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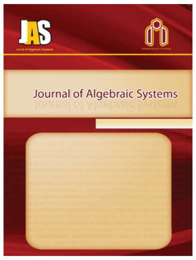

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ON SKEW GENERALIZED TRIANGULAR MATRIX RINGS

M. HABIBI* AND K. PAYKAN

ABSTRACT. In this article, we study skew monoid rings in which the monoid used in their structure is a quotient of a free monoid. We study annihilator conditions of them and describe conditions for transferring some properties from the base ring R to these extensions. Interesting examples are provided for properties that are not transferred from R to these extensions.

1. Introduction

Let F be a free monoid with identity element e generated by $U = \{u_1, \ldots, u_t\}$. We add 0 to F and consider M as a factor of F which setting certain monomials in U to 0. Suppose that n is a minimum natural number such that $\alpha^n = 0$, for any $e \neq \alpha \in M$. Let R be a unitary ring with an endomorphism σ. The first Author and Moussavi in [7] studied the skew monoid ring

$$R[M;\sigma] = \{\sum_{g\in M} r_g g \ : \ r_g \in R\}$$

with usual addition and multiplication which is skewed by the following rule:

$$u_i r = \sigma(r) u_i \qquad \forall\, i = 1, \ldots, t$$

Thus we can display any member α of $R[M;\sigma]$ as follows:

$$\alpha = ae + \sum\nolimits_{1\leq k\leq t} a_k u_k + \sum\nolimits_{1\leq k_1,k_2\leq t} a_{k_1k_2} u_{k_1} u_{k_2}$$
$$+ \cdots + \sum\nolimits_{1\leq k_1,\ldots,k_{n-1}\leq t} a_{k_1\cdots k_{n-1}} u_{k_1} \cdots u_{k_{n-1}}.$$

In [7], the authors studied on properties that the ring $R[M;\sigma]$ inherits from the base ring R. They expressed prime and maximal one-sided ideals of these rings in terms of R and by using this characterization, they described various radicals of $R[M;\sigma]$ in terms of R. Next, in [18] the authors continue studying on these rings.

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One of the techniques in making non-commutative rings is to skew the rings by the endomorphisms. With the help of this method, the authors [5] introduced a *skew triangular matrix ring* as a set of all triangular matrices with usual addition and multiplication which is skewed by the following rule:

$$E_{ij}r = \sigma^{j-i}(r)E_{ij}, \qquad \forall\, 1 \leq i \leq j \leq n$$

where E_{ij} is the elementary matrix. They denoted it by $T_n(R,\sigma)$. In fact for each (a_{ij}) and (b_{ij}) in $T_n(R,\sigma)$, we have

$$(a_{ij})(b_{ij}) = (c_{ij}) \qquad c_{ij} = a_{ii}b_{ij} + a_{i,i+1}\sigma(b_{i+1,j}) + \cdots + a_{ij}\sigma^{j-i}(b_{jj}),$$

for each $i \leq j$. Just note that to define the well-defined multiplication, we need $\sigma(1) = 1$. A variety of subrings of $T_n(R,\sigma)$, like $S(R,n,\sigma)$, $A(R,n,\sigma)$, $B(R,n,\sigma)$ and $T(R,n,\sigma)$ were studied in several articles. (See Section 2 for details of these rings.) The common feature of all these subrings is that the elements have a constant main diagonal. Note that in case $n = 2$ and $\sigma = id_R$, the ring $T(R,n,\sigma)$ is the trivial extension $T(R,R)$.

The interesting thing here is that all mentioned subrings can be viewed as special cases of the above skew monoid ring construction. Henceforth we call $R[M;\sigma]$ a *skew generalized triangular matrix ring*. Although this structure is much wider than the skew triangular matrix rings, but any result on skew generalized triangular matrix rings has its counterpart for each of these subrings and so this property makes skew generalized triangular matrix rings a useful tool for unifying results on the ring extensions listed above. Indeed, this structure is important in two ways. The first is that as an algebraic structure it can be studied and recognized, and to be effective in producing a class of examples with the properties that studied in researches; and the second is due to the handling of the famous subrings of the skew triangular matrix rings, as a special case, to integrate the results about them.

