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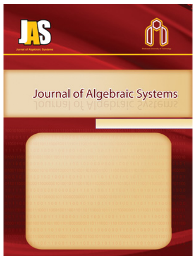

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FIXED POINTS AND CUT-HOMOMORPHISMS GENERATED BY ACTIONS OF A BE-ALGEBRA ON ITS SUBALGEBRA

M. SAMBASIVA RAO

ABSTRACT. The concept of actions of a BE-algebra on its subalgebra is introduced and certain properties of these actions are derived. The notion of cut-homomorphisms is introduced and proved that the class of all cut-homomorphisms forms an ordered BE-algebra. Properties of fixed points of cut-homomorphisms are investigated and a set of equivalent conditions is given for any two cut-homomorphisms are equal in the sense of mappings.

INTRODUCTION

The notion of BE-algebras was introduced and extensively studied by H. S. Kim and Y. H. Kim in [4]. These classes of BE-algebras were introduced as a generalization of the class of BCK-algebras of K. Iseki and S. Tanaka [3]. Some properties of filters of BE-algebras were studied by S. S. Ahn and Y. H. Kim in [1] and by B. L. Meng in [5]. In [11], A. Walendziak discussed some properties of commutative BE-algebras. He also investigated the relationship between BE-algebras, implicative algebras and J-algebras. In 2012, A. Rezaei, and A. Borumand Saeid [7], stated and proved the first, second and third isomorphism theorems in self distributive BE-algebras. Later, these authors [6] introduced the notion of commutative ideals in a BE-algebra. In 2013, A. Borumand Saeid, A. Rezaei and R. A. Borzooei [2] extensively studied the properties of some types of filters in BE-algebras and established relations among them. In 2016, the authors [10] characterized self-distributive BE-algebras, commutative BE-algebras and implicative BE-algebras with the help of left and right self maps. In [9], the author investigated certain significant properties of self-maps and endomorphisms.

In this article, the notion of an action of a BE-algebra on a given subalgebra is introduced. Certain properties of the actions generated by direct products and endomorphisms of BE-algebras are investigated. The notion of permutable actions is introduced in a BE-algebra and then proved that their composition is again an action of the BE-algebra. An ordering is

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introduced on the set of all actions and then derived that this set is partially ordered whenever the respective BE-algebra is commutative. It is also proved that the collection of all actions of a BE-algebra on a given subalgebra forms a semi-lattice. The concept of subcuts of subalgebras and filters of cuts are introduced. It is proved that the set of all subcuts of a given cut forms a partially ordered semi-lattice. The notion of cut-homomorphisms is introduced in BE-algebras and then it is proved that the collection of all cut-homomorphisms forms a BE-algebra which is homomorphic to the given BE-algebra. Further, it is proved that the set of all idempotent cut-homomorphisms forms an upper semi-lattice.

In the final section, the notion of fixed points of a cut-endomorphism is introduced in BE-algebras. A necessary and sufficient condition is given for a cut-endomorphism to have a fixed point. A set of equivalent conditions is given for any two cut-homomorphisms to be equal in the sense of mappings. Finally, some properties of fixed points and images of a cut-endomorphism are investigated.

1. Preliminaries

In this section, certain definitions and results are presented which are taken mostly from [4], [5], and [8] for the ready reference.

Definition 1.1. [4] An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:

(1) $x * x = 1$,
(2) $x * 1 = 1$,
(3) $1 * x = x$,
(4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

A BE-algebra X is called *self-distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A BE-algebra X is called *transitive* if

$$y * z \leq (x * y) * (x * z)$$

for all $x, y, z \in X$. Every self-distributive BE-algebra is transitive. A BE-algebra $(X, *, 1)$ is said to be *commutative* [8] if $(x * y) * y = (y * x) * x$ for all $x, y \in X$. In this case, we consider $(y * x) * x$ as $x \vee y$. In a commutative BE-algebra X, it is clear that $x \vee y = y \vee x$ for all $x, y \in X$. We introduce a relation $\leq$ on X by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$.

Theorem 1.2. [5] *Let X be a transitive BE-algebra and $x, y, z \in X$. Then*

(1) $1 \leq x$ *implies* $x = 1$,

(2) $y \leq z$ *implies* $x * y \leq x * z$ *and* $z * x \leq y * x$.

Definition 1.3. [4] A non-empty subset F of a BE-algebra X is called a *filter* of X if, for all $x, y \in X$, it satisfies the following properties:

(1) $1 \in F$,
(2) $x \in F$ and $x * y \in F$ imply that $y \in F$.

A subset S of a BE-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. Clearly every subalgebra of a BE-algebra contains the element 1. It is clear that every filter of a BE-algebra is a subalgebra. A mapping f from a BE-algebra $(X, *, 1)$ into a BE-algebra $(Y, \circ, 1')$ is called a *homomorphism* if $f(x * y) = f(x) \circ f(y)$ for all $x, y \in X$. It is clear that $f(1) = 1$ whenever f is a homomorphism. A homomorphism of BE-algebra into itself is called an *endomorphism*.

2. Actions of BE-algebras

In this section, the notions of an action and a permutable action of a BE-algebra on a given subalgebra is introduced. Certain properties of these actions and the cuts of the BE-algebras are investigated.

Definition 2.1. Let $(X, *, 1)$ be a BE-algebra and S is a subalgebra of X. A mapping $\sigma : X \times S \to S$ is called an *action* of X on S if it satisfies the following properties:

(C1) $\sigma(a, 1) = 1$ for all $a \in X$,
(C2) $\sigma(1, x) = x$ for all $x \in S$,
(C3) $\sigma(a, x * y) = \sigma(a, x) * \sigma(a, y)$ for all $a \in X$ and $x, y \in S$,
(C4) $\sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x))$ for all $a, b \in X$ and $x \in S$.

An action σ of a BE-algebra X on its subalgebra S is called *idempotent* if $\sigma(a, \sigma(a, x)) = \sigma(a, x)$ for all $a \in X$ and $x \in S$. Further, σ is called a *complete action* if along with (C1)-(C4), it satisfies the following:

$$\sigma(a * b, x) = \sigma(a, x) * \sigma(b, x)$$

for all $a, b \in X$ and $x \in S$. In all the cases, we call the pair (σ, S) an S-cut of the BE-algebra X.

Example 2.2. (1) Let X be a self-distributive BE-algebra and S be a subalgebra of X. For any $a \in X$ and $x \in S$, define a mapping $\sigma : X \times S \to S$ by $\sigma(a, x) = a * x$. Then σ is an action of X on S. Therefore (σ, S) is an S-cut of the BE-algebra X.

