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DOMINATION NUMBER AND IDENTIFYING CODE NUMBER OF THE SUBDIVISION GRAPHS

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Abstract. Let $G = (V, E)$ be a simple graph. A set C of vertices of G is an identifying code of G if for every two vertices x and y the sets $N_G[x] \cap C$ and $N_G[y] \cap C$ are non-empty and different. Given a graph G, the smallest size of an identifying code of G is called the identifying code number of G and is denoted by $\gamma^{ID}(G)$. In this paper, we prove that the identifying code number of the subdivision of a graph G of order n is at most n. Also, we prove that the identifying code number of the subdivision of graphs K_n, $K_{r,s}$ and $C_P(s)$ are n, $r+s$ and $2s$, respectively. Finally, we conjecture that for every graph G of order n the identifying code number of the subdivision of G is n.

1. Introduction

In this paper, all graphs are assumed to be finite, simple and undirected. We will often use the notation $G = (V, E)$ to denote the graph with a non-empty vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of a graph is the number of vertices in graph and the *size* of a graph is the number of edges in graph. An edge of G with endpoints u and v is denoted by $\{u, v\}$ (which sometimes and for convenient we write it as uv), and we write $u \sim v$ to indicate that two vertices u and v are adjacent in G. For every vertex $x \in V(G)$, the *open neighborhood* of vertex x is denoted by $N_G(x)$ and defined as $N_G(x) = \{y \in V(G) \ : \ x \sim y\}$. Also the *close neighborhood* of vertex $x \in V(G)$, is $N_G[x] = N_G(x) \cup \{x\}$. The *degree* of a vertex $x \in V(G)$ is $\deg_G(x) = \big|N_G(x)\big|$. The *maximum degree* and *minimum degree* of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A *complete bipartite* graph is a special kind of bipartite graph in which every vertex of the first part is adjacent to each vertex of the second part. The complete bipartite graph of order $r+s$ is denoted by $K_{r,s}$. For any $X \subseteq V(G)$, the *induced subgraph* on X, is denoted by $G[X]$. The *Cocktail party* graph of order $2s$ denoted by $C_P(s)$ is obtained by removing s disjoint edges from the complete graph K_{2s}. The *subdivision graph* of G is the graph obtained by inserting an additional vertex

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into each edge of G, denoted by $S(G)$. Sometimes the new vertex inserted into the edge $\{v_i, v_j\}$ is denoted by v_{ij}. A subset D of the vertices of G is a *dominating set* of G if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D. The *domination number* of G, which is denoted by $\gamma(G)$, is the minimum size of a dominating set of G. A set C is called a *separating set* of G, if for each pair $u,\ v$ of vertices of G, $N_G[u] \cap C \neq N_G[v] \cap C$ (equivalently, $(N_G[u] \triangle N_G[v]) \cap C \neq \emptyset$). If a dominating set C in G is a separating set of G, then we say that C is an *identifying code* of G and if G has an identifying code, then we say that G is an *identifiable graph*. Given a graph G, the smallest size of an identifying code of G is called the *identifying code number* of G and denoted by $\gamma^{ID}(G)$. If for two distinct vertices x and y, $N_G[x] = N_G[y]$, then they are called *twins*. It is noteworthy that a graph G is identifiable if and only if G is twin-free. In recent years much attention drawn to the domination theory which is a very interesting branch of graph theory. The concept of domination expanded to other parameters of domination such as 2-rainbow domination, signed domination, Roman domination, total Roman domination, and identifying code. For more details, we refer the reader to [1, 4, 6, 15, 16, 17, 18, 21].

The identifying code concept was introduced by Karpovsky et al. [14] in 1998. Later, several various families of graphs have been studied such as cycles and paths [5, 10], trees [3], triangular and square grids [13] and, triangle-free graphs [8]. Also identifying codes have found applications in various fields. These applications include sensor network monitoring [7], identifying codes in random networks [9], communication networks [20] and, the structural analysis of RNA proteins [11].

This paper deals with the study of the subdivision of some graphs. In Section 2, we show that $\gamma(S(G)) \leq |V(G)| - 1$. Specially, we prove that $\gamma(S(K_n)) = n - 1, \gamma(S(C_P(s))) = 2s - 1$, and $\gamma(S(K_{r,s})) = r + s - 1$. In Section 3, we show that the identifying code number of the subdivision of a graph G of order n is at most n. Also, we prove that the identifying code number of the subdivision graphs $S(K_n)$, $S(K_{1,n})$, $S(K_{r,s})$ and $S(C_P(s))$ are $n, n+1$, $r+s$ and $2s$, respectively. According to this facts, we conjecture that for every graph G of order n the identifying code number of the subdivision of G is n.