This article is divided into three parts. In Section 2, we present the famous examples of the skew generalized triangular matrix rings and some well-known results presented about them so far. In Section 3, we state a sufficient condition for the skew generalized triangular matrix ring to be Armendariz for the case $n = 3$ and provide a counterexample for $n \geq 3$. In Section 4, we study the annihilator conditions of skew generalized triangular matrix rings and prove that two properties of being 2-primal and NI are transferred from the base ring to the skew generalized triangular matrix ring, and vice versa. Also, by applying restrictions on the base ring, we express a sufficient condition for the equivalence of being ideal-symmetric and reflexive for two rings R and $R[M;\sigma]$.

2. Examples of skew generalized triangular matrix rings

We start this section by mentioning the subrings of the skew triangular matrix ring $T_n(R,\sigma)$ that are introduced in [8].

1: The subring of the triangular matrices with constant main diagonal is denoted by $S(R,n,\sigma)$

2: The subring of the following triangular matrices is denoted by $A(R,n,\sigma)$.

$$\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=1}^{n-j+1} a_j E_{i,i+j-1} + \sum_{j=\lfloor \frac{n}{2} \rfloor+1}^{n} \sum_{i=1}^{n-j+1} a_{i,i+j-1} E_{i,i+j-1};$$

For example:

$$A(R,3,\sigma) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a,b,c,d \in R \right\};$$

$$A(R,4,\sigma) = \left\{ \begin{pmatrix} a_1 & a_2 & a & b \\ 0 & a_1 & a_2 & c \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_1 \end{pmatrix} \mid a_1,a_2,a,b,c \in R \right\}.$$

3: The subring of the following triangular matrices is denoted by $B(R,n,\sigma)$.

$$A + rE_{1k} \qquad A \in A(R,n,\sigma) \text{ and } r \in R \qquad n = 2k \geq 4$$

4: The subring of the triangular matrices with constant diagonals is denoted by $T(R,n,\sigma)$.

It is well-known that $T(R,n,\sigma) \cong R[x;\sigma]/\langle x^n \rangle$, where $R[x;\sigma]$ is the skew polynomial ring with multiplication subject to the condition $xr = \sigma(r)x$ for each $r \in R$, and $\langle x^n \rangle$ is the ideal generated by x^n.

In the following, some of the obtained results regarding these subrings in the previous works, which are generalized in this article, are stated.

1: [1, Theorem 5] A ring R is reduced if and only if $T(R,n,id_R)$ is Armendariz.

2: [19, Proposition 2.5] If R is a reduced ring, then the trivial extension $T(R,R)$ is Armendariz.

3: [11, Proposition 2] If R is a reduced ring, then $S(R,3,id_R)$ is Armendariz.

4: [11, Example 3] $S(R,n,id_R)$ is not Armendariz, for $n \geq 4$.

5: [3, Theorem 2.8] If R is a semiprime ring, then $T(R, n, id_R)$ is ideal-symmetric.

As we promised, in the following example, we show that all of the above subrings can be examples of the skew generalized triangular matrix ring $R[M;\sigma]$.

Example 2.1. Let R be a ring with an endomorphism σ with $\sigma(1) = 1$. By changing the monoid M, the following rings are made.

1: Suppose U is a subset of $T_n(R;\sigma)$ consisting of all $E_{i,i+1}$ matrices. Then we have

$$R[M;\sigma] = \left\{ \begin{pmatrix} a & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ 0 & a & a_{2,3} & \cdots & a_{2,n} \\ 0 & 0 & a & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & a \end{pmatrix} \mid a, a_{i,j} \in R \right\}$$

and hence $R[M;\sigma] = S(M, n, \sigma)$.

2: Suppose $U = \{u, E_{1,\lfloor\frac{n}{2}\rfloor+1}, E_{2,\lfloor\frac{n}{2}\rfloor+2}, \ldots, E_{n-\lfloor\frac{n}{2}\rfloor,n}\}$, where

$$u = E_{1,2} + \cdots + E_{n-1,n}$$

Then we have

$$R[M;\sigma] = \left\{ \begin{pmatrix} a_1 & \cdots & a_k & a_{1,k+1} & \cdots & a_{1,n} \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_{n-k,n} \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_k \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & a_1 \end{pmatrix} \mid a_l, a_{i,j} \in R \right\}$$

and hence $R[M;\sigma] = A(R, n, \sigma)$.

