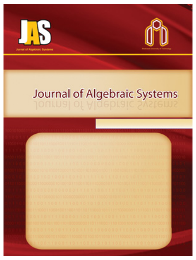
Journal of Algebraic Systems



ISSN:(2345-5128)(2345-511X)
Journal Homepage: www.jas.shahroodut.ac.ir

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To cite this article: S. Ahmadi, E. Vatandoost and A. Behtoei* (15 October 2024): Domination number and identifying code number of the subdivision graphs, Journal of Algebraic Systems, DOI: 10.22044/jas.2023.12257.1649

To link to this article: https://doi.org/10.22044/jas.2023.12257.1649

Published online: 15 October 2024

Journal of Algebraic Systems, vol. xx, no. xx, (202x), pp xx-xx
https://doi.org/10.22044/jas.2023.12257.1649

DOMINATION NUMBER AND IDENTIFYING CODE NUMBER OF THE SUBDIVISION GRAPHS

S. AHMADI, E. VATANDOOST AND A. BEHTOEI*

Abstract. Let $G = (V, E)$ be a simple graph. A set C of vertices of G is an identifying code of G if for every two vertices x and y the sets $N_G[x] \cap C$ and $N_G[y] \cap C$ are non-empty and different. Given a graph G, the smallest size of an identifying code of G is called the identifying code number of G and is denoted by $\gamma^{ID}(G)$. In this paper, we prove that the identifying code number of the subdivision of a graph G of order n is at most n. Also, we prove that the identifying code number of the subdivision of graphs K_n, $K_{r,s}$ and $C_P(s)$ are n, $r+s$ and $2s$, respectively. Finally, we conjecture that for every graph G of order n the identifying code number of the subdivision of G is n.

1. Introduction

In this paper, all graphs are assumed to be finite, simple and undirected. We will often use the notation $G = (V, E)$ to denote the graph with a non-empty vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of a graph is the number of vertices in graph and the *size* of a graph is the number of edges in graph. An edge of G with endpoints u and v is denoted by $\{u, v\}$ (which sometimes and for convenient we write it as uv), and we write $u \sim v$ to indicate that two vertices u and v are adjacent in G. For every vertex $x \in V(G)$, the *open neighborhood* of vertex x is denoted by $N_G(x)$ and defined as $N_G(x) = \{y \in V(G) \ : \ x \sim y\}$. Also the *close neighborhood* of vertex $x \in V(G)$, is $N_G[x] = N_G(x) \cup \{x\}$. The *degree* of a vertex $x \in V(G)$ is $\deg_G(x) = \big|N_G(x)\big|$. The *maximum degree* and *minimum degree* of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A *complete bipartite* graph is a special kind of bipartite graph in which every vertex of the first part is adjacent to each vertex of the second part. The complete bipartite graph of order $r+s$ is denoted by $K_{r,s}$. For any $X \subseteq V(G)$, the *induced subgraph* on X, is denoted by $G[X]$. The *Cocktail party* graph of order $2s$ denoted by $C_P(s)$ is obtained by removing s disjoint edges from the complete graph K_{2s}. The *subdivision graph* of G is the graph obtained by inserting an additional vertex

Published online: 15 October 2024
MSC(2010): Primary: 05C69; Secondary: 05C75.
Keywords: Identifying code; Identifying code number; Subdivision; Domination.
Received: 7 September 2022, Accepted: 30 July 2023.
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into each edge of G, denoted by $S(G)$. Sometimes the new vertex inserted into the edge $\{v_i, v_j\}$ is denoted by v_{ij}. A subset D of the vertices of G is a *dominating set* of G if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D. The *domination number* of G, which is denoted by $\gamma(G)$, is the minimum size of a dominating set of G. A set C is called a *separating set* of G, if for each pair $u,\ v$ of vertices of $G, N_G[u] \cap C \neq N_G[v] \cap C$ (equivalently, $(N_G[u] \triangle N_G[v]) \cap C \neq \emptyset$). If a dominating set C in G is a separating set of G, then we say that C is an *identifying code* of G and if G has an identifying code, then we say that G is an *identifiable graph*. Given a graph G, the smallest size of an identifying code of G is called the *identifying code number* of G and denoted by $\gamma^{ID}(G)$. If for two distinct vertices x and y, $N_G[x] = N_G[y]$, then they are called *twins*. It is noteworthy that a graph G is identifiable if and only if G is twin-free. In recent years much attention drawn to the domination theory which is a very interesting branch of graph theory. The concept of domination expanded to other parameters of domination such as 2-rainbow domination, signed domination, Roman domination, total Roman domination, and identifying code. For more details, we refer the reader to [1, 4, 6, 15, 16, 17, 18, 21].