2. Domination number of $S(G)$

In this section, the domination number of $S(G)$ for some graphs is investigated.

Lemma 2.1. *Let G be a graph of order $n \geq 2$ which contains the path P_ℓ as an induced subgraph. Then $\gamma(S(G)) \leq n - \left\lceil \frac{\ell-1}{3} \right\rceil$.*

Proof. Let

$$V(P_\ell) = \{v_1, v_2, \ldots, v_\ell\},\ E(P_\ell) = \{e_i = \{v_i, v_{i+1}\}\ :\ 1 \leq i \leq \ell - 1\}$$

and $N_{S(G)}(v_{ii+1}) = \{v_i, v_{i+1}\}$ for $1 \leq i \leq \ell - 1$ where $v_{ii+1} \in V(S(G))$ is the new vertex of degree two inserted on the edge e_i.
If $\ell = 3k$, then

$$D = (V(G) \setminus V(P_\ell)) \cup \left\{ v_i \in V(P_\ell)\ :\ i = 3t,\ 1 \leq t \leq \left\lceil \frac{\ell-1}{3} \right\rceil \right\} \cup \left\{ v_{ii+1}\ :\ i = 3t+1,\ 0 \leq t \leq \left\lfloor \frac{\ell-1}{3} \right\rfloor \right\}$$

is a dominating set for $S(G)$. Thus $\gamma(S(G)) \leq |D| = n - \frac{\ell}{3}$.
If $\ell = 3k+1$, then

$$D = (V(G) \setminus V(P_\ell)) \cup \left\{ v_i \in V(P_\ell)\ :\ i = 3t,\ 1 \leq t \leq \left\lceil \frac{\ell-1}{3} \right\rceil \right\} \cup \left\{ v_{ii+1}\ :\ i = 3t+1,\ 0 \leq t \leq \left\lfloor \frac{\ell-4}{3} \right\rfloor \right\} \cup \{v_\ell\}$$

is a dominating set for $S(G)$. Thus $\gamma(S(G)) \leq |D| = n - \frac{\ell-1}{3}$.
If $\ell = 3k+2$, then

$$D = (V(G) \setminus V(P_\ell)) \cup \left\{ v_i \in V(P_\ell)\ :\ i = 3t,\ 1 \leq t \leq \left\lfloor \frac{\ell-1}{3} \right\rfloor \right\} \cup \left\{ v_{ii+1}\ :\ i = 3t+1,\ 0 \leq t \leq \left\lfloor \frac{\ell-1}{3} \right\rfloor \right\}$$

is a dominating set for $S(G)$. Thus $\gamma(S(G)) \leq |D| = n - \frac{\ell+1}{3}$. □

Theorem 2.2. *If $n \geq 2$ is a positive integer, then $\gamma(S(K_n)) = n - 1$.*

Proof. Suppose $n \in \{2, 3\}$. It is easy to see that $\gamma(S(K_2)) = 1$ and $\gamma(S(K_3)) = 2$. Let $n \geq 4$ and $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and

$$V(S(K_n)) = V(K_n) \cup \{v_{ij}\ :\ v_iv_j \in E(K_n)\}.$$

Note that $N_{S(K_n)}(v_i) = \{v_{ij} \ : \ v_j \in N_{K_n}(v_i)\}$ and $N_{S(K_n)}(v_{ij}) = \{v_i, v_j\}$. By Lemma 2.1, $\gamma(S(K_n)) \leq n-1$. Now let D be a dominating set of $S(K_n)$ with minimum cardinality. Without loss of generality, we assume that $v_1 \notin D$. Since D is a dominating set for $S(K_n)$, so v_1 is dominated by a vertex in D. Hence, there is $2 \leq j \leq n$ such that $v_{1j} \in D$. Without loss of generality, Suppose that $v_{12} \in D$. Since D is a dominating set of $S(G)$, for every $3 \leq j \leq n$ we have $v_{1j} \in D$ or $v_j \in D$. So, $|D| \geq n-1$. Therefore, $\gamma(S(K_n)) = n-1$. □