(2) Let X be a BE-algebra and S be a subalgebra of X. For any $a \in X$ and $x \in S$, define a mapping $\sigma : X \times S \to S$ by $\sigma(a, x) = x$. It can be routinely verified that σ is an action of X on S. Hence (σ, S) is an S-cut of X. Further, we see that σ is not complete. For consider $a, b \in X$ and $1 \neq c \in S$ be such that $a * b = 1$. Then

$$\sigma(a * b, c) = c \text{ and } \sigma(a, c) * \sigma(b, c) = c * c = 1$$

Therefore σ is an action of X on S which is not complete.

Proposition 2.3. *Let X be a BE-algebra and S be a subalgebra of X. Let $\mu : X \to S$ be a homomorphism satisfying the following:*

(1) *$\mu(a) * (x * y) = (\mu(a) * x) * (\mu(a) * y)$ for all $a \in X$ and $x, y \in S$,*
(2) *$\mu(a * b) * x = (\mu(a) * x) * (\mu(b) * x)$ for all $a, b \in X$ and $x \in S$.*

For any $a \in X$ and $x \in S$, define a mapping $\sigma_\mu : X \times S \to S$ by $\sigma_\mu(a, x) = \mu(a) * x$. Then σ_μ is a complete action of X on S.

Proof. Since S is a subalgebra of X, we get σ_μ is well-defined. Then

(C1) For any $a \in X$, we get $\sigma_\mu(a, 1) = \mu(a) * 1 = 1$.
(C2) For any $x \in S$, we have $\sigma_\mu(1, x) = \mu(1) * x = 1 * x = x$.
(C3) Let $a \in X$ and $x, y \in S$. Then, we get

$$\begin{aligned} \sigma_\mu(a, x * y) &= \mu(a) * (x * y) \\ &= (\mu(a) * x) * (\mu(a) * y) \\ &= \sigma_\mu(a, x) * \sigma_\mu(a, y). \end{aligned}$$

(C4) Let $a, b \in X$ and $x \in S$. Then, we get

$$\begin{aligned} \sigma_\mu(a, \sigma_\mu(b, x)) &= \sigma_\mu(a, \mu(b) * x) \\ &= \mu(a) * (\mu(b) * x) \\ &= \mu(b) * (\mu(a) * x) \\ &= \mu(b) * \sigma_\mu(a, x) \\ &= \sigma_\mu(b, \sigma_\mu(a, x)). \end{aligned}$$

Thus σ_μ is an action of X on S. Hence (σ_μ, S) is an S-cut of X. Further, assume that μ satisfies the condition. Let $a, b \in X$ and $x \in S$. Then

$$\begin{aligned} \sigma_\mu(a * b, x) &= \mu(a * b) * x \\ &= (\mu(a) * x) * (\mu(b) * x) \\ &= \sigma_\mu(a, x) * \sigma_\mu(b, x) \end{aligned}$$

Therefore σ_μ is a complete action of X on S. □

Proposition 2.4. *Let S_1 and S_2 be the subalgebras of the BE-algebras X_1 and X_2 respectively. Suppose σ_1 and σ_2 are the actions of X_1 on S_1 and X_2 on S_2 respectively. For all $(a, b) \in X_1 \times X_2$ and $(x, y) \in S_1 \times S_2$, define a mapping $\sigma_1 \times \sigma_2 : (X_1 \times X_2) \times (S_1 \times S_2) \to S_1 \times S_2$ by*

$$(\sigma_1 \times \sigma_2)((a, b), (x, y)) = (\sigma_1(a, x), \sigma_2(b, y)).$$

Then $\sigma_1 \times \sigma_2$ is an action of the product algebra $X_1 \times X_2$ on the subalgebra $S_1 \times S_2$. Therefore $(\sigma_1 \times \sigma_2, S_1 \times S_2)$ is a cut of $X_1 \times X_2$.

Proof. Clearly $\sigma_1 \times \sigma_2$ is well-defined. Note that $S_1 \times S_2$ is a subalgebra of $X_1 \times X_2$ with element $(1, 1)$. Now, the properties (C1)-(C4) can be routinely verified by using point-wise operations. □

Proposition 2.5. *Let S be a subalgebra of a BE-algebra X. Suppose $\mu : X \times S \to S$ is a mapping. For all $a, b \in X$ and $x, y \in S$, define a mapping $\sigma_\mu : X^2 \to S^2$ (where $X^2 = X \times X$ and $S^2 = S \times S$) by*

$$\sigma_\mu((a, b), (x, y)) = (\mu(a, x), \mu(b, y))$$

Then μ is an action of X on S if and only if σ_μ is an action of X^2 on S^2. Moreover, (μ, S) is a cut of X if and only if (σ_μ, S^2) is a cut of X^2.

Proof. Clearly σ_μ is well-defined. Note that S^2 is a subalgebra of X^2 with element $(1, 1)$. Assume that μ is an action of X on S. Then (C1) Let $(a, b) \in X^2$. Then $\sigma_\mu((a, b), (1, 1)) = (\mu(a, 1), \mu(b, 1)) = (1, 1)$. (C2) Let $(x, y) \in S^2$. Then, we get

$$\sigma_\mu((1, 1), (x, y)) = (\mu(1, x), \mu(1, y)) = (x, y).$$

(C3) Let $(a, b) \in X^2$ and $(x, y), (z, w) \in S^2$. Since μ is an action of X on S, we get the following consequence:

$$\begin{aligned}
\sigma_\mu((a, b), (x, y) * (z, w)) &= \sigma_\mu((a, b), (x * z, y * w)) \\
&= (\mu(a, x * z), \mu(b, y * w)) \\
&= (\mu(a, x) * \mu(a, z), \mu(b, y) * \mu(b, w)) \\
&= (\mu(a, x), \mu(b, y)) * (\mu(a, z), \mu(b, w)) \\
&= \sigma_\mu((a, b), (x, y)) * \sigma_\mu((a, b), (z, w))
\end{aligned}$$

(C4) Let $(a,b),(c,d) \in X^2$ and $(x,y) \in S^2$. Since μ is an action of X on S, we get the following consequence:

$$\begin{aligned}\sigma_\mu((a,b),\sigma_\mu((c,d),(x,y)) &= \sigma_\mu((a,b),(\mu(c,x),\mu(d,y)))\\ &= (\mu(a,\mu(c,x)),\mu(b,\mu(d,y)))\\ &= (\mu(c,\mu(a,x)),\mu(d,\mu(b,y)))\\ &= \sigma_\mu((c,d),(\mu(a,x),\mu(b,y)))\\ &= \sigma_\mu((c,d),\sigma_\mu((a,b),(x,y))\end{aligned}$$

Hence σ_μ is an action of X^2 on S^2. Therefore (σ_μ, S^2) is a cut of X^2.

