_{\mathfrak{p}_j}\subseteq \mathfrak{q}R_{\mathfrak{p}_j},$$

which implies $\operatorname{Ann}_R N\subseteq\mathfrak{q}$. Therefore, $\Lambda\subseteq \operatorname{Supp} N$.

On the other hand, by the definition there exist elements $y_1,...,y_t\in I$, such that

$$\operatorname{Rad}(I+\operatorname{Ann}_R N/\operatorname{Ann}_R N)=\operatorname{Rad}((y_1,...,y_t)R+\operatorname{Ann}_R N/\operatorname{Ann}_R N).$$

By the *Prime Avoidance Theorem*,

$$I\nsubseteq\left(\bigcup_{\mathfrak{q}\in\Lambda\setminus V(I)}\mathfrak{q}\right)\bigcup\left(\bigcup_{\mathfrak{p}\in\operatorname{Ass}_R N}\mathfrak{p}\right),$$

which shows that

$$(y_1,...,y_t)R+\operatorname{Ann}_R N\nsubseteq\left(\bigcup_{\mathfrak{q}\in\Lambda\setminus V(I)}\mathfrak{q}\right)\bigcup\left(\bigcup_{\mathfrak{p}\in\operatorname{Ass}_R N}\mathfrak{p}\right).$$

But, $\operatorname{Ann}_R N\subseteq\left(\bigcap_{\mathfrak{q}\in\Lambda\setminus V(I)}\mathfrak{q}\right)\bigcap\left(\bigcap_{\mathfrak{p}\in\operatorname{Ass}_R N}\mathfrak{p}\right)$, and consequently

$$(y_1,...,y_t)R\nsubseteq\left(\bigcup_{\mathfrak{q}\in\Lambda\setminus V(I)}\mathfrak{q}\right)\bigcup\left(\bigcup_{\mathfrak{p}\in\operatorname{Ass}_R N}\mathfrak{p}\right).$$

Therefore, by [21, Exercise 16.8] there is $a\in(y_2,\ldots,y_t)R$ such that

$$y_1+a\notin\left(\bigcup_{\mathfrak{q}\in\Lambda\setminus V(I)}\mathfrak{q}\right)\bigcup\left(\bigcup_{\mathfrak{p}\in\operatorname{Ass}_R N}\mathfrak{p}\right).$$

Let $x:=y_1+a$. Then $x\in I$ and

$$\operatorname{Rad}(I+\operatorname{Ann}_R N/\operatorname{Ann}_R N)=\operatorname{Rad}((x,y_2,...,y_t)R+\operatorname{Ann}_R N/\operatorname{Ann}_R N).$$

Now it is easy to see that

$$\begin{aligned}&\operatorname{Rad}(I+\operatorname{Ann}_R N/xN/\operatorname{Ann}_R N/xN)\\&=\operatorname{Rad}((y_2,...,y_t)R+\operatorname{Ann}_R N/xN/\operatorname{Ann}_R N/xN),\end{aligned}$$

and hence $\operatorname{ara}(I+\operatorname{Ann}_R N/xN/\operatorname{Ann}_R N/xN)\leq t-1$. The short exact sequence

$$0\longrightarrow N\xrightarrow{x}N\longrightarrow N/xN\longrightarrow 0,$$

induces an exact sequence

$$\operatorname{Tor}_{i+1}^R(N,D_I(R))\xrightarrow{x}\operatorname{Tor}_{i+1}^R(N,D_I(R))\longrightarrow\operatorname{Tor}_{i+1}^R(N/xN,D_I(R))$$

$$\longrightarrow \operatorname{Tor}_i^R(N, D_I(R)) \xrightarrow{x} \operatorname{Tor}_i^R(N, D_I(R)),$$

for all $i \in \mathbb{N}_0$. Consequently, for each $0 \leq i \leq v$, we have the short exact sequence,

$$0 \longrightarrow U_{i+1} \longrightarrow \operatorname{Tor}_{i+1}^R(N/xN, D_I(R)) \longrightarrow (0 :_{\operatorname{Tor}_i^R(N,D_I(R))} x) \longrightarrow 0,$$

where $U_{i+1} := \operatorname{Tor}_{i+1}^R(N, D_I(R))/x \operatorname{Tor}_{i+1}^R(N, D_I(R))$.