Theorem 2.3. *Let $s \geq 3$ be a positive integer. Then $\gamma(S(C_P(s))) = 2s-1$.*

Proof. Suppose that $V(C_P(s)) = \{v_1, v_2, \ldots, v_{2s}\}$ in which

$$\{v_1v_{s+1}, v_2v_{s+2}, ..., v_sv_{2s}\} \cap E(C_P(s)) = \emptyset$$

and

$$V(S(C_P(s))) = V(C_P(s)) \cup \{v_{ij} \ : \ v_iv_j \in E(C_P(s))\}.$$

Note that $N_{S(C_P(s))}(v_i) = \{v_{ij} \ : \ v_j \in N_{C_P(s)}(v_i)\}$ and $N_{S(C_P(s))}(v_{ij}) = \{v_i, v_j\}$. By Lemma 2.1, $\gamma(S(C_P(s)) \leq 2s-1$.

Now let D be a dominating set of $S(C_P(s))$ with minimum cardinality. Since

$$|D| = \gamma(S(C_P(s))) \leq 2s-1,$$

without loss of generality, we can assume that $v_1 \notin D$. Since D is a dominating set for $S(C_P(s))$, v_1 must be dominated by a vertex in D. Thus, there exists $j \in \{2, 3, ..., 2s\} \setminus \{s+1\}$ such that $v_{1j} \in D$. Without loss of generality, suppose that $v_{12} \in D$. Simlarly, for each $j \in \{3, ..., 2s\} \setminus \{s+1\}$ the vertex v_{1j} must be dominated by D and hence, $v_{1j} \in D$ or $v_j \in D$. Therefore, $|D| \geq 2s-2$.

If $v_{s+1} \in D$, then $|D| \geq 2s-1$ and the proof is complete. Otherwise, and in order to dominate v_{s+1}, there exist $j \in \{1, 2, \ldots, 2s\} \setminus \{1, s+1\}$ such that $v_{s+1\,j} \in D$. However, $\gamma(S(C_P(s))) \geq 2s-1$. Therefore, $\gamma(S(C_P(s))) = 2s-1$. □

Theorem 2.4. *If r, s are two integers such that $r \geq 2$ and $s \geq 2$, then $\gamma(S(K_{r,s})) = r+s-1$.*

Proof. Assume that $V(K_{r,s}) = \{x_1, \ldots, x_r\} \cup \{y_1, \ldots, y_s\}$ and

$$V(S(K_{r,s})) = V(K_{r,s}) \cup \{z_{ij} \ : \ 1 \leq i \leq r, 1 \leq j \leq s\}$$

in which z_{ij} is the new vertex inserted on the edge x_iy_j to obtain the subdivision graph $S(K_{r,s})$ from $K_{r,s}$. By Lemma 2.1,

$$\gamma(S(K_{r,s})) \leq r+s-1.$$

Now let T be a dominating set of minimum cardinality in $S(K_{r,s})$. If $\{x_1,\ldots,x_r,y_1,\ldots,y_s\} \subseteq T$, then $\gamma(S(K_{r,s})) = |T| \geq r+s$, which is a contradiction. Therefore, there exists at least one vertex in $\{x_1,\ldots,x_r,y_1,\ldots,y_s\}$ which is not in T. Wihtout loss of generality, (and by renaming the vertices if it is necessary) assume that

$$\{x_1,\ldots,x_r,y_1,\ldots,y_s\} \setminus T = \{x_1,x_2,\ldots,x_{t_1},y_1,y_2,\ldots,y_{t_2}\}.$$

Thus, $\{x_{t_1+1},\ldots,x_r,y_{t_2+1},\ldots,y_s\} \subseteq T$ and hence $|T| \geq (r-t_1)+(s-t_2)$. Since T is a dominating set, each vertex in $\{x_1,\ldots,x_{t_1},y_1,\ldots,y_{t_2}\}$ should be dominated by a vertex in T, and by considering the structure of $S(K_{r,s})$, these dominating vertices must be in the set $\{z_{ij} \ : \ 1 \leq i \leq r, 1 \leq j \leq s\}$. Also, since t_1t_2 vertices in $\{z_{ij} \mid 1 \leq i \leq t_1, 1 \leq j \leq t_2\}$ must be dominated by T, we should have $\{z_{ij} \ : \ 1 \leq i \leq t_1, 1 \leq j \leq t_2\} \subseteq T$. Therefore, $|T| \geq (r-t_1)+(s-t_2)+t_1t_2$. Note that

$$\begin{aligned} |T| &\geq (r-t_1)+(s-t_2)+t_1t_2 \\ &= (r+s)+(t_1-1)(t_2-1)-1. \end{aligned}$$