Conversely, assume that σ_μ is an action of X^2 on S^2. (C1) Let $a \in X$. Then $(\mu(a,1),\mu(1,1)) = \sigma_\mu((a,1),(1,1)) = (1,1)$. Hence $\mu(a,1) = 1$. (C2) Let $x \in S$. Then $(\mu(1,x),\mu(1,1)) = \sigma_\mu((1,1),(1,x)) = (1,1)$. Hence $\mu(1,x) = 1$. (C3) Let $a \in X$ and $x,y \in S$. Since σ_μ is an action of X^2 on S^2, we get that

$$\begin{aligned}(\mu(a,x*y),\mu(1,1)) &= \sigma_\mu((a,1),(x*y,1))\\ &= \sigma_\mu((a,1),(x,1)*(y,1))\\ &= \sigma_\mu((a,1),(x,1))*\sigma_\mu((a,1),(y,1))\\ &= (\mu(a,x),\mu(1,1))*(\mu(a,y),\mu(1,1))\\ &= (\mu(a,x)*\mu(a,y),\mu(1,1))\end{aligned}$$

Hence $\mu(a,x*y) = \mu(a,x)*\mu(a,y)$. (C4) Let $a,b \in X$ and $x \in S$. Since σ_μ is an action of X^2 on S^2. Then

$$\begin{aligned}(\mu(a,\mu(b,x)),1) &= (\mu(a,\mu(b,x)),\mu(1,1))\\ &= \sigma_\mu((a,1),(\mu(b,x),1))\\ &= \sigma_\mu((a,1),(\mu(b,x),\mu(1,1)))\\ &= \sigma_\mu((a,1),\sigma_\mu((b,1),(x,1)))\\ &= \sigma_\mu((b,1),\sigma_\mu((a,1),(x,1)))\\ &= \sigma_\mu((b,1),(\mu(a,x),\mu(1,1)))\\ &= \sigma_\mu((b,1),(\mu(a,x),1))\\ &= (\mu(b,\mu(a,x)),\mu(1,1))\\ &= (\mu(b,\mu(a,x)),1)\end{aligned}$$

Hence $\mu(a,\mu(b,x)) = \mu(b,\mu(a,x))$. Thus μ is an action of X on S. □

Proposition 2.6. *Let X be a BE-algebra and S be a subalgebra of X. Suppose $\mu : X \to X$ is an endomorphism and $\sigma : X \times S \to S$ is a* *mapping. For $a \in X$ and $x \in S$, define a mapping $\sigma_\mu : X \times S \to S$ by*

$$\sigma_\mu(a, x) = \sigma(\mu(a), x)$$

If σ is a complete action of X on S, then σ_μ is a complete action of X on S. Further, the converse is also true whenever μ is surjective.

Proof. Assume that σ is a complete action of X on S. Then, we get

(C1) For any $a \in X$, we get that $\sigma_\mu(a, 1) = \sigma(\mu(a), 1) = 1$.

(C2) For any $x \in S$, we get $\sigma_\mu(1, x) = \sigma(\mu(1), x) = \sigma(1, x) = x$.

(C3) Let $a \in X$ and $x, y \in S$. Since σ is an action of X on S and μ is an endomorphism, we get that

$$\begin{aligned}\sigma_\mu(a, x * y) &= \sigma(\mu(a), x * y)\\ &= \sigma(\mu(a), x) * \sigma(\mu(a), y)\\ &= \sigma_\mu(a, x) * \sigma_\mu(a, y)\end{aligned}$$

(C4) Let $a, b \in X$ and $x \in S$. Since σ is an action of X on S, we get

$$\begin{aligned}\sigma_\mu(a, \sigma_\mu(b, x)) &= \sigma_\mu(a, \sigma(\mu(b), x))\\ &= \sigma(\mu(a), \sigma(\mu(b), x))\\ &= \sigma(\mu(b), \sigma(\mu(a), x))\\ &= \sigma_\mu(b, \sigma(\mu(a), x))\\ &= \sigma_\mu(b, \sigma_\mu(a, x))\end{aligned}$$

(C5) Let $a, b \in X$ and $x \in S$. Since σ is a complete action, we get

$$\begin{aligned}\sigma_\mu(a * b, x) &= \sigma(\mu(a * b), x)\\ &= \sigma(\mu(a) * \mu(b), x)\\ &= \sigma(\mu(a), x) * \sigma(\mu(b), x)\\ &= \sigma_\mu(a, x) * \sigma_\mu(b, x)\end{aligned}$$

Therefore σ_μ is a complete action of X on S. To prove the converse, let us suppose that μ is a surjective mapping. Assume that σ_μ is a complete action of X on S. Then, we get

(C1) Let $a \in X$. Since μ is surjective, there exists $b \in X$ such that $\mu(b) = a$. Now, $\sigma(a, 1) = \sigma(\mu(b), 1)) = \sigma_\mu(b, 1) = 1$.

(C2) For any $x \in S$, we get $\sigma(1, x) = \sigma(\mu(1), x) = \sigma_\mu(1, x) = x$.

(C3) Let $a \in X$ and $x, y \in S$. Since μ is surjective, there exist $b \in X$ such that $\mu(b) = a$. Since σ_μ is an action of X on S, we get

$$\begin{aligned}\sigma(a, x * y) &= \sigma(\mu(b), x * y)\\ &= \sigma_\mu(b, x * y)\\ &= \sigma_\mu(b, x) * \sigma_\mu(b, y)\\ &= \sigma(\mu(b), x) * \sigma(\mu(b), y)\\ &= \sigma(a, x) * \sigma(a, y)\end{aligned}$$

(C4) Let $a, b \in X$ and $x \in S$. Since μ is surjective, there exist $a_0, b_0 \in X$ such that $\mu(a_0) = a$ and $\mu(b_0) = b$. Since σ_μ is an action,

$$\begin{aligned}\sigma(a, \sigma(b, x)) &= \sigma(\mu(a_0), \sigma(\mu(b_0), x)\\ &= \sigma(\mu(a_0), \sigma_\mu(b_0, x))\\ &= \sigma_\mu(a_0, \sigma_\mu(b_0, x))\\ &= \sigma_\mu(b_0, \sigma_\mu(a_0, x))\\ &= \sigma_\mu(b_0, \sigma(\mu(a_0), x))\\ &= \sigma(\mu(b_0), \sigma(\mu(a_0), x))\\ &= \sigma(b, \sigma(a, x))\end{aligned}$$

(C5) Let $a, b \in X$ and $x \in S$. Since μ is a surjective mapping, there exist $a_0, b_0 \in X$ such that $\mu(a_0) = a$ and $\mu(b_0) = b$. Since σ_μ is an action of X on S, we get

$$\begin{aligned}\sigma(a * b, x) &= \sigma(\mu(a_0) * \mu(b_0), x)\\ &= \sigma(\mu(a_0 * b_0), x)\\ &= \sigma_\mu(a_0 * b_0, x)\\ &= \sigma_\mu(a_0, x) * \sigma_\mu(b_0, x)\\ &= \sigma(\mu(a_0), x) * \sigma(\mu(b_0), x)\\ &= \sigma(a, x) * \sigma(b, x)\end{aligned}$$