The identifying code concept was introduced by Karpovsky et al. [14] in 1998. Later, several various families of graphs have been studied such as cycles and paths [5, 10], trees [3], triangular and square grids [13] and, triangle-free graphs [8]. Also identifying codes have found applications in various fields. These applications include sensor network monitoring [7], identifying codes in random networks [9], communication networks [20] and, the structural analysis of RNA proteins [11].

This paper deals with the study of the subdivision of some graphs. In Section 2, we show that $\gamma(S(G)) \leq |V(G)| - 1$. Specially, we prove that $\gamma(S(K_n)) = n - 1, \gamma(S(C_P(s))) = 2s - 1$, and $\gamma(S(K_{r,s})) = r + s - 1$. In Section 3, we show that the identifying code number of the subdivision of a graph G of order n is at most n. Also, we prove that the identifying code number of the subdivision graphs $S(K_n)$, $S(K_{1,n})$, $S(K_{r,s})$ and $S(C_P(s))$ are $n, n+1$, $r+s$ and $2s$, respectively. According to this facts, we conjecture that for every graph G of order n the identifying code number of the subdivision of G is n.

2. Domination number of $S(G)$

In this section, the domination number of $S(G)$ for some graphs is investigated.

Lemma 2.1. *Let G be a graph of order $n \geq 2$ which contains the path P_ℓ as an induced subgraph. Then $\gamma(S(G)) \leq n - \left\lceil \frac{\ell-1}{3} \right\rceil$.*

Proof. Let

$$V(P_\ell) = \{v_1, v_2, \dots, v_\ell\},\ E(P_\ell) = \{e_i = \{v_i, v_{i+1}\}\ :\ 1 \leq i \leq \ell - 1\}$$

and $N_{S(G)}(v_{ii+1}) = \{v_i, v_{i+1}\}$ for $1 \leq i \leq \ell - 1$ where $v_{ii+1} \in V(S(G))$ is the new vertex of degree two inserted on the edge e_i.
If $\ell = 3k$, then

$$D = (V(G) \setminus V(P_\ell)) \cup \left\{ v_i \in V(P_\ell)\ :\ i = 3t,\ 1 \leq t \leq \left\lceil \frac{\ell-1}{3} \right\rceil \right\} \cup \left\{ v_{ii+1}\ :\ i = 3t+1,\ 0 \leq t \leq \left\lfloor \frac{\ell-1}{3} \right\rfloor \right\}$$

is a dominating set for $S(G)$. Thus $\gamma(S(G)) \leq |D| = n - \frac{\ell}{3}$.
If $\ell = 3k+1$, then

$$D = (V(G) \setminus V(P_\ell)) \cup \left\{ v_i \in V(P_\ell)\ :\ i = 3t,\ 1 \leq t \leq \left\lceil \frac{\ell-1}{3} \right\rceil \right\} \cup \left\{ v_{ii+1}\ :\ i = 3t+1,\ 0 \leq t \leq \left\lfloor \frac{\ell-4}{3} \right\rfloor \right\} \cup \{v_\ell\}$$

is a dominating set for $S(G)$. Thus $\gamma(S(G)) \leq |D| = n - \frac{\ell-1}{3}$.
If $\ell = 3k+2$, then

$$D = (V(G) \setminus V(P_\ell)) \cup \left\{ v_i \in V(P_\ell)\ :\ i = 3t,\ 1 \leq t \leq \left\lfloor \frac{\ell-1}{3} \right\rfloor \right\} \cup \left\{ v_{ii+1}\ :\ i = 3t+1,\ 0 \leq t \leq \left\lfloor \frac{\ell-1}{3} \right\rfloor \right\}$$

is a dominating set for $S(G)$. Thus $\gamma(S(G)) \leq |D| = n - \frac{\ell+1}{3}$. □

Theorem 2.2. *If $n \geq 2$ is a positive integer, then $\gamma(S(K_n)) = n - 1$.*