If we have $t_2 = 0$, then $\{y_1,y_2,\ldots,y_s\} \subseteq T$ and $\{x_1,x_2,\ldots,x_{t_1}\} \cap T = \emptyset$. Since T is a dominating set, there exist vertices

$$\{z_{1j_1},z_{2j_2},\ldots,z_{t_1j_{t_1}}\} \subseteq T$$

which dominate vertices in $\{x_1,x_2,\ldots,x_{t_1}\}$. This implies that

$$|T| \geq (r-t_1)+s+t_1 = r+s,$$

which is a contradiction, because we show that $\gamma(S(K_{r,s})) \leq r+s-1$. Thus, $t_2 \geq 1$ and similarly we can show that $t_1 \geq 1$. Therefore,

$$\gamma(S(K_{r,s})) = |T| \geq (r+s)+(t_1-1)(t_2-1)-1$$

in which $t_1 \geq 1$ and $t_2 \geq 1$. The minimum value of the statement $(r+s)+(t_1-1)(t_2-1)-1$ with the conditions $t_1 \geq 1$, $t_2 \geq 1$ occurs just when $t_1 = 1$ or $t_2 = 1$ and this leads to the value $r+s-1$. Without loss of generality, assume that $t_2 = 1$ (the case $t_1 = 1$ will be similarly done). Since vertices $x_1,x_2,\ldots,x_{t_1}$ are dominated by T, and by considering the structure of $S(K_{r,s})$, there exist t_1 vertices of the from z_{ij} in T. If none of these vertices is adjacent to y_1, then for dominating y_1 an extra vertex in $\{z_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq s\}$ is required which implies that $|T| \geq r+s$, a contradiction. Hence, we can assume that $\{x_1,x_2,\ldots,x_{t_1}\}$ are dominated by $\{z_{1j_1},\ldots,z_{t_1j_{t_1}}\}$ and $z_{1j_1} = z_{11}$ (i.e. y_1 is dominated by z_{11}). Since the set $\{x_{t_1+1},\ldots,x_r,y_2,\ldots,y_s,z_{1j_1},z_{t_1j_{t_1}}\}$ is a subset of T and it is a dominating set of cardinality $r+s-1$, and by the minimality of T, we have $\gamma(S(K_{r,s})) = |T| = r+s-1$. □

3. Identifying code number of $S(G)$

In this section, we determine the identifying code number of the subdivision graphs $S(K_n)$, $S(C_P(s))$, $S(K_{1,n})$ and, $S(K_{r,s})$ with $r \geq 2$ and $s \geq 2$. At first, consider the following results.

Theorem 3.1. [12] *Let $n \geq 2$ be a positive integer. Then $\gamma^{ID}(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.*

Theorem 3.2. ([5], [10]) *Let $n \geq 4$ be a positive integer. Then*

$$\gamma^{ID}(C_n) = \begin{cases} 3 & \text{if } n = 4, 5 \\ \frac{n}{2} & \text{if } n \geq 6 \text{ is even} \\ \frac{n+3}{2} & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

It is clear that for each $n \geq 3$, $S(P_n) \cong P_{2n-1}$ and $S(C_n) \cong C_{2n}$. Thus, by Theorems 3.1 and 3.2, $\gamma^{ID}(S(P_n)) = n$ and $\gamma^{ID}(S(C_n)) = n$. In the following two results, we show that $|V(G)|$ is generally an upper bound for $\gamma^{ID}(S(G))$.

Proposition 3.3. *Let G be a graph of order $n \geq 2$. Then $S(G)$ is an identifiable graph.*

Proof. Let a and b be two arbitrary and distinct vertices in $S(G)$. If a is not adjacent to b in $S(G)$, then $N_{S(G)}[a] \neq N_{S(G)}[b]$. Let a be adjacent to b in $S(G)$. Since $S(G)$ is a bipartite graph and $V(G)$ is a partite set in it, without loss of generality, we can assume that $a \in V(G)$ and hence, there exists $c \in V(G) \setminus \{a\}$ such that $N_{S(G)}(b) = \{a, c\}$. Since $c \notin N_{S(G)}[a]$, we have $N_{S(G)}[a] \neq N_{S(G)}[b]$. Therefore $S(G)$ is twin-free and hence, an identifiable graph.

