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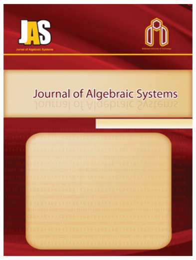

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I. Nikoufar* and Z. Baghernezhad Shayan

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SOME OPERATOR INEQUALITIES IN HILBERT C^*-MODULES VIA THE OPERATOR PERSPECTIVE

I. NIKOUFAR* AND Z. BAGHERNEZHAD SHAYAN

Abstract. Some Hilbert C^*-module versions of Hölder-McCarthy and Hölder type inequalities and their complementary on a Hilbert C^*-module are obtained by Seo [22]. The purpose of this paper is to extend these results for some operator convex (resp. concave) functions on a Hilbert C^*-module via the operator perspective approach. By choosing some elementary functions, we reach some new types of inequalities in Hilbert C^*-modules.

1. Introduction

A family of inequalities concerning inner products of vectors and functions began with Cauchy. The extensions and generalizations later led to the inequalities of Schwarz, Minkowski, and Hölder. The well-known Hölder inequality is one of the most important inequalities in functional analysis. Hölder's inequality for sequences of numbers asserts that if $x_i, y_i \in \mathbb{C}$ $(i = 1, ..., n)$, then

$$\sum_{i=1}^{n} |x_i||y_i| \leq (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} |y_i|^q)^{\frac{1}{q}}$$

for all positive real numbers p and q such that $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. The reverse inequality holds where $0 < p < 1$ or $p < 0$.

Many authors have studied non-commutative versions of the Hölder inequality and its inverses. Ando and Hiai in [1] discussed the norm and the matrix Hölder inequality and Bourin et al. in [2] showed the operator geometric mean version. Seo in [22] demonstrated a Hilbert C^*-module version of Hölder-McCarthy inequality and its complementary and Hölder type inequalities and their reverses on a Hilbert C^*-module, and so on. A new Cauchy–Schwarz inequality in the framework of semi–inner product C^*-modules over unital C^*-algebras is proved under the operator geometric mean and the polar decomposition [7].

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*Corresponding author.

The non-commutative perspective of the continuous real-valued function f was defined in [5] by setting

$$P_f(A,B) = B^{1/2} f(B^{-1/2}AB^{-1/2})B^{1/2}$$

for every self-adjoint operator A and every positive invertible operator B on a Hilbert space $\mathcal{H}$ where the spectrum of the operator $B^{-1/2}AB^{-1/2}$ is in the domain of the function f. When B is positive, one may set

$$P_f(A,B) = \lim_{\epsilon\to 0} P_f(A, B+\epsilon I)$$

if the limit exists in the strong operator topology. The main results of [6] were generalized in [5] for the non-commutative case, and the necessary and sufficient conditions for joint convexity (resp. concavity) of the perspective and generalized perspective functions were proved. For some applications we refer the interested readers to [17, 16, 15, 14] and references therein.

Let A and B be positive invertible operators. The operator α-geometric mean is the perspective of the function t^α which was defined by Kubo and Ando [10]. Indeed,

$$\begin{aligned} A\#_\alpha B &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}} \\ &= P_{t^\alpha}(B,A). \end{aligned}$$

The relative operator entropy is the perspective of the function $\log t$. Fujii and Kamei introduced this type of operator entropy in [8]. Indeed,

$$\begin{aligned} S(A,B) &= A^{\frac{1}{2}}\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \\ &= P_{\log t}(B,A). \end{aligned}$$

Furuta defined the generalized relative operator entropy in [9], and this type of operator entropy is the perspective of the function $t^q \log t$, $q \in \mathbb{R}$, i. e.,

$$\begin{aligned} S_q(A,B) &= A^{1/2}(A^{-1/2}BA^{-1/2})^q(\log A^{-1/2}BA^{-1/2})A^{1/2} \\ &= P_{t^q \log t}(B,A). \end{aligned}$$

Note that for $q = 0$ this reduces to $S_0(A,B) = S(A,B)$. Using this type of operator entropy, Furuta obtained the parametric extension of the operator Shannon inequality and its reverse one, see also [18, 19].