By the inductive assumption, the R-modules $\operatorname{Tor}_{i+1}^R(N/xN, D_I(R))$ are I-cominimax, for all $i \in \mathbb{N}_0$. Also, by Lemma 3.7, obviously the $R_{\mathfrak{p}_j}$-module $(U_{i+1})_{\mathfrak{p}_j}$ is of finite length, for all integers $1 \leq j \leq n$ and $0 \leq i \leq v$, because if $qR_{\mathfrak{p}} \in \operatorname{Att} \operatorname{Tor}_{i+1}^R(N, D_I(R))_{\mathfrak{p}} \cap V(xR_{\mathfrak{p}})$ then $x \in q$, $q \in \Lambda$ and $I \subseteq q$. Also the $R_{\mathfrak{p}}$-module $\operatorname{Tor}_i^{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})) = (\operatorname{Tor}_i^R(N, D_I(R)))_{\mathfrak{p}}$, is Artinian and $IR_{\mathfrak{p}}$-cofinite. As $I \subseteq q$, so

$$\operatorname{Tor}_{i+1}^R(N, D_I(R))_{\mathfrak{p}}/qR_{\mathfrak{p}} \operatorname{Tor}_{i+1}^R(N, D_I(R))_{\mathfrak{p}}$$

is of finite length which shows that $qR_{\mathfrak{p}} \in \{\mathfrak{p}R_{\mathfrak{p}}\}$ and so there exists a finitely generated submodule $U_{i+1,j}$ of U_{i+1} such that $(U_{i+1})_{\mathfrak{p}_j} = (U_{i+1,j})_{\mathfrak{p}_j}$. Set $U'_{i+1} := U_{i+1,1} + \cdots + U_{i+1,n}$, for all $0 \leq i \leq v$. Then for each $0 \leq i \leq v$, U'_{i+1} is a finitely generated submodule of U_{i+1} such that

$$\operatorname{Supp}_R U_{i+1}/U'_{i+1} \subseteq \Omega_v \setminus \{\mathfrak{p}_1, ..., \mathfrak{p}_n\} \subseteq \operatorname{Max} R.$$

For each $0 \leq i \leq v$, set $W_{i+1} := \operatorname{Tor}_{i+1}^R(N/xN, D_I(R))$. Then, for each $0 \leq i \leq v$, there is an exact sequence

$$0 \longrightarrow U_{i+1}/U'_{i+1} \longrightarrow W_{i+1}/W'_{i+1} \longrightarrow (0 :_{\operatorname{Tor}_i^R(N,D_I(R))} x) \longrightarrow 0,$$

for some finitely generated submodule W'_{i+1} of W_{i+1}.

We will show that U_{i+1} is a minimax R-module, for all $0 \leq i \leq v$. To do this, we notice that for each $0 \leq i \leq v$, W_{i+1}/W'_{i+1} is I-cominimax and hence $\operatorname{Hom}_R(R/I, U_{i+1}/U'_{i+1})$ is a minimax R-module. But

$$\operatorname{Supp} U_{i+1}/U'_{i+1} \subseteq \operatorname{Max} R$$

and U_{i+1}/U'_{i+1} is I-torsion, and therefore the R-module U_{i+1}/U'_{i+1} is Artinian. That is U_{i+1} is a minimax R-module. Consequently, for each $0 \leq i \leq v$, the R-module $(0 :_{\operatorname{Tor}_i^R(N,D_I(R))} x)$ is also I-cominimax. Moreover, from the exact sequence

$$0 \longrightarrow (N \otimes_R D_I(R))/x(N \otimes_R D_I(R)) \longrightarrow (N/xN) \otimes D_I(R) \longrightarrow 0,$$

and inductive assumption, it follows that the following R-module

$$\operatorname{Ext}_R^j(R/I, (N \otimes_R D_I(R))/x(N \otimes_R D_I(R)))$$

is minimax, for all $j \in \mathbb{N}_0$. Now, since $x \in I$ and the R-modules

$$(0 :_{\operatorname{Tor}_i^R(N,D_I(R))} x), \ \operatorname{Tor}_i^R(N, D_I(R))/x \operatorname{Tor}_i^R(N, D_I(R))$$

are I-cominimax, for all $1 \leq i \leq v$, from [23, Corollary 3.4] it follows that $\operatorname{Tor}_i^R(N, D_I(R))$ is I-cominimax, for all $1 \leq i \leq v$. Furthermore, since the R-module $(0 :_{N \otimes_R D_I(R)} x)$ is I-cominimax and the R-module $\operatorname{Ext}_R^j(R/I, (N \otimes_R D_I(R))/x(N \otimes_R D_I(R)))$ is minimax, for all $j \in \mathbb{N}_0$, by applying the method which is used already in the proof of [23, Corollary 3.4], it can be seen that the R-module $\operatorname{Ext}_R^j(R/I, N \otimes_R D_I(R))$ is finitely generated, for all $j \in \mathbb{N}_0$.