□

Theorem 3.4. *If G is a graph of order $n \geq 2$, then $\gamma^{ID}(S(G)) \leq n$.*

Proof. Let $C = V(G)$. It is clear that C is a dominating set for $S(G)$. We claim that C is an identifying code for $S(G)$. For this purpose, let x and y be two distince vertices in $V(S(G))$. We consider the following cases:

Case 1) $\{x, y\} \subseteq V(G)$: Since $S(G)$ is a bipartite graph with $V(G)$ as a partite set, x is not adjacent to y in $S(G)$. Thus $x \in N_{S(G)}[x] \cap C$ and $x \notin N_{S(G)}[y] \cap C$. Hence, $N_{S(G)}[x] \cap C \neq N_{S(G)}[y] \cap C$.

Case 2) $\{x, y\} \cap V(G) = \emptyset$: In this case, there exists a vertex $a \in V(G)$ such that $x \sim a$ and $a \nsim y$. Thus $a \in N_{S(G)}[x] \cap C$ and $a \notin N_{S(G)}[y] \cap C$ which implies that $N_{S(G)}[x] \cap C \neq N_{S(G)}[y] \cap C$.

Case 3) $x \notin V(G)$ and $y \in V(G)$: Assume that $N_{S(G)}(x) = \{a, b\}$ in which $ab \in E(G)$. If $y \notin \{a, b\}$, then $a \in N_{S(G)}[x] \cap C$ and $a \notin N_{S(G)}[y] \cap C$. If $y = a$, then $b \in N_{S(G)}[x] \cap C$ and $b \notin N_{S(G)}[y] \cap C$. However

$$N_{S(G)}[x] \cap C \neq N_{S(G)}[y] \cap C.$$

These facts show that C is an identifying code for $S(G)$ and hence,

$$\gamma^{ID}(S(G)) \leq |C| = n.$$

□

Now we use the following result to show that $\gamma^{ID}(S(C_P(s))) = 2s$ and $\gamma^{ID}(S(K_n)) = n$.

Theorem 3.5. [19] *If G is a graph of order n and maximum degree Δ, then $\gamma^{ID}(G) \geq \frac{2n}{\Delta+2}$.*

Theorem 3.6. *For each $n \geq 2$, $\gamma^{ID}(S(K_n)) = n$.*

Proof. By Theorem 3.4, we have $\gamma^{ID}(S(K_n)) \leq n$. Since $|V(S(K_n))| = \frac{n(n+1)}{2}$, Theorem 3.5 implies that $\gamma^{ID}(S(K_n)) \geq n$. Thus, $\gamma^{ID}(S(K_n)) = n$. □

Theorem 3.7. *Let $s \geq 2$ be a positive integer. Then $\gamma^{ID}(S(C_P(s))) = 2s$.*

Proof. It is clear that $S(C_P(s))$ is of order $2s^2$. By Theorem 3.4,

$$\gamma^{ID}(S(C_P(s))) \leq 2s.$$

By Theorem 3.5, $\gamma^{ID}(S(C_P(s))) \geq \frac{2(2s^2)}{(2s-2)+2} = 2s$. Therefore,

$$\gamma^{ID}(S(C_P(s))) = 2s.$$

□

In the following, among some other useful results, we show that for $n \geq 4$, $V(K_n)$ is the unique optimum identifying code in $S(K_n)$.

Theorem 3.8. *Let G be an identifiable graph of order n and C be an identifying code of $S(G)$. Then we have*

$$|C| \geq \max\left\{\deg_G(x) + \left\lceil \frac{\deg_G(x)}{2} \right\rceil \ : \ x \in V(G) \setminus C \right\}.$$

Specially, if $\gamma^{ID}(S(G)) < n$, then $\gamma^{ID}(S(G)) \geq \delta(G) + \left\lceil \frac{\delta(G)}{2} \right\rceil$.