3: Suppose $n = 2k$ and $U = \{u, E_{1,k}, E_{1,k+1}, E_{2,k+2}, \ldots, E_{n-k,n}\}$. Then we have

$$R[M;\sigma] = \left\{ \begin{pmatrix} a_1 & \cdots & a_{1,k} & a_{1,k+1} & \cdots & a_{1,n} \\ 0 & \ddots & \ddots & a_k & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_{n-k,n} \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_k \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & a_1 \end{pmatrix} \mid a_l, a_{i,j} \in R \right\}$$

and hence $R[M;\sigma] = B(R, n, \sigma)$.

4: Suppose $U = \{u\}$. Then we have $R[M;\sigma] = T(R, n, \sigma)$.

For additional information see [18]. Now the following question arises.

Question: Is there a known structure in noncommutative algebra that could be a different example of these extensions alongside the above examples?

3. On Armendariz skew generalized triangular matrix rings

It is a natural rule that for studying ring extensions, most of the time, some conditions must be applied to σ. One of these conditions is:

$$a\sigma(a) = 0 \Longrightarrow a = 0 \qquad \forall a \in R \qquad\qquad (\dagger)$$

As a pioneer, this restriction was imposed by Krempa [12] and a ring R with property $(\dagger)$ is called *σ-rigid*. Afterwards, Hashemi and Moussavi [10] imposed the following restriction on σ and introduced σ-compatible rings:

$$a\sigma(b) = 0 \Longleftrightarrow ab = 0 \qquad \forall a, b \in R$$

They proved that R is σ-rigid if and only if R is σ-compatible and reduced [10, Lemma 2.2].

Next, Nasr-Isfahani and Moussavi [17] introduced weakly rigid rings for studying other annihilator conditions of R. A ring R is called *σ-weakly rigid* if for each $a, b \in R$, $aRb = 0$ if and only if $a\sigma(Rb) = 0$. Clearly, prime rings with any automorphism σ are examples of σ-weakly rigid rings. Also R is σ-rigid if and only if R is σ-weakly rigid and reduced. Moreover, this property is well behaved in transferring to the full and upper triangular matrix rings by [17, Theorems 2.6 and 2.7].

Armendariz in [2] proved that if R is a reduced ring, then $f(x)g(x) = 0$ in $R[x]$ implies that $ab = 0$ for all $a \in coef(f(x))$ and $b \in coef(g(x))$. In [19] a ring R with the above property is called *Armendariz*. So reduced rings are Armendariz. But, there are many examples that show that the converse of this result is not true, in general. The most important family of non-reduced Armendariz rings was introduced by Anderson and Camillo [1, Theorem 5]. They showed that $R[x]/(x^n)$ is an Armendariz ring if and only if R is reduced.

In the following, we prove that if $n = 3$, then R is σ-rigid if and only if $R[M;\sigma]$ is Armendariz. First note that we can extend an endomorphism σ

to $\bar{\sigma} : R[x] \to R[x]$ by defining

$$\bar{\sigma}(a_0 + a_1x + \cdots + a_nx^n) = \sigma(a_0) + \sigma(a_1)x + \cdots + \sigma(a_n)x^n.$$

Also, it is not hard to check that R is σ-rigid if and only if $R[x]$ is $\bar{\sigma}$-rigid.

Lemma 3.1. *Let R be an arbitrary ring with an endomorphism σ. Then we have $R[M;\sigma][x] \cong R[x][M;\bar{\sigma}]$.*

Proof. Let $\alpha_0 + \cdots + \alpha_m x^m$ be an element of $R[M;\sigma][x]$ and

$$\begin{aligned}
\alpha_0 &= a^{(0)}e + \sum\nolimits_{1\le k\le t} a_k^{(0)}u_k + \sum\nolimits_{1\le k_1,k_2\le t} a_{k_1k_2}^{(0)}u_{k_1}u_{k_2} \\
&\quad + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} a_{k_1\cdots k_{n-1}}^{(0)}u_{k_1}\cdots u_{k_{n-1}} \\
&\vdots \\
\alpha_m &= a^{(m)}e + \sum\nolimits_{1\le k\le t} a_k^{(m)}u_k + \sum\nolimits_{1\le k_1,k_2\le t} a_{k_1k_2}^{(m)}u_{k_1}u_{k_2} \\
&\quad + \cdots + \sum\nolimits_{1\le k_1,\ldots,k_{n-1}\le t} a_{k_1\cdots k_{n-1}}^{(m)}u_{k_1}\cdots u_{k_{n-1}}
\end{aligned}$$