Therefore σ is a complete action of X on S. □

Definition 2.7. Let $(X, *, 1)$ be a BE-algebra and S be a subalgebra of X. Two actions σ_i and σ_j of X on S are said to be *permutable* if for all $a, b \in X$ and $x \in S$, the following property holds:

$$\sigma_i(a, \sigma_j(b, x)) \ = \ \sigma_i(b, \sigma_j(a, x)).$$

Example 2.8. Let $X = \{1, a, b, c\}$ be the given set. Define a binary operation $*$ on X as given in the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	a	1	c
c	1	a	1	1

Clearly $(X, *, 1)$ is a BE-algebra. Consider the subalgebra $S = \{b, 1\}$. Define two mappings σ_1 and σ_2 from $X \times S$ into S as given by

$$\sigma_1(x, y) \;=\; x * y \qquad \sigma_2(x, y) \;=\; y$$

for all $x \in X$ and $y \in S$. Clearly σ_1 and σ_2 are actions of X on S. It can be easily verified that σ_1 and σ_2 are permutable actions of X on S.

Proposition 2.9. *Let X be a BE-algebra and S is a subalgebra of X. Let σ_1 and σ_2 be two permutable actions of X on S. Define the composition of the actions σ_1 and σ_2 as*

$$(\sigma_1 \circ \sigma_2)(a, x) \;=\; \sigma_1(a, \sigma_2(a, x))$$

for all $a \in X$ and $x \in S$. Then $\sigma_1 \circ \sigma_2$ is an action of X on S.

Proof. (C1) and (C2) are clear. To prove (C3), let $a \in X$ and $x, y \in S$. Then, we get the following:

$$\begin{aligned}(\sigma_1 \circ \sigma_2)(a, x * y) &= \sigma_1(a, \sigma_2(a, x * y)) \\ &= \sigma_1(a, \sigma_2(a, x) * \sigma_2(a, y)) \\ &= \sigma_1(a, \sigma_2(a, x)) * \sigma_1(a, \sigma_2(a, y)) \\ &= (\sigma_1 \circ \sigma_2)(a, x) * (\sigma_1 \circ \sigma_2)(a, y)\end{aligned}$$

(C4). Let $a, b \in X$ and $x \in S$. Since σ_1 and σ_2 are permutable, we get

$$\begin{aligned}(\sigma_1 \circ \sigma_2)(a, (\sigma_1 \circ \sigma_2)(b, x)) &= (\sigma_1 \circ \sigma_2)(a, \sigma_1(b, \sigma_2(b, x))) \\ &= \sigma_1(a, \sigma_2(a, \sigma_1(b, \sigma_2(b, x)))) \\ &= \sigma_1(a, \sigma_2(b, \sigma_1(a, \sigma_2(b, x)))) \\ &= \sigma_1(b, \sigma_2(a, \sigma_1(a, \sigma_2(b, x)))) \\ &= \sigma_1(b, \sigma_2(a, \sigma_1(b, \sigma_2(a, x)))) \\ &= \sigma_1(b, \sigma_2(b, \sigma_1(a, \sigma_2(a, x)))) \\ &= \sigma_1(b, \sigma_2(b, (\sigma_1 \circ \sigma_2)(a, x))) \\ &= (\sigma_1 \circ \sigma_2)(b, (\sigma_1 \circ \sigma_2)(a, x))\end{aligned}$$

Therefore $\sigma_1 \circ \sigma_2$ is an action of X on S. □

Theorem 2.10. *Let X be a BE-algebra with given ordering $\leq$. Suppose that S is a subalgebra of X and $\Sigma(X)$ denotes the set of all actions of X on S. For any $\sigma_1, \sigma_2 \in \Sigma(X)$, define an ordering $\leq_\sigma$ on the set of all actions of X on S as given by*

$$\sigma_1 \leq_\sigma \sigma_2 \text{ if and only if } \sigma_1(a,x) \leq \sigma_2(a,x)$$

for all $a \in X$ and $x \in S$. Then $\leq_\sigma$ is a BE-ordering on $\Sigma(X)$. Further, if X is commutative, then $\leq_\sigma$ is a partial ordering on $\Sigma(X)$.

Proof. Since S is a subalgebra of X and $\leq$ is a BE-ordering on X, it is clear that $\leq_\sigma$ is reflexive. Further, if X is commutative, then it is transitive. Hence $\leq_\sigma$ is transitive on $\Sigma(X)$. Since X is commutative, we get that $\leq$ is anti-symmetric and hence $\leq_\sigma$ is anti-symmetric on $\Sigma(X)$. Therefore $\leq_\sigma$ is a partial ordering on $\Sigma(X)$. □

Suppose that X is a commutative BE-algebra and S is a subalgebra of X. Let σ_1 and σ_2 be two actions of X on S. Due to the commutativity of the subalgebra S, we get

$$\sigma_1(a,x) \vee \sigma_2(a,x) = (\sigma_2(a,x) * \sigma_1(a,x)) * \sigma_1(a,x)$$

for any $a \in X$ and $x \in S$. Further, we have

$$\sigma_1(a,x) \vee \sigma_2(a,x) = \sigma_2(a,x) \vee \sigma_1(a,x).$$

Lemma 2.11. *Let X be a BE-algebra and S a commutative subalgebra of X. If σ is an action of X on S, then $\sigma(a, x \vee y) = \sigma(a,x) \vee \sigma(a,y)$ for any $a \in X$ and $x, y \in S$.*

Proof. Routine verification. □

Proposition 2.12. *Suppose X is a BE-algebra and S a commutative subalgebra of X. Let σ_1 and σ_2 be two permutable actions of X on S. Define the supremum of the actions σ_1 and σ_2 as given under*

$$(\sigma_1 \sqcup \sigma_2)(a,x) = \sigma_1(a,x) \vee \sigma_2(a,x)$$

for all $a \in X$ and $x \in S$. Then $\sigma_1 \sqcup \sigma_2$ is an action of X on S. Further $\sigma_1 \sqcup \sigma_2$ is a complete action of X on S whenever both σ_1 and σ_2 are complete actions of X on S.