Yanagi et al. defined the notion of Tsallis relative operator entropy in [23]. The operator perspective corresponding to the function $\frac{t^\lambda - 1}{\lambda}$, for $0 < \lambda \le 1$, is the Tsallis relative operator entropy between positive invertible operators A and B. Indeed,

$$\begin{aligned} T_\lambda(A,B) &= \frac{A^{1/2}(A^{-1/2}BA^{-1/2})^\lambda A^{1/2} - A}{\lambda} \\ &= P_{\frac{t^\lambda - 1}{\lambda}}(B,A). \end{aligned}$$

It is usual to denote the Tsallis relative operator entropy by $T_\lambda(A,B)$ again for $\lambda \in \mathbb{R}\backslash\{0\}$. Some operator inequalities related to the Tsallis relative operator entropy were proved in [24, 17]. The concept of entropy not only is used for determining the difference between two states of a dynamic system but also play an important role in different subjects such as statistical mechanics, information theory, etc.

Mond and Pečarić [13] proved an operator version of the Jensen's inequality for a convex function f on an interval I, a self–adjoint operator A on a Hilbert space $\mathcal{H}$ with spectrum in I, and a unit vector $x \in \mathcal{H}$ as follows

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle. \tag{1.1}$$

This inequality holds in a C^*-algebra $\mathcal{A}$ when the inner product replaces by a state on $\mathcal{A}$, i. e.,

$$f(\varphi(a)) \leq \varphi(f(a))$$

for every self–adjoint element $a \in \mathcal{A}$. Davis [4] showed that if Φ is a completely positive linear map on $B(\mathcal{H})$ and f an operator convex function on an interval I, then

$$f(\Phi(a)) \leq \Phi(f(a)) \tag{1.2}$$

holds for every self–adjoint operator a on $\mathcal{H}$ whose spectrum is contained in I. Choi removed the restriction to a completely positive linear map [3] who showed that (1.2), the so-called Choi–Davis–Jensen inequality, remains valid for all positive unital linear maps Φ. This inequality will play an essential role in our main results.

In this paper, we investigate a Hilbert C^*-module version of the Hölder-McCarthy inequality associated with an operator perspective and its complementary. As an application, we obtain some Hölder type inequalities and their reverses on Hilbert C^*-modules via the operator perspective of some well-known functions. By choosing some elementary functions, our results recover some known results from [22] with concise and simple proofs.

2. Preliminaries

The notion of Hilbert C^*-module is a generalization of the notion of Hilbert space by allowing the inner product to take values in a possibly more general C^*-algebra $\mathcal{A}$ instead of the complex field $\mathbb{C}$. Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{X}$ be a complex linear space which is a right $\mathcal{A}$-module with a scalar multiplication satisfying $\lambda(ax) = (\lambda a)x = a(\lambda x)$ for $x \in \mathcal{X}$, $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. The space $\mathcal{X}$ is called a pre-Hilbert $\mathcal{A}$-module or inner product $\mathcal{A}$-module if there exists a C^*-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ with the following properties

(i) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for all $x, y, z \in \mathcal{X}, \alpha, \beta \in \mathbb{C}$,
(ii) $\langle x, ya \rangle = \langle x, y \rangle a$ for all $x, y \in \mathcal{X}, a \in \mathcal{A}$,
(iii) $\langle y, x \rangle = \langle x, y \rangle^*$ for all $x, y \in \mathcal{X}$,
(iv) $\langle x, x \rangle \geq 0$ for all $x \in \mathcal{X}$ and if $\langle x, x \rangle = 0$, then $x = 0$.