It is easy to show that $\varphi : R[M;\sigma][x] \to R[x][M;\bar{\sigma}]$ given by

$$\begin{aligned}
\sum\nolimits_{i=0}^{m} \alpha_i x^i \to\ & fe + \sum\nolimits_{1\le k\le t} f_ku_k + \sum\nolimits_{1\le k_1,k_2\le t} f_{k_1k_2}u_{k_1}u_{k_2} \\
&+ \cdots + \sum\nolimits_{1\le k_1,\ldots,k_{n-1}\le t} f_{k_1\cdots k_{n-1}}u_{k_1}\cdots u_{k_{n-1}},
\end{aligned}$$

where

$$\begin{aligned}
f &= a^{(0)} + a^{(1)}x + \cdots + a^{(m)}x^m \\
f_k &= a_k^{(0)} + a_k^{(1)}x + \cdots + a_k^{(m)}x^m && 1 \le k \le t \\
f_{k_1k_2} &= a_{k_1k_2}^{(0)} + a_{k_1k_2}^{(1)}x + \cdots + a_{k_1k_2}^{(m)}x^m && 1 \le k_1, k_2 \le t \\
&\vdots \\
f_{k_1\cdots k_{n-1}} &= a_{k_1\cdots k_{n-1}}^{(0)} + a_{k_1\cdots k_{n-1}}^{(1)}x + \cdots + a_{k_1\cdots k_{n-1}}^{(m)}x^m && 1 \le k_1, \ldots, k_{n-1} \le t
\end{aligned}$$

is an isomorphism and we are done. □

Theorem 3.2. *Let R be an arbitrary ring with an endomorphism σ. If R is σ-rigid, then $R[M;\sigma]$ is Armendariz for any M with $n = 3$.*

Proof. Let R be σ-rigid and $F(x), G(x)$ be two polynomials in $R[M;\sigma][x]$ with $F(x)G(x) = 0$. Consider

$$F(x) = \alpha_0 + \alpha_1x + \cdots + \alpha_rx^r, \qquad G(x) = \beta_0 + \beta_1x + \cdots + \beta_sx^s,$$

where

$$\alpha_i = a^{(i)}e + \sum_{1\le k\le t} a_k^{(i)}u_k + \sum_{1\le k_1,k_2\le t} a_{k_1k_2}^{(i)}u_{k_1}u_{k_2} \qquad \forall\, 0\le i\le r$$

and

$$\beta_j = b^{(j)}e + \sum_{1\le k\le t} b_k^{(j)}u_k + \sum_{1\le k_1,k_2\le t} b_{k_1k_2}^{(j)}u_{k_1}u_{k_2} \qquad \forall\ 0\le j\le s$$

So, according to Lemma 3.1, we have

$$\begin{aligned}&(fe + \sum_{1\le k\le t} f_k u_k + \sum_{1\le k_1,k_2\le t} f_{k_1k_2}u_{k_1}u_{k_2})\\ &\times (ge + \sum_{1\le k\le t} g_k u_k + \sum_{1\le k_1,k_2\le t} g_{k_1k_2}u_{k_1}u_{k_2}) = 0,\end{aligned}$$

where

$$\begin{aligned} f &= a^{(0)} + a^{(1)}x + \cdots + a^{(r)}x^r,\\ f_k &= a_k^{(0)} + a_k^{(1)}x + \cdots + a_k^{(r)}x^r,\\ f_{k_1k_2} &= a_{k_1k_2}^{(0)} + a_{k_1k_2}^{(1)}x + \cdots + a_{k_1k_2}^{(r)}x^r,\\ g &= b^{(0)} + b^{(1)}x + \cdots + b^{(s)}x^s,\\ g_k &= b_k^{(0)} + b_k^{(1)}x + \cdots + b_k^{(s)}x^s,\\ g_{k_1k_2} &= g_{k_1k_2}^{(0)} + g_{k_1k_2}^{(1)}x + \cdots + g_{k_1k_2}^{(s)}x^s.\end{aligned}$$

Therefore, we have the following equations:

(1) $fg = 0$;

(2) $fg_k + f_k\bar{\sigma}(g) = 0 \qquad 1\le k\le t$;

(3) $fg_{k_1k_2} + f_{k_1}\bar{\sigma}(g_{k_2}) + f_{k_1k_2}\bar{\sigma}^2(g) = 0 \qquad 1\le k_1,k_2\le t$.