Proof. Suppose $n \in \{2, 3\}$. It is easy to see that $\gamma(S(K_2)) = 1$ and $\gamma(S(K_3)) = 2$. Let $n \geq 4$ and $V(K_n) = \{v_1, v_2, \dots, v_n\}$ and

$$V(S(K_n)) = V(K_n) \cup \{v_{ij}\ :\ v_i v_j \in E(K_n)\}.$$

Note that $N_{S(K_n)}(v_i) = \{v_{ij} \ : \ v_j \in N_{K_n}(v_i)\}$ and $N_{S(K_n)}(v_{ij}) = \{v_i, v_j\}$. By Lemma 2.1, $\gamma(S(K_n)) \leq n-1$. Now let D be a dominating set of $S(K_n)$ with minimum cardinality. Without loss of generality, we assume that $v_1 \notin D$. Since D is a dominating set for $S(K_n)$, so v_1 is dominated by a vertex in D. Hence, there is $2 \leq j \leq n$ such that $v_{1j} \in D$. Without loss of generality, Suppose that $v_{12} \in D$. Since D is a dominating set of $S(G)$, for every $3 \leq j \leq n$ we have $v_{1j} \in D$ or $v_j \in D$. So, $|D| \geq n-1$. Therefore, $\gamma(S(K_n)) = n-1$. □

Theorem 2.3. *Let $s \geq 3$ be a positive integer. Then $\gamma(S(C_P(s))) = 2s-1$.*

Proof. Suppose that $V(C_P(s)) = \{v_1, v_2, \ldots, v_{2s}\}$ in which

$$\{v_1v_{s+1}, v_2v_{s+2}, ..., v_sv_{2s}\} \cap E(C_P(s)) = \emptyset$$

and

$$V(S(C_P(s))) = V(C_P(s)) \cup \{v_{ij} \ : \ v_iv_j \in E(C_P(s))\}.$$

Note that $N_{S(C_P(s))}(v_i) = \{v_{ij} \ : \ v_j \in N_{C_P(s)}(v_i)\}$ and $N_{S(C_P(s))}(v_{ij}) = \{v_i, v_j\}$. By Lemma 2.1, $\gamma(S(C_P(s)) \leq 2s-1$.

Now let D be a dominating set of $S(C_P(s))$ with minimum cardinality. Since

$$|D| = \gamma(S(C_P(s))) \leq 2s-1,$$

without loss of generality, we can assume that $v_1 \notin D$. Since D is a dominating set for $S(C_P(s))$, v_1 must be dominated by a vertex in D. Thus, there exists $j \in \{2, 3, ..., 2s\} \setminus \{s+1\}$ such that $v_{1j} \in D$. Without loss of generality, suppose that $v_{12} \in D$. Simlarly, for each $j \in \{3, ..., 2s\} \setminus \{s+1\}$ the vertex v_{1j} must be dominated by D and hence, $v_{1j} \in D$ or $v_j \in D$. Therefore, $|D| \geq 2s-2$.

If $v_{s+1} \in D$, then $|D| \geq 2s-1$ and the proof is complete. Otherwise, and in order to dominate v_{s+1}, there exist $j \in \{1, 2, \ldots, 2s\} \setminus \{1, s+1\}$ such that $v_{s+1\,j} \in D$. However, $\gamma(S(C_P(s))) \geq 2s-1$. Therefore, $\gamma(S(C_P(s))) = 2s-1$. □

Theorem 2.4. *If r, s are two integers such that $r \geq 2$ and $s \geq 2$, then $\gamma(S(K_{r,s})) = r+s-1$.*

Proof. Assume that $V(K_{r,s}) = \{x_1, \ldots, x_r\} \cup \{y_1, \ldots, y_s\}$ and

$$V(S(K_{r,s})) = V(K_{r,s}) \cup \{z_{ij} \ : \ 1 \leq i \leq r, 1 \leq j \leq s\}$$

in which z_{ij} is the new vertex inserted on the edge x_iy_j to obtain the subdivision graph $S(K_{r,s})$ from $K_{r,s}$. By Lemma 2.1,

$$\gamma(S(K_{r,s})) \leq r+s-1.$$

Now let T be a dominating set of minimum cardinality in $S(K_{r,s})$. If $\{x_1,\ldots,x_r,y_1,\ldots,y_s\}\subseteq T$, then $\gamma(S(K_{r,s}))=|T|\geq r+s$, which is a contradiction. Therefore, there exists at least one vertex in $\{x_1,\ldots,x_r,y_1,\ldots,y_s\}$ which is not in T. Wihtout loss of generality, (and by renaming the vertices if it is necessary) assume that

$$\{x_1,\ldots,x_r,y_1,\ldots,y_s\}\setminus T=\{x_1,x_2,\ldots,x_{t_1},y_1,y_2,\ldots,y_{t_2}\}.$$