Finally, as $v \in \mathbb{N}$ is an arbitrary integer, it is concluded that the R-module $\operatorname{Tor}_i^R(N, D_I(R))$ is I-cominimax, for all $i \in \mathbb{N}$. This completes the inductive step. □

Now, we are ready to establish the second main result of this paper.

Theorem 3.10. Let I be an ideal of a ring R such that $H_I^i(R) \in \mathscr{C}^1(R, I)_{com}$ for each integer $i \geq 2$. Then $I \in \mathscr{I}'(R)$.

Proof. Let $\overline{R} := R/\Gamma_I(R)$ and $\overline{I} = I\overline{R}$. We know that if $I \in \mathscr{I}'(R)$, then $\overline{I} \in \mathscr{I}'(\overline{R})$. Conversely, if $\overline{I} \in \mathscr{I}'(\overline{R})$ then for each finitely generated R-module M we have $JM \subsetneq \Gamma_I(M)$, where $J := \Gamma_I(R)$ and hence for each $i \in \mathbb{N}$ we have $H_I^i(M) \simeq H_I^i(M/JM) \simeq H_{\overline{I}}^i(M/JM)$. Thus, from Lemma 2.3 we get $I \in \mathscr{I}'(R)$. On the other hand, for each $i \geq 1$ we have

$$H_I^i(R) \simeq H_I^i(\overline{R}) \simeq H_{\overline{I}}^i(\overline{R}).$$

Hence, by using the Lemma 2.3 we can see that $H_I^i(R) \in \mathscr{C}^1(R, I)_{com}$, for all $i \geq 2$, if and only if $H_{\overline{R}}^i(\overline{R}) \in \mathscr{C}^1(\overline{R}, \overline{I})_{com}$, for all $i \geq 2$. So, by passing to the quotient ring $\overline{R}$, we can make the additional assumption that $\Gamma_I(R) = 0$.

Now, let N be a finitely generated R-module and set $W := N \otimes_R D_I(R)$. From the assumption, $H_I^i(R) \in \mathscr{C}^1(R, I)_{com}$ for all $i \geq 2$, by Proposition 3.9 it follows that the R-module $\operatorname{Ext}_R^j(R/I, W)$ is minimax, for all $j \in \mathbb{N}_0$.

By the assumption $\mathrm{c}^1(I, R) \leq 1$ and Lemma 2.2, for each finitely generated R-module U we have $\mathrm{c}^1(I, U) \leq \mathrm{c}^1(I, R) \leq 1$. Since by [11, Theorem 3.4.10], for each $i \in \mathbb{N}_0$, the local cohomology functor $H_I^i(-)$ commutes with direct limits, and W can be viewed as the direct limit of its finitely generated submodules, we have $\mathrm{c}^1(I, W) \leq 1$ and $\dim H_I^i(W) \leq 1$, for all integers $i \geq 2$.

Let $\mathfrak{p}$ be a prime ideal of R with $\dim R/\mathfrak{p} \geq 2$. As by the assumption the R-module $H_I^i(R)$ is in $\mathscr{C}^1(R, I)$, for all $i \geq 2$, we see that

$$H_{IR_\mathfrak{p}}^i(R_\mathfrak{p}) \simeq (H_I^i(R))_\mathfrak{p} = 0, \ \text{for all } i \geq 2.$$

Thus, by [11, Lemma 6.3.1] the $IR_{\mathfrak{p}}$-transform functor $D_{IR_{\mathfrak{p}}}(-)$ is exact. Hence, by applying [11, Exercise 6.1.8] we conclude that

$$W_{\mathfrak{p}} = (D_I(R) \otimes_R N)_{\mathfrak{p}} \simeq D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \simeq D_{IR_{\mathfrak{p}}}(N_{\mathfrak{p}}).$$

Thus, by using [11, Corollary 2.2.8] we achieve the isomorphisms

$$(H^i_I(W))_{\mathfrak{p}} \simeq H^i_{IR_{\mathfrak{p}}}(W_{\mathfrak{p}}) \simeq H^i_{IR_{\mathfrak{p}}}(D_{IR_{\mathfrak{p}}}(N_{\mathfrak{p}})) = 0, \text{ for } i = 0, 1.$$

Therefore, $\dim H^i_I(W) \leq 1$, for $i = 0, 1$. Consequently, for all $i \in \mathbb{N}_0$, we have $\dim H^i_I(W) \leq 1$ and the R-module $\mathrm{Ext}^i_R(R/I, W)$ is minimax. Now, by induction on n we prove that the R-module $H^n_I(W)$ is I-cominimax for all $n \in \mathbb{N}_0$.