By multiplying g in Eq. (2) from the left side, we obtain

$$gfg_k + gf_k\bar{\sigma}(g) = 0.$$

Also, Eq. (1) implies that $gf = 0$, since $R[x]$ is $\bar{\sigma}$-rigid and hence reduced. Therefore $gf_k\bar{\sigma}(g) = 0$ and consequently $gf_kg = 0$, by $\bar{\sigma}$-compatibility of $R[x]$. Thus $(gf_k)^2 = 0$ and so $gf_k = 0$. Until now, we get

$$f_kg = g_kf = 0 \qquad \forall k = 1,\dots,t$$

Next, by multiplying g in Eq. (3) from the left side, we get

$$gfg_{k_1k_2} + gf_{k_1}\bar{\sigma}(g_{k_2}) + gf_{k_1k_2}\bar{\sigma}^2(g) = 0$$

and similar to the above argument, we have

$$gf_{k_1k_2} = 0 \qquad \forall\ 1 \le k_1, k_2 \le t$$

This implies that

$$fg_{k_1k_2} + f_{k_1}\bar{\sigma}(g_{k_2}) = 0. \qquad (*)$$

Now, by multiplying g_{k_2} in Eq. $(*)$ from the left side, we have

$$g_{k_2}fg_{k_1k_2} + g_{k_2}f_{k_1}\bar{\sigma}(g_{k_2}) = 0,$$

and thus $g_{k_2}f_{k_1}\bar{\sigma}(g_{k_2}) = 0$. Therefore,

$$f_{k_1}g_{k_2} = g_{k_2}f_{k_1} = 0 \qquad \forall\ 1 \le k_1, k_2 \le t$$

and consequently

$$fg_{k_1k_2} = 0 \qquad \forall\ 1 \le k_1, k_2 \le t$$

So, we obtain multiplying each summand of

$$fe + \sum\nolimits_{1\le k\le t} f_k u_k + \sum\nolimits_{1\le k_1,k_2\le t} f_{k_1k_2}u_{k_1}u_{k_2}$$

to each summand of

$$ge + \sum\nolimits_{1\le k\le t} g_k u_k + \sum\nolimits_{1\le k_1,k_2\le t} g_{k_1k_2}u_{k_1}u_{k_2}$$

is equal to zero. So,

$$\begin{aligned} a^{(i)}b^{(j)} &= a^{(i)}b_k^{(j)} = a^{(i)}b_{k_1k_2}^{(j)} = 0 \\ a_k^{(i)}b^{(j)} &= a_k^{(i)}b_k^{(j)} = a_k^{(i)}b_{k_1k_2}^{(j)} = 0 \\ a_{k_1k_2}^{(i)}b^{(j)} &= a_{k_1k_2}^{(i)}b_k^{(j)} = a_{k_1k_2}^{(i)}b_{k_1k_2}^{(j)} = 0 \end{aligned}$$

for each $0 \le i \le r$ and $0 \le j \le s$, since $R[x]$ is Armendariz. Thus $\alpha_i\beta_j = 0$, for each $0 \le i \le r$ and $0 \le j \le s$, by compatibility of σ. Hence $R[M;\sigma]$ is Armendariz and the proof is complete. $\square$

Now, in the following example, we show that Theorem 3.2 does not remain true, if $n \ge 4$.

Example 3.3. Let R be a σ-rigid ring with $\sigma(1) = 1$ (in particular reduced rings with $\sigma = \mathrm{id}_R$) and M be a free monoid generated by $\{u, v, w\}$ with 0 added and the following relations:

$$u^2 = v^2 = w^2 = uw = vu = wv = wu = 0$$

Thus

$$R[M;\sigma] = \{a + bu + cv + dw + fuv + gvw + huvw \ :\ a, b, c, d, f, g, h \in R\}$$

is not Armendariz. This is because

$$[u+(u-uv)x][w+(w+vw)x]=0,$$

but $u(w+vw)=uvw\neq 0$.