Thus, $\{x_{t_1+1},\ldots,x_r,y_{t_2+1},\ldots,y_s\}\subseteq T$ and hence $|T|\geq(r-t_1)+(s-t_2)$. Since T is a dominating set, each vertex in $\{x_1,\ldots,x_{t_1},y_1,\ldots,y_{t_2}\}$ should be dominated by a vertex in T, and by considering the structure of $S(K_{r,s})$, these dominating vertices must be in the set $\{z_{ij}\ :\ 1\leq i\leq r,1\leq j\leq s\}$. Also, since t_1t_2 vertices in $\{z_{ij}\mid 1\leq i\leq t_1,1\leq j\leq t_2\}$ must be dominated by T, we should have $\{z_{ij}\ :\ 1\leq i\leq t_1,1\leq j\leq t_2\}\subseteq T$. Therefore, $|T|\geq(r-t_1)+(s-t_2)+t_1t_2$. Note that

$$\begin{aligned}|T|&\geq(r-t_1)+(s-t_2)+t_1t_2\\&=(r+s)+(t_1-1)(t_2-1)-1.\end{aligned}$$

If we have $t_2=0$, then $\{y_1,y_2,\ldots,y_s\}\subseteq T$ and $\{x_1,x_2,\ldots,x_{t_1}\}\cap T=\emptyset$. Since T is a dominating set, there exist vertices

$$\{z_{1j_1},z_{2j_2},\ldots,z_{t_1j_{t_1}}\}\subseteq T$$

which dominate vertices in $\{x_1,x_2,\ldots,x_{t_1}\}$. This implies that

$$|T|\geq(r-t_1)+s+t_1=r+s,$$

which is a contradiction, because we show that $\gamma(S(K_{r,s}))\leq r+s-1$. Thus, $t_2\geq 1$ and similarly we can show that $t_1\geq 1$. Therefore,

$$\gamma(S(K_{r,s}))=|T|\geq(r+s)+(t_1-1)(t_2-1)-1$$

in which $t_1\geq 1$ and $t_2\geq 1$. The minimum value of the statement $(r+s)+(t_1-1)(t_2-1)-1$ with the conditions $t_1\geq 1$, $t_2\geq 1$ occurs just when $t_1=1$ or $t_2=1$ and this leads to the value $r+s-1$. Without loss of generality, assume that $t_2=1$ (the case $t_1=1$ will be similarly done). Since vertices $x_1,x_2,\ldots,x_{t_1}$ are dominated by T, and by considering the structure of $S(K_{r,s})$, there exist t_1 vertices of the from z_{ij} in T. If none of these vertices is adjacent to y_1, then for dominating y_1 an extra vertex in $\{z_{ij}\mid 1\leq i\leq r,1\leq j\leq s\}$ is required which implies that $|T|\geq r+s$, a contradiction. Hence, we can assume that $\{x_1,x_2,\ldots,x_{t_1}\}$ are dominated by $\{z_{1j_1},\ldots,z_{t_1j_{t_1}}\}$ and $z_{1j_1}=z_{11}$ (i.e. y_1 is dominated by z_{11}). Since the set $\{x_{t_1+1},\ldots,x_r,y_2,\ldots,y_s,z_{1j_1},z_{t_1j_{t_1}}\}$ is a subset of T and it is a dominating set of cardinality $r+s-1$, and by the minimality of T, we have $\gamma(S(K_{r,s})) = |T| = r+s-1$. □

3. Identifying code number of $S(G)$

In this section, we determine the identifying code number of the subdivision graphs $S(K_n)$, $S(C_P(s))$, $S(K_{1,n})$ and, $S(K_{r,s})$ with $r \geq 2$ and $s \geq 2$. At first, consider the following results.