Proof. (C1), (C2) and (C3) can be routinely verified. To prove (C4), let $a \in X$ and $x \in S$. For simplicity of the representation, in the following, we use the notation $\sigma_i(a_x) = \sigma_i(a, x)$ and $\sigma_i(b_x) = \sigma_i(b, x)$ for $i = 1, 2$. Since σ_1 and σ_2 are permutable, we get

$$\begin{aligned}(\sigma_1 \sqcup \sigma_2)(a, (\sigma_1 \sqcup \sigma_2)(b_x)) &= (\sigma_1 \sqcup \sigma_2)(a, \sigma_1(b_x) \vee \sigma_2(b_x)) \\ &= \sigma_1(a, \sigma_1(b_x) \vee \sigma_2(b_x)) \\ &\quad \vee \sigma_2(a, \sigma_1(b_x) \vee \sigma_2(b_x)) \\ &= \sigma_1(a, \sigma_1(b_x)) \vee \sigma_1(a, \sigma_2(b_x))) \\ &\quad \vee \sigma_2(a, \sigma_1(b_x)) \vee \sigma_2(a, \sigma_2(b_x)) \\ &= \sigma_1(b, \sigma_1(a_x)) \vee \sigma_1(b, \sigma_2(a_x))) \\ &\quad \vee \sigma_2(b, \sigma_1(a_x)) \vee \sigma_2(b, \sigma_2(a_x)) \\ &= \sigma_1(b, \sigma_1(a_x) \vee \sigma_2(a_x)) \vee \sigma_2(b, \sigma_1(a_x) \vee \sigma_2(a_x)) \\ &= \sigma_1(b, (\sigma_1 \sqcup \sigma_2)(a_x)) \vee \sigma_2(b, (\sigma_1 \sqcup \sigma_2)(a_x)) \\ &= (\sigma_1 \sqcup \sigma_2)(b, (\sigma_1 \sqcup \sigma_2)(a_x))\end{aligned}$$

Hence $\sigma_1 \sqcup \sigma_2$ is an action of X on S. Further, suppose that σ_1, σ_2 are complete actions of X on S. It can be routinely verified that $\sigma_1 \sqcup \sigma_2$ is a complete action of X on S. □

The following theorem is a direct consequence of the above results.

Theorem 2.13. *Suppose $(X, *, 1)$ is a BE-algebra and S a commutative subalgebra of X. Let $\Sigma(X)$ be the set of all permutable actions of X on S. Then $(\Sigma(X), \sqcup)$ is a semi-lattice with partial ordering $\leq_\sigma$. Therefore $(\Sigma(X), \leq_\sigma)$ is a partially order set.*

3. Cut-homomorphisms

In this section, the concept of subcuts of subalgebras of a BE-algebra is introduced. The notion of cut-homomorphisms is introduced in BE-algebras. It is proved that the collection of all idempotent cut-homomorphism forms an upper semi-lattice.

Definition 3.1. Let S be a subalgebra of a BE-algebra X. Suppose (σ, S) is a cut of a BE-algebra X. A subalgebra S' of S is said to a *subcut* of S if S' is closed under action by elements of X. In this case, we simply call (σ, S') a subcut of (σ, S).

Definition 3.2. Suppose that S is a subalgebra of a BE-algebra X and (σ, S) is a cut of X. A subcut (σ, F) of the cut (σ, S) is called a *filter* of (σ, S) if it satisfies the following properties:

(1) F is a filter of S,
(2) $\sigma(x,y) \leq x$ for all $x \in S$ and $1 \neq y \in F$.

In this case, we simply call that F is a filter of the cut (σ, S). For any subalgebra S of X, it is clear that $\{1\}$ is a filter of any cut (σ, S). A filter F of a cut (σ, S) is called *proper* if $F \neq S$.

Example 3.3. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

It can be routinely seen that $(X, *, 1)$ is a BE-algebra. Consider the subalgebra $S = \{a, b, 1\}$. Define a mapping $\sigma : X \times S \longrightarrow S$ by $\sigma(a, 1) = 1$, $\sigma(1, x) = x$ for all $a \in X, x \in S$, and

$$\sigma(c, a) = a, \ \sigma(c, b) = b \ \text{ and } \ \sigma(c, c) = 1$$

Note that σ is an action of X on S. Consider the set $F = \{1, a\}$. Then, it can be easily verified that F is a filter of the cut (σ, S).

Proposition 3.4. *Let S be a subalgebra of a BE-algebra $(X, *, 1)$ and σ is an action of X on S. Then the set-intersection of any two filters of the cut (σ, S) is again a filter of (σ, S).*

Proof. Given that σ an action of X on S. Let (σ, F_1) and (σ, F_2) be two filters of the cut (σ, S). Clearly $F_1 \cap F_2$ is a filter of S. Therefore the intersection of F_1 and F_2 is a filter of the cut (σ, S). □

The following corollaries are direct consequences of Proposition 3.4.

Corollary 3.5. *Let S be a subalgebra of a BE-algebra X and σ is an action of X on S. Suppose $\{F_\alpha\}_{\alpha \in \Delta}$ is an indexed family of filters of the cut (σ, S). Then the set intersection $\bigcap\limits_{\alpha \in \Delta} F_\alpha$ is a filter of (σ, S).*

Corollary 3.6. *Let S be a subalgebra of a BE-algebra X. If σ is an action of X on S, then the intersection of all filters of (σ, S) is $\{1\}$.*

Proposition 3.7. *Let S be a commutative subalgebra of a BE-algebra X. Suppose σ_1 and σ_2 are two actions of X on S. If F is filter of both the cuts (σ_1, S) and (σ_2, S), then F is a filter of $(\sigma_1 \sqcup \sigma_2, S)$.*

Proof. By Proposition 2.12, $\sigma_1 \sqcup \sigma_2$ is an action of X on S. Let $x \in S$ and $1 \neq y \in F$. Since F is filter of (σ_1, S) and (σ_2, S), we get

$$\sigma_1(x, y) \leq x \text{ and } \sigma_2(x, y) \leq x.$$

Hence $(\sigma_1 \sqcup \sigma_2)(x, y) = \sigma_1(x, y) \vee \sigma_2(x, y) \leq x$ for all $x \in S$ and $1 \neq y \in F$. Therefore F is a filter of the cut $(\sigma_1 \sqcup \sigma_2, S)$. □

Theorem 3.8. *Let S be a commutative subalgebra of a BE-algebra X. Suppose $\{\sigma_i\}_{i\in\Delta}$ is an indexed family of actions of X on S. For any filter F of S, the set $\{(\sigma_i, F)\}_{i\in\Delta}$ of all subcuts forms a partially ordered semi-lattice with respect to the operation $\sqcup$.*

Proof. By Proposition 3.7 and Theorem 2.10, it follows. □

Definition 3.9. Let F and G be two filters of a BE-algebra X. Suppose σ and μ are actions of X on F and G respectively. For the cuts (σ, F) and (μ, G) of X, the mapping $f : (\sigma, F) \longrightarrow (\mu, G)$ is called a *cut-homomorphism* if it satisfies the following properties:

(H1) $f(x * y) = f(x) * f(y)$ for all $x, y \in F$,
(H2) $f(\sigma(a, x)) = \mu(a, f(x))$ for all $a \in X$ and $x \in F$.