For $n = 0$, by Lemma 3.2, the R-modules $\mathrm{Hom}_R(R/I, \Gamma_I(W))$ and $\mathrm{Ext}^1_R(R/I, \Gamma_I(W))$ are minimax. Therefore, by Lemma 3.3, the R-module $\Gamma_I(W)$ is I-cominimax.

Suppose, inductively, that $n > 0$ and the result has been proved for smaller values of n. Then by Lemma 3.2, the R-modules $\mathrm{Hom}_R(R/I, H^n_I(W))$ and $\mathrm{Ext}^1_R(R/I, H^n_I(W))$ are minimax and so by Lemma 4.3 the R-module $H^n_I(W)$ is I-cominimax. This completes the inductive step.

According to [11, Remark 2.2.7], there is an exact sequence

$$0 \longrightarrow R \longrightarrow D_I(R) \longrightarrow H^1_I(R) \longrightarrow 0,$$

which induces the exact sequence

$$\mathrm{Tor}^R_1(N, H^1_I(R)) \xrightarrow{f} N \xrightarrow{g} N \otimes_R D_I(R) \longrightarrow N \otimes_R H^1_I(R) \longrightarrow 0,$$

whence, we get the following exact sequence

$$0 \longrightarrow \operatorname{im} g \longrightarrow N \otimes_R D_I(R) \longrightarrow N \otimes_R H^1_I(R) \longrightarrow 0. \quad (4.10.1)$$

Since

$$\operatorname{Supp} \operatorname{im} f \subseteq \operatorname{Supp} \mathrm{Tor}^R_1(N, H^1_I(R)) \subseteq \operatorname{Supp} H^1_I(R) \subseteq V(I),$$

it follows that $\ker g = \operatorname{im} f \subseteq \Gamma_I(N)$ and hence

$$\begin{aligned} \operatorname{im} g/\Gamma_I(\operatorname{im} g) &\simeq \big(N/\ker g\big)/\Gamma_I(N/\ker g) \\ &= \big(N/\ker g\big)/\big(\Gamma_I(N)/\ker g\big) \\ &\simeq N/\Gamma_I(N). \end{aligned}$$

Thus,

$$H^i_I(\operatorname{im} g) \simeq H^i_I(\operatorname{im} g/\Gamma_I(\operatorname{im} g)) \simeq H^i_I(N/\Gamma_I(N)) \simeq H^i_I(N), \text{ for all } i \in \mathbb{N}.$$

Moreover, for each integer $i \geq 2$, from the exact sequence (4.10.1) we get an exact sequence

$$H_I^{i-1}(N \otimes_R H_I^1(R)) \longrightarrow H_I^i(\operatorname{im} g) \longrightarrow H_I^i(W) \longrightarrow H_I^i(N \otimes_R H_I^1(R)),$$

which yields the isomorphism $H_I^i(\operatorname{im} g) \simeq H_I^i(W)$, for each $i \geq 2$. (Note that for each $j \in \mathbb{N}$ we have $H_I^j(N \otimes_R H_I^1(R)) = 0$, because the R-module $N \otimes_R H_I^1(R)$ is I-torsion). So, we have $H_I^i(W) \simeq H_I^i(\operatorname{im} g) \simeq H_I^i(N)$, for all $i \geq 2$. Now, we are in a position to deduce that for all $i \geq 2$, the R-module $H_I^i(N)$ is I-cominimax. Because the R-module $H_I^0(N)$ is finitely generated with support in $V(I)$, it follows that $H_I^0(N)$ is I-cominimax. Therefore, for each integer $i \neq 1$ the R-module $H_I^i(N)$ is I-cominimax. Hence, by [23, Proposition 3.11] the R-module $H_I^1(N)$ is I-cominimax too. This means that $I \in \mathscr{I}'(R)$. □

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