The space $\mathcal{X}$ is called a (right) Hilbert $\mathcal{A}$-module or a Hilbert C^*-module if it is complete with respect to the norm $||x|| = \sqrt{||\langle x, x \rangle||}$ for $x \in \mathcal{X}$, where the latter norm denotes the C^*-norm of $\mathcal{A}$. If $\mathcal{X}$ satisfies all conditions for an inner product $\mathcal{A}$-module except for the second part of (iv), then we call $\mathcal{X}$ a semi inner product $\mathcal{A}$-module. For instance, every inner product space is a right Hilbert $\mathbb{C}$-module, and every right ideal I of the C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with the C^*-valued inner product $\langle a, b \rangle = ab^*$ for $a, b \in I$. For more details about Hilbert C^*-modules we refer [11].

From now on let $\mathcal{B}(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H and $\mathcal{A}$ a C^*-subalgebra of $\mathcal{B}(H)$. Note that an element x of the Hilbert $\mathcal{A}$-module $\mathcal{X}$ is nonsingular if the element $\langle x, x \rangle \in \mathcal{A}$ is invertible and it is unital if $\langle x, x \rangle = 1$.

We review the basic concepts of adjointable operators on a Hilbert C^*-module. Throughout this paper let $\mathcal{X}$ be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. Let $End_{\mathcal{A}}(\mathcal{X})$ denote the set of all bounded $\mathbb{C}$-linear $\mathcal{A}$-homomorphism from $\mathcal{X}$ to $\mathcal{X}$ and $T \in End_{\mathcal{A}}(\mathcal{X})$. We say that T is adjointable if there exists a $T^* \in End_{\mathcal{A}}(\mathcal{X})$ such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle$$

for all $x, y \in \mathcal{X}$. Let $L(\mathcal{X})$ denote the set of all adjointable operators from $\mathcal{X}$ to $\mathcal{X}$. Then, $L(\mathcal{X})$ is a C^*-algebra and its norm is defined by

$$||T|| = \sup\{||\langle Tx, Tx \rangle||^{\frac{1}{2}} : ||x|| < 1\}.$$

The operator T is a positive element of $L(\mathcal{X})$ if and only if $\langle x, Tx\rangle \geq 0$ for all $x \in X$, cf. [21].

3. HÖLDER-McCarthy INEQUALITY

In this section, we motivate to find a Hilbert C^*-module version of the Hölder-McCarthy inequality associated with an operator perspective and its complementary.

Theorem 3.1. *Let T be a positive operator in $L(\mathcal{X})$ and let x be a nonsingular element of $\mathcal{X}$. If f is operator concave, then*

$$\langle x, f(T)x\rangle \leq P_f\big(\langle x, Tx\rangle, \langle x, x\rangle\big). \tag{3.1}$$

The reverse inequality holds when f is an operator convex function.

Proof. For a nonsingular element x of $\mathcal{X}$, put

$$\phi_x(X) = \big\langle x\langle x, x\rangle^{-\frac{1}{2}}, Xx\langle x, x\rangle^{-\frac{1}{2}}\big\rangle, \quad X \in L(\mathcal{X}).$$

So,

$$\phi_x(I) = \big\langle x\langle x, x\rangle^{-\frac{1}{2}}, x\langle x, x\rangle^{-\frac{1}{2}}\big\rangle = I$$

and ϕ_x is a positive linear map from $L(\mathcal{X})$ to $\mathcal{A}$. By applying operator concavity of f and [3, 4] one can deduce

$$\phi_x\big(f(T)\big) \leq f\big(\phi_x(T)\big).$$

Hence,

$$\big\langle x\langle x, x\rangle^{-\frac{1}{2}}, f(T)x\langle x, x\rangle^{-\frac{1}{2}}\big\rangle \leq f\big(\langle x\langle x, x\rangle^{-\frac{1}{2}}, Tx\langle x, x\rangle^{-\frac{1}{2}}\rangle\big)$$

and so

$$\langle x, x\rangle^{-\frac{1}{2}}\langle x, f(T)x\rangle\langle x, x\rangle^{-\frac{1}{2}} \leq f\big(\langle x, x\rangle^{-\frac{1}{2}}\langle x, Tx\rangle\langle x, x\rangle^{-\frac{1}{2}}\big).$$