Theorem 3.1. [12] *Let $n \geq 2$ be a positive integer. Then $\gamma^{ID}(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.*

Theorem 3.2. ([5], [10]) *Let $n \geq 4$ be a positive integer. Then*

$$\gamma^{ID}(C_n) = \begin{cases} 3 & \text{if } n = 4, 5 \\ \frac{n}{2} & \text{if } n \geq 6 \text{ is even} \\ \frac{n+3}{2} & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

It is clear that for each $n \geq 3$, $S(P_n) \cong P_{2n-1}$ and $S(C_n) \cong C_{2n}$. Thus, by Theorems 3.1 and 3.2, $\gamma^{ID}(S(P_n)) = n$ and $\gamma^{ID}(S(C_n)) = n$. In the following two results, we show that $|V(G)|$ is generally an upper bound for $\gamma^{ID}(S(G))$.

Proposition 3.3. *Let G be a graph of order $n \geq 2$. Then $S(G)$ is an identifiable graph.*

Proof. Let a and b be two arbitrary and distinct vertices in $S(G)$. If a is not adjacent to b in $S(G)$, then $N_{S(G)}[a] \neq N_{S(G)}[b]$. Let a be adjacent to b in $S(G)$. Since $S(G)$ is a bipartite graph and $V(G)$ is a partite set in it, without loss of generality, we can assume that $a \in V(G)$ and hence, there exists $c \in V(G) \setminus \{a\}$ such that $N_{S(G)}(b) = \{a, c\}$. Since $c \notin N_{S(G)}[a]$, we have $N_{S(G)}[a] \neq N_{S(G)}[b]$. Therefore $S(G)$ is twin-free and hence, an identifiable graph.

□

Theorem 3.4. *If G is a graph of order $n \geq 2$, then $\gamma^{ID}(S(G)) \leq n$.*

Proof. Let $C = V(G)$. It is clear that C is a dominating set for $S(G)$. We claim that C is an identifying code for $S(G)$. For this purpose, let x and y be two distince vertices in $V(S(G))$. We consider the following cases:

Case 1) $\{x, y\} \subseteq V(G)$: Since $S(G)$ is a bipartite graph with $V(G)$ as a partite set, x is not adjacent to y in $S(G)$. Thus $x \in N_{S(G)}[x] \cap C$ and $x \notin N_{S(G)}[y] \cap C$. Hence, $N_{S(G)}[x] \cap C \neq N_{S(G)}[y] \cap C$.

Case 2) $\{x, y\} \cap V(G) = \emptyset$: In this case, there exists a vertex $a \in V(G)$ such that $x \sim a$ and $a \nsim y$. Thus $a \in N_{S(G)}[x] \cap C$ and $a \notin N_{S(G)}[y] \cap C$ which implies that $N_{S(G)}[x] \cap C \neq N_{S(G)}[y] \cap C$.

Case 3) $x \notin V(G)$ and $y \in V(G)$: Assume that $N_{S(G)}(x) = \{a, b\}$ in which $ab \in E(G)$. If $y \notin \{a, b\}$, then $a \in N_{S(G)}[x] \cap C$ and $a \notin N_{S(G)}[y] \cap C$. If $y = a$, then $b \in N_{S(G)}[x] \cap C$ and $b \notin N_{S(G)}[y] \cap C$. However

$$N_{S(G)}[x] \cap C \neq N_{S(G)}[y] \cap C.$$

These facts show that C is an identifying code for $S(G)$ and hence,

$$\gamma^{ID}(S(G)) \leq |C| = n.$$

□

Now we use the following result to show that $\gamma^{ID}(S(C_P(s))) = 2s$ and $\gamma^{ID}(S(K_n)) = n$.

Theorem 3.5. [19] *If G is a graph of order n and maximum degree Δ, then $\gamma^{ID}(G) \geq \frac{2n}{\Delta+2}$.*

Theorem 3.6. *For each $n \geq 2$, $\gamma^{ID}(S(K_n)) = n$.*

Proof. By Theorem 3.4, we have $\gamma^{ID}(S(K_n)) \leq n$. Since $|V(S(K_n))| = \frac{n(n+1)}{2}$, Theorem 3.5 implies that $\gamma^{ID}(S(K_n)) \geq n$. Thus, $\gamma^{ID}(S(K_n)) = n$. □

Theorem 3.7. *Let $s \geq 2$ be a positive integer. Then $\gamma^{ID}(S(C_P(s))) = 2s$.*

Proof. It is clear that $S(C_P(s))$ is of order $2s^2$. By Theorem 3.4,

$$\gamma^{ID}(S(C_P(s))) \leq 2s.$$

By Theorem 3.5, $\gamma^{ID}(S(C_P(s))) \geq \frac{2(2s^2)}{(2s-2)+2} = 2s$. Therefore,

$$\gamma^{ID}(S(C_P(s))) = 2s.$$

□

In the following, among some other useful results, we show that for $n \geq 4$, $V(K_n)$ is the unique optimum identifying code in $S(K_n)$.