Proof. Let x be a vertex in $V(G) \setminus C$ and assume that $\deg_G(x) = d$, $N_G(x) = \{y_1, y_2, \ldots, y_d\}$ and $N_{S(G)}(x) = \{e_1, e_2, \ldots, e_d\}$. Since C is a dominating set of $S(G)$ and $x \notin C$, there exists $1 \leq i \leq d$ such that $e_i \in C$. Also, each vertex in $\{e_1, e_2, ..., e_d\} \setminus \{e_i\}$ must be dominated by C and hence, for each $j \in \{1, 2, \ldots, d\} \setminus \{i\}$ we have $C \cap \{e_i, y_i\} \neq \emptyset$. Therefore, $|C \cap \{y_1, \ldots, y_d, e_1, \ldots, e_d\}| \geq d$. Note that for each $1 \leq i \leq d$, two vertices e_i and y_i are adjacent in $S(G)$ and hence, $e_i \in N_{S(G)}[y_i]$ and $y_i \in N_{S(G)}[e_i]$. Now since C is an identifying code of $S(G)$, for each $i \in \{1, 2, \ldots, d\}$ we should have $N_{S(G)}[e_i] \cap C \neq N_{S(G)}[y_i] \cap C$. Therefore, for each $i \in \{1, 2, \ldots, d\}$ there exists a vertex $z_i \notin \{e_1, e_2, ..., e_d\}$ such that $z_i \in C \cap N_{S(G)}(y_i)$. Note that when $y_i y_j$ is an edge in G and z is the (new) vertex of degree two in $S(G)$ with $N_{S(G)}(z) = \{y_i, y_j\}$, then z can play both of the roles of z_i and z_j in our above statement and hence, we may have $z_i = z_j$. Thus, by pairing z_i's in the worst case, we have $|\{z_1, z_2, \ldots, z_d\}| \geq \left\lceil \frac{\deg_G(x)}{2} \right\rceil$. This implies that

$$\begin{aligned} |C| &\geq |C \cap \{y_1, \ldots, y_d, e_1, \ldots, e_d\}| + |C \cap \{z_1, z_2, ..., z_d\}| \\ &\geq d + \left\lceil \frac{d}{2} \right\rceil, \end{aligned}$$

which completes the proof. Note that when $\gamma^{ID}(S(G)) < n$, there exists at least one vertex $x \in V(G) \setminus C$ and we know that $\deg_G(x) \geq \delta(G)$. □

Corollary 3.9. *Let G be a triangle-free identifiable graph of order n and C be an identifying code of $S(G)$. Then,*

$$|C| \geq 2 \max \{\deg_G(x) \ : \ x \in V(G) \setminus C \ \}.$$

Specially, if $\gamma^{ID}(S(G)) < n$, then $\gamma^{ID}(S(G)) \geq 2\,\delta(G)$.

Proof. The proof is a direct consequence of the proof of Theorem 3.8 by considering the fact $y_i y_j \notin E(G)$ when G is triangle-free, which itself implies that $z_i \neq z_j$ for each $i \neq j$. □

Corollary 3.10. *Let $n \geq 2$ be an integer and C be an identifying code of $S(K_n)$. If $V(K_n) \nsubseteq C$, then $|C| \geq (n-1) + \left\lceil \frac{(n-1)}{2} \right\rceil$.*

Proof. By Theorem 3.8, the proof is straightforward. □

Note that $\gamma^{ID}(S(K_2)) = \gamma^{ID}(P_3) = 2$ and $\gamma^{ID}(S(K_3)) = \gamma^{ID}(C_6) = 3$. In addition, P_3 has a unique identifying code of size two, but C_6 has at least two different identifying codes of size three.

Corollary 3.11. *Let $n \geq 4$ be an integer and C be an identifying code of $S(K_n)$ with minimum size (i.e., $|C| = \gamma^{ID}(S(K_n))$). Then, $C = V(K_n)$.*

Proof. By Theorem 3.4, we have $|C| = \gamma^{ID}(S(K_n)) \leq n$. This fact and Corollary 3.10 implies that $V(K_n) \setminus C = \emptyset$. Thus, $V(K_n) \subseteq C$ and hence, $|C| \geq n$. This facts imply that $\gamma^{ID}(S(K_n)) = |C| = n$ and $C = V(G)$. □

Finally, we determine the identifying code number of complete bipartite graphs and according to obtained results, we propose a conjecture.