A bijective cut-homomorphism is called a *cut-isomorphism*. A cut-homomorphism from a cut (σ, F) into itself is called a *cut-endomorphism*.

Proposition 3.10. *The composition of any two cut-homomorphisms of a BE-algebra is again a cut-homomorphism.*

Proof. Let F, G and H be three filters of a BE-algebra X. Suppose σ, μ and δ be three actions of X on F, G and H respectively. Let $f : (\sigma, F) \longrightarrow (\mu, G)$ and $g : (\mu, G) \longrightarrow (\delta, H)$ be two cut-homomorphisms. Clearly $g \circ f : (\sigma, F) \longrightarrow (\delta, H)$ is a cut-homomorphism. □

Definition 3.11. Let (σ, F) be a cut of a BE-algebra X. For any $a \in X$, define a self-map $\sigma_a : (\sigma, F) \longrightarrow (\sigma, F)$ by $\sigma_a(x) = \sigma(a, x)$ for all $x \in F$.

Proposition 3.12. *Let (σ, F) be a cut of a BE-algebra X. For any $a \in X$, the mapping $\sigma_a : (\sigma, F) \longrightarrow (\sigma, F)$ defined above is a cut-endomorphism.*

Proof. Let $a \in X$. For any $x, y \in F$, we get

$$\sigma_a(x * y) = \sigma(a, x * y) = \sigma(a, x) * \sigma(a, y) = \sigma_a(x) * \sigma_a(y).$$

For any $b \in X$ and $x \in F$, we get

$$\sigma_a(\sigma(b, x)) = \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x)) = \sigma(b, \sigma_a(x)).$$

Therefore σ_a is a cut-endomorphism. □

Lemma 3.13. *Let (σ, S) be a cut of a BE-algebra X. For any $a \in X$,*

(1) *for any $x \in (\sigma, S)$, $\sigma_a(x) = x * \sigma_a(x)$,*
(2) *for any $x \in (\sigma, S)$, $x = \sigma_a(x) * x$ whenever σ is complete,*
(3) *if F is a filter of (σ, S), then $(\sigma, \sigma_a(F))$ is a subcut of (σ, S).*

Proof. (1) Let $a \in X$. For any $x \in (\sigma, S)$, we get

$$\sigma_a(x) = \sigma(a, x) = \sigma(1 * a, x) = \sigma(1, x) * \sigma(a, x) = x * \sigma_a(x)$$

(2) Let $a \in X$ and σ is complete. For any $x \in (\sigma, S)$, we get

$$x = \sigma(1, x) = \sigma(a * 1, x) = \sigma(a, x) * \sigma(1, x) = \sigma_a(x) * x$$

(3) Let F be a filter of (σ, S). Let $\sigma_a(x), \sigma_a(y) \in \sigma_a(F)$ where $x, y \in F$. Then

$$\sigma_a(x) * \sigma_a(y) = \sigma(a, x) * \sigma(a, y) = \sigma(a, x * y) = \sigma_a(x * y) \in \sigma_a(F)$$

because of $x * y \in F$. Therefore $\sigma_a(F)$ is a subalgebra of X, for any $b \in X$ and $\sigma_a(x) \in \sigma_a(F)$. Then $x \in F$ and

$$\sigma(b, \sigma_a(x)) = \sigma(b, \sigma(a, x)) = \sigma(a, \sigma(b, x)) = \sigma_a(\sigma(b, x)) \in \sigma_a(F)$$

since $\sigma(b, x) \in F$. Hence $\sigma : X \times \sigma_a(F) \longrightarrow \sigma_a(F)$ is an action of X on $\sigma_a(F)$. Therefore $(\sigma, \sigma_a(F))$ is a subcut of (σ, S). □

Lemma 3.14. *Let (σ, S) be a cut of a BE-algebra X. For any $a, b \in X$,*

(1) *σ_1 is the identity map on (σ, S),*
(2) *σ_a is order preserving on (σ, S),*
(3) *$\sigma_a \circ \sigma_b = \sigma_b \circ \sigma_a$.*

Proof. (1) For any $x \in S$, we get that $\sigma_1(x) = \sigma(1, x) = x$.

(2) Let $x, y \in S$ and $x \leq y$. Then

$$\sigma_a(x) * \sigma_a(y) = \sigma_a(x * y) = \sigma_a(1) = \sigma(a, 1) = 1.$$

Hence $\sigma_a(x) \leq \sigma_a(y)$. Therefore σ_a is order preserving.

(3) For any $x \in S$, we have

$$\begin{aligned}(\sigma_a \circ \sigma_b)(x) &= \sigma_a(\sigma_b(x)) \\ &= \sigma_a(\sigma(b, x)) \\ &= \sigma(a, \sigma(b, x)) \\ &= \sigma(b, \sigma(a, x)) \\ &= \sigma(b, \sigma_a(x)) \\ &= \sigma_b(\sigma_a(x)) \\ &= (\sigma_b \circ \sigma_a)(x).\end{aligned}$$

Therefore $\sigma_a \circ \sigma_b = \sigma_a \circ \sigma_b$. □

Theorem 3.15. *Let (σ, S) be a cut of a BE-algebra X. Then the collection $\mathcal{M} = \{\sigma_a \mid a \in X\}$ is a BE-algebra and there is an onto homomorphism from X into $\mathcal{M}$.*

Proof. For any $a, b \in X$, define a binary operation $\odot$ on $\mathcal{M}$ as $(\sigma_a \odot \sigma_b)(x) = \sigma_{a*b}(x)$ for all $x \in S$. For any $a \in X$, we have

$$(\sigma_a \odot \sigma_1)(x) = \sigma_{a*1}(x) = \sigma_1(x)$$

for all $x \in S$. Hence $\sigma_a \odot \sigma_1 = \sigma_1$. Again, we have

$$(\sigma_1 \odot \sigma_a)(x) = \sigma_{1*a}(x) = \sigma_a(x)$$

for all $x \in S$. Hence $\sigma_1 \odot \sigma_a = \sigma_a$. Also $(\sigma_a \odot \sigma_a)(x) = \sigma_{a*a}(x) = \sigma_1(x)$. Hence $\sigma_a \odot \sigma_a = \sigma_1$. Similarly, we can prove that $\sigma_a \odot (\sigma_b \odot \sigma_c) = \sigma_b \odot (\sigma_a \odot \sigma_c)$ for any $\sigma_a, \sigma_b, \sigma_c \in \mathcal{M}$. Therefore $(\mathcal{M}, \odot, \sigma_1)$ is a BE-algebra where σ_1 is the top element.