This means

$$\begin{aligned}\langle x, f(T)x\rangle &\leq \langle x, x\rangle^{\frac{1}{2}} f\big(\langle x, x\rangle^{-\frac{1}{2}}\langle x, Tx\rangle\langle x, x\rangle^{-\frac{1}{2}}\big)\langle x, x\rangle^{\frac{1}{2}} \\ &= P_f\big(\langle x, Tx\rangle, \langle x, x\rangle\big).\end{aligned}$$

□

Remark 3.2. An easy consequence of Theorem 3.1 is a Mond and Pečarić type inequality like (1.1) in Hilbert C^*-modules. Indeed, one can deduce an operator version of Jensen's inequality in the Hilbert C^*-module framework by replacing $\langle x, x\rangle = 1$ in (3.1).

Let T be a positive operator in $L(\mathcal{X})$ and let x be a unital nonsingular element of $\mathcal{X}$. If f is operator concave, then

$$\langle x, f(T)x\rangle \leq f(\langle x, Tx\rangle).$$

The reverse inequality holds when f is operator convex.

Corollary 3.3. *Under the hypotheses of Theorem 3.1, if f^{-1} exists, then*

$$\langle x, Tx\rangle \leq P_f\big(\langle x, f^{-1}(T)x\rangle, \langle x, x\rangle\big). \tag{3.2}$$

The reverse inequality holds when f is operator convex.

Proof. Replace $f^{-1}(T)$ instead of T in (3.1) to deduce the result. □

We can deduce a Hilbert C^*-module version of the Hölder–McCarthy inequality by choosing a suitable function f in Theorem 3.1. We show that by our approach, the proofs are short and simple. In particular, if we consider $f(t) = t^{\frac{1}{p}}$, then we deduce [22, Theorem 3.1] as follows.

Corollary 3.4. *Let T be a positive operator in $L(\mathcal{X})$ and let x be a nonsingular element of $\mathcal{X}$.*

(i) *If $p \geq 1$, then $\langle x, Tx\rangle \leq \langle x, x\rangle \#_{\frac{1}{p}} \langle x, T^p x\rangle$.*
(ii) *If $p \leq -1$ or $\frac{1}{2} \leq p \leq 1$, then $\langle x, x\rangle \#_{\frac{1}{p}} \langle x, T^p x\rangle \leq \langle x, Tx\rangle$.*

Proof. Consider $f(t) = t^{\frac{1}{p}}$ and note that $f^{-1}(t) = t^p$.

(i) Since $p \geq 1$, we have $0 < \frac{1}{p} \leq 1$ and so f is operator concave and the desired result comes by Corollary 3.3 and the inequality (3.2).

(ii) Since $p \leq -1$ or $\frac{1}{2} \leq p \leq 1$, we have $-1 \leq \frac{1}{p} < 0$ or $1 \leq \frac{1}{p} \leq 2$. This implies the function f is operator convex and the result follows by Corollary 3.3, and the reverse of (3.2). □

We now declare some operator inequalities associated with the operator perspective of some elementary functions.

Corollary 3.5. *Let U be a positive operator in $L(\mathcal{X})$ and let x be a nonsingular element of $\mathcal{X}$. If $\lambda \in (0, 1]$, then*

$$\langle x, (U + I)^\lambda x\rangle \leq \langle x, x\rangle + \lambda T_\lambda(\langle x, x\rangle, \langle x, (U + I)x\rangle). \tag{3.3}$$

The reverse inequality holds for $\lambda \in [-1, 0)$.