Theorem 3.8. *Let G be an identifiable graph of order n and C be an identifying code of $S(G)$. Then we have*

$$|C| \geq \max\left\{\deg_G(x) + \left\lceil \frac{\deg_G(x)}{2} \right\rceil \ : \ x \in V(G) \setminus C \right\}.$$

Specially, if $\gamma^{ID}(S(G)) < n$, then $\gamma^{ID}(S(G)) \geq \delta(G) + \left\lceil \frac{\delta(G)}{2} \right\rceil$.

Proof. Let x be a vertex in $V(G) \setminus C$ and assume that $\deg_G(x) = d$, $N_G(x) = \{y_1, y_2, \ldots, y_d\}$ and $N_{S(G)}(x) = \{e_1, e_2, \ldots, e_d\}$. Since C is a dominating set of $S(G)$ and $x \notin C$, there exists $1 \leq i \leq d$ such that $e_i \in C$. Also, each vertex in $\{e_1, e_2, ..., e_d\} \setminus \{e_i\}$ must be dominated by C and hence, for each $j \in \{1, 2, \ldots, d\} \setminus \{i\}$ we have $C \cap \{e_i, y_i\} \neq \emptyset$. Therefore, $|C \cap \{y_1, \ldots, y_d, e_1, \ldots, e_d\}| \geq d$. Note that for each $1 \leq i \leq d$, two vertices e_i and y_i are adjacent in $S(G)$ and hence, $e_i \in N_{S(G)}[y_i]$ and $y_i \in N_{S(G)}[e_i]$. Now since C is an identifying code of $S(G)$, for each $i \in \{1, 2, \ldots, d\}$ we should have $N_{S(G)}[e_i] \cap C \neq N_{S(G)}[y_i] \cap C$. Therefore, for each $i \in \{1, 2, \ldots, d\}$ there exists a vertex $z_i \notin \{e_1, e_2, ..., e_d\}$ such that $z_i \in C \cap N_{S(G)}(y_i)$. Note that when $y_i y_j$ is an edge in G and z is the (new) vertex of degree two in $S(G)$ with $N_{S(G)}(z) = \{y_i, y_j\}$, then z can play both of the roles of z_i and z_j in our above statement and hence, we may have $z_i = z_j$. Thus, by pairing z_i's in the worst case, we have $|\{z_1, z_2, \ldots, z_d\}| \geq \left\lceil \frac{\deg_G(x)}{2} \right\rceil$. This implies that

$$\begin{aligned} |C| &\geq |C \cap \{y_1, \ldots, y_d, e_1, \ldots, e_d\}| + |C \cap \{z_1, z_2, ..., z_d\}| \\ &\geq d + \left\lceil \frac{d}{2} \right\rceil, \end{aligned}$$

which completes the proof. Note that when $\gamma^{ID}(S(G)) < n$, there exists at least one vertex $x \in V(G) \setminus C$ and we know that $\deg_G(x) \geq \delta(G)$. □

Corollary 3.9. *Let G be a triangle-free identifiable graph of order n and C be an identifying code of $S(G)$. Then,*

$$|C| \geq 2 \max \{\deg_G(x) \ : \ x \in V(G) \setminus C \ \}.$$

Specially, if $\gamma^{ID}(S(G)) < n$, then $\gamma^{ID}(S(G)) \geq 2\,\delta(G)$.

Proof. The proof is a direct consequence of the proof of Theorem 3.8 by considering the fact $y_i y_j \notin E(G)$ when G is triangle-free, which itself implies that $z_i \neq z_j$ for each $i \neq j$. □

Corollary 3.10. *Let $n \geq 2$ be an integer and C be an identifying code of $S(K_n)$. If $V(K_n) \nsubseteq C$, then $|C| \geq (n-1) + \left\lceil \frac{(n-1)}{2} \right\rceil$.*

Proof. By Theorem 3.8, the proof is straightforward. □

Note that $\gamma^{ID}(S(K_2)) = \gamma^{ID}(P_3) = 2$ and $\gamma^{ID}(S(K_3)) = \gamma^{ID}(C_6) = 3$. In addition, P_3 has a unique identifying code of size two, but C_6 has at least two different identifying codes of size three.