Since $(\mathcal{M}, \odot, \sigma_1)$ is a BE-algebra, define a mapping $\Omega : X \longrightarrow \mathcal{M}$ as $\Omega(a) = \sigma_a$ for all $a \in X$. Clearly Ω is well-defined. For any $a, b \in X$, we get $\Omega(a * b) = \sigma_{a*b} = \sigma_a \odot \sigma_b = \Omega(a) \odot \Omega(b)$. Therefore Ω is a homomorphism. Let $\sigma_a \in \mathcal{M}$. For this $a \in X$, it is clear that $\Omega(a) = \sigma_a$. Therefore Ω is an onto homomorphism. □

In a self-distributive BE-algebra X with subalgebra S, the action $\sigma : X \times S \longrightarrow S$ defined by $\sigma(a, x) = a * x$ for all $a \in X$ and $x \in S$ is observed as the idempotent action.

Theorem 3.16. *Let σ be an idempotent action of a BE-algebra X on its subalgebra S. Then the collection $K' = \{\sigma_a \mid a \in X\}$ is an upper semi-lattice with top element σ_1.*

Proof. For any $a, b \in X$, define a binary operation $\wedge$ on K' as $\sigma_a \wedge \sigma_b = \sigma_a \circ \sigma_b$. For any $a \in X$, we have

$$\begin{aligned}(\sigma_a \wedge \sigma_a)(x) &= (\sigma_a \circ \sigma_a)(x)\\ &= \sigma_a(\sigma_a(x))\\ &= \sigma_a(\sigma(a, x))\\ &= \sigma(a, \sigma(a, x))\\ &= \sigma(a, x)\\ &= \sigma_a(x)\end{aligned}$$

for all $x \in S$. Hence $\sigma_a \wedge \sigma_a = \sigma_a$. Let $a, b \in X$. By Lemma 3.14(3), we get $\sigma_a \wedge \sigma_b = \sigma_a \circ \sigma_b = \sigma_b \circ \sigma_a = \sigma_b \wedge \sigma_a$. Since the composition of self mappings is associative, it is concluded that $(K', \wedge)$ is a semi-lattice. For any $a \in X$ and $x \in S$, we get

$$(\sigma_a \wedge \sigma_1)(x) = (\sigma_a \circ \sigma_1)(x) = \sigma_a(\sigma_1(x)) = \sigma_a(x).$$

Hence $\sigma_a \wedge \sigma_1 = \sigma_a$. Similarly, $\sigma_1 \wedge \sigma_a = \sigma_a$. Therefore $(K', \wedge)$ is a semi-lattice where σ_1 as the top element. □

4. Fixed points of cut-endomorphisms

In this section, the concept of fixed points of a cut-endomorphism is introduced in BE-algebras. A necessary and sufficient condition is given for a cut-endomorphism to have a fixed point. Properties of fixed points and images of a cut-endomorphism are investigated.

Definition 4.1. Let S be a subalgebra of a BE-algebra X. Suppose (σ, S) be a cut of X and $f : (\sigma, S) \longrightarrow (\sigma, S)$ is a cut-endomorphism. An element $x \in S$ is called a *fixed point* of f if $f(x) = x$.

Example 4.2. Consider the subalgebra $S = \{a, b, 1\}$ of the BE-algebra X which is given in Example 3.3. Define a self-mapping $f : S \longrightarrow S$ as given by

$$f(x) = \begin{cases} 1 & \text{if } x = 1\\ a & \text{otherwise}\end{cases}$$

It can be easily noticed that f is a cut-endomorphism on S. Under this self mapping f, the elements 1 and a of the subalgebra S are fixed points but not the element b because of $f(b) = a$.

Theorem 4.3. *Let S be a subalgebra of BE-algebra X and (σ, S) be a cut of X. For any $a \in X$, the cut-endomorphism σ_a has a fixed point in S if and only if there exists a constant mapping $g : S \longrightarrow S$ such that $g(\sigma(a, x)) = \sigma(a, g(x))$ for all $x \in S$.*

Proof. Assume that σ_a has a fixed point, say c. For this $c \in S$, define a constant map $g : S \longrightarrow S$ by $g(x) = c$ for all $x \in S$. Then, we get $g(\sigma(a, x)) = c$ and $\sigma(a, g(x)) = \sigma(a, c) = \sigma_a(c) = c$ for all $x \in S$. Therefore $g(\sigma(a, x)) = \sigma(a, g(x))$ for all $x \in S$.

Conversely, assume that there exists $c \in S$ and a constant mapping $h : S \longrightarrow S$ such that $h(x) = c$ and $h(\sigma(a, x)) = \sigma(a, h(x))$ for all $x \in S$. Hence $\sigma_a(c) = \sigma(a, c) = \sigma(a, h(x)) = h(\sigma(a, x)) = c$. Therefore c is a fixed point of σ_a. □

Proposition 4.4. *Let (σ, S) be a cut of a BE-algebra X. For any $a \in X$, the class of all fixed points of σ_a given by*

$$Fix(\sigma_a) = \{x \in S \mid \sigma_a(x) = x\}$$

is a subcut of (σ, S).

Proof. Let $a \in X$. Since $\sigma_a(1) = 1$, we get $1 \in Fix(\sigma_a)$. Let $x, y \in Fix(\sigma_a)$. Then, we get $\sigma_a(x) = x$ and $\sigma_a(y) = y$. Hence

$$\sigma_a(x * y) = \sigma(a, x * y) = \sigma(a, x) * \sigma(a, y) = \sigma_a(x) * \sigma_a(y) = x * y.$$

Thus $x * y \in Fix(\sigma_a)$. Therefore $Fix(\sigma_a)$ is a uni-subalgebra of (σ, S). For any $b \in X$ and $x \in Fix(\sigma_a)$. Then $\sigma(a, x) = \sigma_a(x) = x$. Hence

$$\sigma_a(\sigma(b, x)) = \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x)) = \sigma(b, x).$$

Hence $\sigma(b, x) \in Fix(\sigma_a)$. Thus $\sigma : X \times Fix(\sigma_a) \longrightarrow Fix(\sigma_a)$ is an action of X on $Fix(\sigma_a)$. Therefore $(\sigma, Fix(\sigma_a))$ is a subcut of (σ, S). □

Let (σ, S) be a cut of a BE-algebra X where S is a subalgebra of X. For any cut-homomorphism σ_a, its image is given as

$$Im(\sigma_a) = \{\sigma_a(x) \mid x \in S\}.$$

Proposition 4.5. *Let (σ, S) be a cut of a BE-algebra X. For any $a \in X$, $Im(\sigma_a)$ is a subalgebra of S.*

Proof. Clearly $Im(\sigma_a)$ is a subsets of S and $1 \in Im(\sigma_a)$. Let $x, y \in Im(\sigma_a)$. Then $x = \sigma_a(x')$ and $y = \sigma_a(y')$ for some $x', y' \in S$. Now

$$x * y = \sigma_a(x') * \sigma_a(y') = \sigma_a(x' * y').$$

Since $x' * y' \in S$, we get $x * y \in Im(\sigma_a)$. Therefore $Im(\sigma_a)$ is a subalgebra of S. □