Similar to the method used in the proof of Theorem 3.2, one can see that for an arbitrary σ-rigid ring R, if $n=2$, then $R[M;\sigma]$ is Armendariz. In this case, one may suspect that if R is Armendariz, then so is $R[M;\sigma]$. But the following example eliminates the possibility, even if $M=\{0,u\}$.

Example 3.4. Let R be a σ-rigid ring with $\sigma(1)=1$. So $T(R,R)$ is Armendariz. Suppose $M=\{0,e,u\}$ (i.e. M be a free monoid generated by $\{u\}$ with 0 added and $u^2=0$). We have $\mathcal{F}(x)\mathcal{G}(x)=0$, where

$$\mathcal{F}(x)=\begin{pmatrix}0&1\\0&0\end{pmatrix}e+\left[\begin{pmatrix}0&1\\0&0\end{pmatrix}e+\begin{pmatrix}-1&0\\0&-1\end{pmatrix}u\right]x$$

and

$$\mathcal{G}(x)=\begin{pmatrix}0&1\\0&0\end{pmatrix}e+\left[\begin{pmatrix}0&1\\0&0\end{pmatrix}e+\begin{pmatrix}1&0\\0&1\end{pmatrix}u\right]x$$

But

$$\left[\begin{pmatrix}0&1\\0&0\end{pmatrix}e\right]\left[\begin{pmatrix}0&1\\0&0\end{pmatrix}e+\begin{pmatrix}1&0\\0&1\end{pmatrix}u\right]\neq 0$$

Therefore, $T(R,R)[M;\bar{\sigma}]$ is not Armendariz.

4. On annihilator conditions of skew generalized triangular matrix rings

We use $\mathbf{P}(R)$, $L\text{-}rad(R)$, $Ni\ell^*(R)$ and $ni\ell(R)$ to denote the prime radical, the Levitsky radical (i.e. sum of all locally nilpotent ideals), the upper nil radical (i.e. sum of all nil ideals) and the set of all nilpotent elements of R, respectively.

A ring R is called 2-*primal* if $\mathbf{P}(R)=ni\ell(R)$. 2-primal rings play an important role in ring theory because they are a generalization of two categories of commutative and reduce rings. Shin in [20, Proposition 1.11] showed that a ring R is 2-primal if and only if every minimal prime ideal P of R is completely prime (i.e. R/P is a domain). Due to Marks [15], a ring R is called an *NI ring* if $Ni\ell^*(R)=ni\ell(R)$. Note that R is an NI ring if and only if $ni\ell(R)$ forms an ideal. It is obvious that 2-primal rings are NI, but the converse need not hold [15, Example 2.2]. Chen and Cui [4] called a ring R to be *weakly 2-primal* if $L\text{-}rad(R) = ni\ell(R)$. It is clear that the following chain of implications hold for any ring:

$$\text{2-primal} \Rightarrow \text{weakly 2-primal} \Rightarrow \text{NI}$$

It is shown in [4] that weakly 2-primal rings are a family of rings for which some known results on semicommutative rings extend naturally. It is also shown in [4, Example 3.1] that there is a weakly 2-primal ring that is not 2-primal.

Theorem 4.1. *Let R be an arbitrary ring with an endomorphism σ. Then:*

- **(1):** *$R[M;\sigma]$ is an NI ring if and only if so is R.*
- **(2):** *$R[M;\sigma]$ is a 2-primal ring if and only if so is R.*
- **(3):** *$R[M;\sigma]$ is a weakly 2-primal ring if and only if so is R.*

Proof. The map $\varphi : R[M;\sigma] \to R/\mathfrak{Rad}(R)$ via $\varphi(\sum_{m\in M} r_m m) = r_e + \mathfrak{Rad}(R)$ is a ring epimorphism with $ker(\varphi) = \mathfrak{Rad}(R[M;\sigma])$, by [7, Theorem 2.13], where $\mathfrak{Rad}(-)$ is one of the radicals $\mathbf{P}$(-), $L\text{-}rad$(-) or $Ni\ell^*$(-). Therefore, we obtain $R/\mathfrak{Rad}(R) \cong R[M;\sigma]/\mathfrak{Rad}(R[M;\sigma])$. Now, the parts (1), (2) and (3) follow from the fact that a ring R is 2-primal if and only if $R/\mathbf{P}(R)$ is reduced, R is NI if and only if $R/Ni\ell^*(R)$ is reduced and R is weakly 2-primal if and only if $R/L\text{-}rad(R)$ is reduced. □