Theorem 3.12. [2] *Let G be a graph of order n. Then $\gamma(G) \leq \gamma^{ID}(G)$.*

Theorem 3.13. *Let $n \geq 2$ be an integer. Then $\gamma^{ID}(S(K_{1,n-1})) = n$.*

Proof. Let $G = K_{1,n-1}$ and assume that

$$V(G) = \{v_1, v_2, \ldots, v_n\},$$

$\deg_G(v_n) = n - 1$, and $V(S(G)) = V(G) \cup \{v_{in} : 1 \leq i \leq n-1\}$. Specially, we have $N_{S(G)}(v_{in}) = \{v_i, v_n\}$. Let D be a dominating set for $S(G)$. Then, $|D \cap \{v_i, v_{in}\}| \geq 1$ for each $i \in \{1, 2, ..., n-1\}$ which implies that $\gamma(S(G)) \geq n-1$. It is easy to see that $\{v_{in} : 1 \leq i \leq n-1\}$ is a dominating set for $S(G)$ and hence, $\gamma(S(G)) \leq n-1$. Therefore, $\gamma(S(G)) = n-1$.

Now Also, let C be an identifying code of $S(G)$ with the minimum cardinality. By Theorem 3.4, $|C| \leq n$. By Theorem 3.12, $|C| \geq n-1$. Suppose on the contrary that $|C| = n-1$. Since C is a dominating set of $S(G)$, we have $|C \cap \{v_i, v_{in}\}| = 1$ for each $i \in \{1, 2, ..., n-1\}$ and since $|C| = n-1$ we must have $v_n \notin C$. In this case, for every $1 \leq i \leq n-1$, we have $N_{S(G)}[v_i] \cap C = N_{S(G)}[v_{in}] \cap C$, which is a contradiction. Therefore, $\gamma^{ID}(S(G)) = n$. □

Theorem 3.14. *For each pair of integers $r, s \geq 1$, we have*

$$\gamma^{ID}(S(K_{r,s})) = r + s.$$

Proof. By Theorem 3.13, we have $\gamma^{ID}(S(K_{1,s})) = 1 + s$ and it can be easily seen that $\gamma^{ID}(S(K_{1,1})) = 2$.

Hence, hereafter assume that $r \geq 2$ and $s \geq 2$. By Theorem 2.4,

$$\gamma(S(K_{r,s})) = r + s - 1$$

and hence, $\gamma^{ID}(S(K_{r,s})) \geq \gamma(S(K_{r,s})) = r + s - 1$. Also, by Theorem 3.4, we have $\gamma^{ID}(S(K_{r,s})) \leq r + s$. Let T be a dominating set of cardinality $r + s - 1$ in $S(K_{r,s})$. By using the proof of Theorem 2.4 and its notations, we should have $t_2 = 1$ (or $t_1 = 1$, similarly). Since y_1 and x_1 are dominated by z_{11} and

$N_{S(G)}[x_1] \cap T = \{z_{11}\} = N_{S(G)}[z_{11}] \cap T$, two vertices x_1 and z_{11} can not be identified by T and hence T is not an identifying code. Since each dominating set of minimum cardinality in $S(K_{r,s})$ has a structure like T, each identifying code in $S(K_{r,s})$ must be of cardinality at least $r + s$, and this completes the proof. □

Conjecture 3.15. Let G be a simple graph of ordr n. Then the identifying code number of $S(G)$ is equal to n.