Lemma 4.6. *Let (σ, S) be a cut of a BE-algebra X. Then $Fix(\sigma_1) = S$. Let $a \in X$. If σ is idempotent, then we have*

(1) *$\sigma_a(x) \in Fix(\sigma_a)$ for all $x \in S$,*
(2) *$\sigma_a(x) \in Im(\sigma_a)$ for all $x \in S$.*

Proof. Let $a \in X$. Clearly $Fix(\sigma_1) \subseteq S$. For any $x \in S$, we get that $\sigma_1(x) = \sigma(1, x) = x$. Hence $x \in Fix(\sigma_1)$. Therefore $S \subseteq Fix(\sigma_1)$. The remaining part is clear. □

Theorem 4.7. *Let σ be an idempotent action of a BE-algebra X on its subalgebra S. For any $a, b \in X$, the following are equivalent:*

(1) *$\sigma_a = \sigma_b$;*
(2) *$Im(\sigma_a) = Im(\sigma_b)$;*
(3) *$Fix(\sigma_a) = Fix(\sigma_b)$.*

Proof. (1) $\Rightarrow$ (2): It is obvious.

(2) $\Rightarrow$ (3): Assume that $Im(\sigma_a) = Im(\sigma_b)$. Let $x \in Fix(\sigma_a)$. Then, we get $x = \sigma_a(x) \in Im(\sigma_a) = Im(\sigma_b)$. Hence $x = \sigma_b(y)$ for some $y \in S$. Since σ is idempotent, we get $\sigma_b(x) = \sigma_b(\sigma_b(y)) = \sigma_b(y) = x$. Thus $x \in Fix(\sigma_b)$. Therefore $Fix(\sigma_a) \subseteq Fix(\sigma_b)$. Similarly, we can obtain that $Fix(\sigma_b) \subseteq Fix(\sigma_a)$. Therefore $Fix(\sigma_a) = Fix(\sigma_b)$.

(3) $\Rightarrow$ (1): Assume that $Fix(\sigma_a) = Fix(\sigma_b)$. Let $x \in S$ be an arbitrary element. Since $\sigma_a(x) \in Fix(\sigma_a) = Fix(\sigma_b)$, we get

$$\sigma_b(\sigma_a(x)) = \sigma_a(x).$$

Also we have $\sigma_b(x) \in Fix(\sigma_b) = Fix(\sigma_a)$. Hence $\sigma_a(\sigma_b(x)) = \sigma_b(x)$. Thus, it yields

$$\sigma_a(x) = \sigma_b(\sigma_a(x)) = (\sigma_b \circ \sigma_a)(x) = (\sigma_a \circ \sigma_b)(x) = \sigma_a(\sigma_b(x)) = \sigma_b(x).$$

Hence σ_a and σ_b are equal in the sense of mappings. Thus $\sigma_a = \sigma_b$. □

Theorem 4.8. *Let σ and μ be two idempotent actions of a BE-algebra X on its subalgebra S. For any $a \in X$, the following are equivalent:*

(1) *$\sigma_a = \mu_a$;*
(2) *$Im(\sigma_a) = Im(\mu_a)$;*
(3) *$Fix(\sigma_a) = Fix(\mu_a)$.*

Proof. (1) ⇒ (2): It is obvious.

(2) ⇒ (3): Assume that $Im(\sigma_a) = Im(\mu_a)$. Let $x \in Fix(\sigma_a)$. Then, we get

$$x = \sigma_a(x) \in Im(\sigma_a) = Im(\mu_a).$$

Hence $x = \mu_a(y)$ for some $y \in S$. Now $\mu_a(x) = \mu_a(\mu_a(y)) = \mu_a(y) = x$. Thus $x \in Fix(\mu_a)$. Therefore $Fix(\sigma_a) \subseteq Fix(\mu_a)$. Similarly, we can obtain that $Fix(\mu_a) \subseteq Fix(\sigma_a)$. Therefore $Fix(\sigma_a) = Fix(\mu_a)$.

(3) ⇒ (1): Assume that $Fix(\sigma_a) = Fix(\mu_a)$. Let $x \in S$. Since $\sigma_a(x) \in Fix(\sigma_a) = Fix(\mu_a)$, we get $\mu_a(\sigma_a(x)) = \sigma_a(x)$. Also we have $\mu_a(x) \in Fix(\mu_a) = Fix(\sigma_a)$. Hence $\sigma_a(\mu_a(x)) = \mu_a(x)$. Thus, it yields

$$\sigma_a(x) = \mu_a(\sigma_a(x)) = (\mu_a \circ \sigma_a)(x) = (\sigma_a \circ \mu_a)(x) = \sigma_a(\mu_a(x)) = \mu_a(x).$$

Hence σ_a and μ_a are equal in the sense of mappings. Thus $\sigma_a = \mu_a$. □

Theorem 4.9. *Let σ be an action of a BE-algebra $(X, *, 1)$ on its subalgebra S. Then the collection $\mathcal{K} = \{Fix(\sigma_a) \mid a \in X\}$ forms a BE-algebra with top element S. Hence there exists an onto homomorphism from $\mathcal{M}$ into $\mathcal{K}$.*

Proof. For any $Fix(\sigma_a), Fix(\sigma_b) \in \mathcal{K}$ where $a, b \in X$, define an operation $\circledast$ on $\mathcal{K}$ by

$$Fix(\sigma_a) \circledast Fix(\sigma_b) = Fix(\sigma_{a*b})$$

By Lemma 4.6(1), we have $Fix(\sigma_1) = S$. It can be routinely verified that $(\mathcal{K}, \circledast, Fix(\sigma_1))$ is a BE-algebra. For any $a \in X$, define

$$g : \mathcal{M} \longrightarrow \mathcal{K}$$

by $g(\sigma_a) = Fix(\sigma_a)$. Clearly g is well-defined and onto. For any $\sigma_a, \sigma_b \in \mathcal{M}$, we get

$$g(\sigma_a \odot \sigma_b) = g(\sigma_{a*b}) = Fix(\sigma_{a*b}) = Fix(\sigma_a) \circledast Fix(\sigma_b) = g(\sigma_a) \circledast g(\sigma_b).$$

Therefore g is a homomorphism. □

Corollary 4.10. *Let σ be an action of a BE-algebra X on its subalgebra S. Then there exists an onto homomorphism from X into $\mathcal{K}$.*

Proof. By Theorem 3.15, Ω is a onto homomorphism from X into $\mathcal{M}$. By Theorem 4.9, we have g is an onto homomorphism from $\mathcal{M}$ into $\mathcal{K}$. Hence $g \circ \Omega$ is the required onto homomorphism from X into $\mathcal{K}$. □

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