Let a, b and c are three arbitrary elements in a ring R. If $abc = 0$ implies that $acb = 0$, the ring R is called *symmetric*. Also, R is called *reversible* if $ab = 0$ equivalent to $ba = 0$. Reversible rings were defined by Cohn in [6]. As an important property of reversible rings, he showed that the Köthe conjecture is true for this class of rings. According to [16], a ring R is called *reflexive* if $aRb = 0$ implies $bRa = 0$. Reversible rings are clearly reflexive. It is shown by [13, Lemma 2.1] that a ring R is reflexive if and only if $IJ = 0$ implies $JI = 0$ for all ideals I and J of R. These arguments naturally give rise to extending the study of symmetric ring property to the lattice of ideals. A generalization of symmetric rings was defined by Camillo, Kwak and Lee in [3]. A ring R is called *ideal-symmetric* if $IJK = 0$ implies $IKJ = 0$ for all ideals I, J and K of R. It is obvious that semiprime rings are ideal-symmetric. It proved by [3, Lemma 1.1(2)] that R is ideal-symmetric if and only if $aRbRc = 0$ implies $aRcRb = 0$.

Liu and Zhao introduced APP rings in [14]. A ring R is called *right APP* if $r_R(aR)$ is left s-unital (i.e. for each $a \in r_R(aR)$ there is an $x \in r_R(aR)$ such that $xa = a$). For more details and examples of APP rings, see [14].

Theorem 4.2. *Let R be a semiprime or right APP ring and σ an automorphism on R. Assume that R is a σ-weakly rigid. Then:*

- **(i):** *R is reflexive if and only if so is $R[M;\sigma]$;*
- **(ii):** *R is ideal-symmetric if and only if so is $R[M;\sigma]$.*

Proof. We only prove (ii), because the proof of the other case is similar. Assume that R is ideal-symmetric. For $i = 1, 2, 3$, consider

$$\alpha_i = a^{(i)}e + \sum\nolimits_{1\leq k\leq t} a_k^{(i)} u_k + \sum\nolimits_{1\leq k_1,k_2\leq t} a_{k_1k_2}^{(i)} u_{k_1}u_{k_2}$$
$$+\cdots+ \sum\nolimits_{1\leq k_1,\dots,k_{n-1}\leq t} a_{k_1\cdots k_{n-1}}^{(i)} u_{k_1}\cdots u_{k_{n-1}}$$

in $T = R[M;\sigma]$ such that $\alpha_1 T\alpha_2 T\alpha_3 = 0$. Hence, by [9, Lemma 2.3], we have $a_1Ra_2Ra_3 = 0$ for all $a_i \in \mathfrak{C}_{\alpha_i}$. Since R is ideal-symmetric it concludes that $a_1Ra_3Ra_2 = 0$. Now, σ-weakly rigidity of R implies that $\alpha_1T\alpha_3T\alpha_2 = 0$. Hence T is ideal-symmetric. Conversely, suppose that T is ideal-symmetric. Let $aRbRc = 0$ for all $a, b, c \in R$. Since R is σ-weakly rigid, $(ae)T(be)T(ce) = 0$. Thus $(ae)T(ce)T(be) = 0$ and so $aRcRb = 0$ for all $a, b, c \in R$. Therefore, R is ideal-symmetric and the result follows. □

Remark 4.3. Example 2.16 in [13] and Remark 2.5(1) in [3] indicate that right APP rings are independent of ideal-symmetric and reflexive rings, respectively.

We finish this article with the following two corollaries which are immediate consequences of Theorem 4.2.

Corollary 4.4. *Let R be a semiprime ring and σ an automorphism on R. If R is σ-weakly rigid, then $R[M;\sigma]$ is ideal-symmetric.*

Corollary 4.5. *Let R be a prime ring and σ an automorphism on R. Then $R[M;\sigma]$ is ideal-symmetric.*

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