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References

1. J. Amjadi, S. Nazari-Moghaddam, S. M. Sheikholeslami and L. Volkmann, Total Roman domination number of trees, *Australas. J. Combin.*, **69** (2017), 271–285.
2. D. Auger, I. Charon, O. Hudry and A. Lobstein, Watching systems in graphs: an extension of identifying codes, *Discret. Appl. Math.*, **161** (2013), 1674–1685.
3. D. Auger, Minimal identifying codes in trees and planar graphs with large girth, *European J. Combin.*, **31** (2010), 1372–1384.
4. A. Behtoei, E. Vatandoost and F. Azizi Rajol Abad, Signed Roman domination number and join of graphs, *J. Algebr. Syst.*, **4** (2016), 65–77.
5. C. Chen, C. Lu and Z. Miao, Identifying codes and locating–dominating sets on paths and cycles, *Discret. Appl. Math.*, **159** (2011), 1540–1547.
6. M. Dettlaff, M. Lemańska, M. Miotk, J. Topp, R. Ziemann and P. Żyliński, Graphs with equal domination and certified domination numbers, *Opuscula Math.*, **39** (2019), 815–827.
7. N. Fazlollahi, D. Starobinski and A. Trachtenberg, Connected identifying codes for sensor network monitoring, *IEEE Wireless Communications and Networking Conference*, (2011), 1026–1031.
8. F. Foucaud, R. Klasing, A. Kosowski and A. Raspaud, On the size of identifying codes in triangle-free graphs, *Discret. Appl. Math.*, **160** (2012), 1532–1546.
9. A. Frieze, R. Martin, J. Moncel, M. Ruszinkó and C. Smyth, Codes identifying sets of vertices in random networks, *Discrete Math.*, **307** (2007), 1094–1107.
10. S. Gravier, J. Moncel and A. Semri, Identifying codes of cycles, *European J. Combin.*, **27** (2006), 767–776.
11. T. Haynes, D. Knisley, E. Seier and Y. Zou, A quantitative analysis of secondary RNA structure using domination-based parameters on trees, *BMC bioinform.*, **7** (2006), 108–115.
12. M. D. Hernando Martin, M. Mora Giné and I. M. Pelayo Melero, Watching systems in complete bipartite graphs, *Jor. de Mat. Disc. Alg.*, **11** (2012), 53–60.
13. I. Honkala and T. Laihonen, On identifying codes in the triangular and square grids, *SIAM J. Comput.*, **33** (2004), 304–312.

14. M. G. Karpovsky, K. Chakrabarty and L. B. Levitin, On a new class of codes for identifying vertices in graphs, *IEEE Trans. Inf.*, **44** (1998), 599–611.
15. Z. Mansouri and D. A. Mojdeh, Outer independent rainbow dominating functions in graphs, *Opuscula Math.*, **40** (2020), 599–615.
16. F. Ramezani, E. D. Rodriguez-Bazan and J. A. Rodriguez-Velazquez, On the Roman domination number of generalized Sierpinski graphs, *Filomat*, **31** (2017), 6515–6528.
17. A. Shaminejad and E. Vatandoost, The identifying code number and functigraphs, *J. Algebr. Syst.*, **10** (2022), 155–166.
18. A. Shaminezhad and E. Vatandoost, On 2-rainbow domination number of functigraph and its complement, *Opuscula Math.*, **40** (2020), 617–627.
19. B. Stanton, *On vertex identifying codes for infinite lattices*, PhD Thesis, Iowa State University, 2011, arXiv: 1102.2643.
20. K. Thulasiraman, M. Xu, Y. Xiao and X. D. Hu, Vertex identifying codes for fault isolation in communication networks, *In Proceedings of the International Conference on Discrete Mathematics and Applications*, Bangalore, 2006.
21. E. Vatandoost and F. Ramezani, On the domination and signed domination numbers of a zero-divisor graph, *Electron. J. Graph Theory Appl. (EJGTA)*, **4** (2016), 148–156.

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DOMINATION NUMBER AND IDENTIFYING CODE NUMBER OF THE SUBDIVISION GRAPHS

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عدد احاطه‌گر و عدد کد شناسایی زیرتقسیم گراف‌ها

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فرض کنید $G=(V,E)$ یک گراف ساده باشد. مجموعه C از رئوس گراف G یک کد شناساگر برای G است هرگاه برای هر دو رأس x و y مجموعه‌های $N_G[x]\cap C$ و $N_G[y]\cap C$ ناتهی و متمایز باشند. برای گراف داده شده‌ی G اندازه کوچکترین کد شناساگر را عدد کد شناسایی گراف G می‌نامند و با $\gamma^{ID}(G)$ نشان می‌دهند. در این مقاله ثابت می‌کنیم که عدد کد شناسایی زیرتقسیم هر گراف از مرتبه n حداکثر برابر n است. هم‌چنین، نشان می‌دهیم که عدد کد شناسایی زیرتقسیم گراف‌های K_n، $K_{r,s}$ و $C_P(s)$ به ترتیب برابر n، $r+s$ و ۲s است. در پایان این حدس را مطرح می‌کنیم که عدد کد شناسایی زیرتقسیم هر گراف G از مرتبه n برابر n است.

کلمات کلیدی: کد شناساگر، عدد کد شناساگر، زیرتقسیم، احاطه‌گری.