Corollary 3.11. *Let $n \geq 4$ be an integer and C be an identifying code of $S(K_n)$ with minimum size (i.e., $|C| = \gamma^{ID}(S(K_n))$). Then, $C = V(K_n)$.*

Proof. By Theorem 3.4, we have $|C| = \gamma^{ID}(S(K_n)) \leq n$. This fact and Corollary 3.10 implies that $V(K_n) \setminus C = \emptyset$. Thus, $V(K_n) \subseteq C$ and hence, $|C| \geq n$. This facts imply that $\gamma^{ID}(S(K_n)) = |C| = n$ and $C = V(G)$. □

Finally, we determine the identifying code number of complete bipartite graphs and according to obtained results, we propose a conjecture.

Theorem 3.12. [2] *Let G be a graph of order n. Then $\gamma(G) \leq \gamma^{ID}(G)$.*

Theorem 3.13. *Let $n \geq 2$ be an integer. Then $\gamma^{ID}(S(K_{1,n-1})) = n$.*

Proof. Let $G = K_{1,n-1}$ and assume that

$$V(G) = \{v_1, v_2, \ldots, v_n\},$$

$\deg_G(v_n) = n-1$, and $V(S(G)) = V(G) \cup \{v_{in} \ : \ 1 \leq i \leq n-1\}$. Specially, we have $N_{S(G)}(v_{in}) = \{v_i, v_n\}$. Let D be a dominating set for $S(G)$. Then, $|D \cap \{v_i, v_{in}\}| \geq 1$ for each $i \in \{1, 2, ..., n-1\}$ which implies that $\gamma(S(G)) \geq n-1$. It is easy to see that $\{v_{in} \ : \ 1 \leq i \leq n-1\}$ is a dominating set for $S(G)$ and hence, $\gamma(S(G)) \leq n-1$. Therefore, $\gamma(S(G)) = n-1$.

Now Also, let C be an identifying code of $S(G)$ with the minimum cardinality. By Theorem 3.4, $|C| \leq n$. By Theorem 3.12, $|C| \geq n-1$. Suppose on the contrary that $|C| = n-1$. Since C is a dominating set of $S(G)$, we have $|C \cap \{v_i, v_{in}\}| = 1$ for each $i \in \{1, 2, ..., n-1\}$ and since $|C| = n-1$ we must have $v_n \notin C$. In this case, for every $1 \leq i \leq n-1$, we have $N_{S(G)}[v_i] \cap C = N_{S(G)}[v_{in}] \cap C$, which is a contradiction. Therefore, $\gamma^{ID}(S(G)) = n$. □

Theorem 3.14. *For each pair of integers $r, s \geq 1$, we have*

$$\gamma^{ID}(S(K_{r,s})) = r + s.$$

Proof. By Theorem 3.13, we have $\gamma^{ID}(S(K_{1,s})) = 1 + s$ and it can be easily seen that $\gamma^{ID}(S(K_{1,1})) = 2$.

Hence, hereafter assume that $r \geq 2$ and $s \geq 2$. By Theorem 2.4,

$$\gamma(S(K_{r,s})) = r + s - 1$$

and hence, $\gamma^{ID}(S(K_{r,s})) \geq \gamma(S(K_{r,s})) = r + s - 1$. Also, by Theorem 3.4, we have $\gamma^{ID}(S(K_{r,s})) \leq r + s$. Let T be a dominating set of cardinality $r + s - 1$ in $S(K_{r,s})$. By using the proof of Theorem 2.4 and its notations, we should have $t_2 = 1$ (or $t_1 = 1$, similarly). Since y_1 and x_1 are dominated by z_{11} and

$N_{S(G)}[x_1] \cap T = \{z_{11}\} = N_{S(G)}[z_{11}] \cap T$, two vertices x_1 and z_{11} can not be identified by T and hence T is not an identifying code. Since each dominating set of minimum cardinality in $S(K_{r,s})$ has a structure like T, each identifying code in $S(K_{r,s})$ must be of cardinality at least $r+s$, and this completes the proof. □

Conjecture 3.15. Let G be a simple graph of ordr n. Then the identifying code number of $S(G)$ is equal to n.

Acknowledgments

The authors warmly thank the anonymous referees for reading this manuscript very carefully and providing numerous valuable corrections and suggestions which improve the quality of this paper.

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