Proof. Consider

$$f(t) = \frac{(t + 1)^\lambda - 1}{\lambda},\ t \geq 0.$$

This function is positive and operator monotone for $\lambda \in [-1,0) \cup (0,1]$, so that it is operator concave. According to Theorem 3.1, one has

$$\begin{aligned}
&\langle x, \frac{(U+I)^\lambda - I}{\lambda} x\rangle \\
&= \langle x, f(U)x\rangle \\
&\leq P_f(\langle x, Ux\rangle, \langle x, x\rangle) \\
&= \frac{1}{\lambda}\Big(\langle x,x\rangle^{\frac{1}{2}}\Big(\langle x,x\rangle^{-\frac{1}{2}}\langle x,Ux\rangle\langle x,x\rangle^{-\frac{1}{2}} + I\Big)^\lambda \langle x,x\rangle^{\frac{1}{2}} - \langle x,x\rangle\Big) \\
&= \frac{1}{\lambda}\Big(\langle x,x\rangle^{\frac{1}{2}}\Big(\langle x,x\rangle^{-\frac{1}{2}}\Big(\langle x,Ux\rangle + \langle x,x\rangle\Big)\langle x,x\rangle^{-\frac{1}{2}}\Big)^\lambda \langle x,x\rangle^{\frac{1}{2}} - \langle x,x\rangle\Big) \\
&= T_\lambda(\langle x,x\rangle, \langle x,Ux\rangle + \langle x,x\rangle) \\
&= T_\lambda(\langle x,x\rangle, \langle x,(U+I)x\rangle).
\end{aligned}$$

Multiplying both sides by $\lambda \in (0,1]$, one can deduce (3.3). Multiplying both sides by $\lambda \in [-1,0)$, one can reach the reverse inequality (3.3). □

Corollary 3.6. *Let U be a positive operator in $L(\mathcal{X})$ and let x be a nonsingular element of $\mathcal{X}$. If $\lambda \in [1,2]$, then*

$$\langle x, U^\lambda x\rangle \geq \langle x,x\rangle + \lambda T_\lambda(\langle x,x\rangle, \langle x,Ux\rangle).$$

Proof. Consider

$$f(t) = \frac{t^\lambda - 1}{\lambda},\ t \geq 0.$$

This function is operator convex for $\lambda \in [1,2]$. According to Theorem 3.1, we get

$$\begin{aligned}
\langle x, \frac{U^\lambda - I}{\lambda} x\rangle &= \langle x, f(U)x\rangle \\
&\geq P_f(\langle x,Ux\rangle, \langle x,x\rangle) \\
&= \frac{1}{\lambda}\Big(\langle x,x\rangle^{\frac{1}{2}}\Big(\langle x,x\rangle^{-\frac{1}{2}}\langle x,Ux\rangle\langle x,x\rangle^{-\frac{1}{2}}\Big)^\lambda \langle x,x\rangle^{\frac{1}{2}} - \langle x,x\rangle\Big) \\
&= T_\lambda(\langle x,x\rangle, \langle x,Ux\rangle),
\end{aligned}$$

whence a simplification shows the result. □

Corollary 3.7. *Let T be a positive operator in $L(\mathcal{X})$ and let x be a nonsingular element of $\mathcal{X}$. Then,*

(i) $\langle x, (\log T)x\rangle \leq S(\langle x,x\rangle, \langle x,Tx\rangle)$.

(ii) $\langle x, (T\log T)x\rangle \geq S_1(\langle x, x\rangle, \langle x, Tx\rangle)$.

Proof. (i) Consider $f(t) = \log t$ and apply Theorem 3.1.

(ii) Consider $f(t) = t\log t$. Since f is operator convex, the reverse inequality in Theorem 3.1 entails the result. □

The Mond–Pećarič method [12, Corollary 2.4] and [12, Corollary 2.5] for the strictly concave and convex differentiable function f, respectively on the interval $[m, M]$ with $m < M$ presents a counterpart for the well known Choi–Davis–Jensen inequality (1.2). We prove a complementary inequality for the Hilbert C^*-module version of the Hölder-McCarthy inequality associated with an operator perspective.

Theorem 3.8. *Let T be a positive operator in $L(\mathcal{X})$ with*

$$mI \leq T \leq MI,$$

$0 < m < M$ and let x be a nonsingular element of $\mathcal{X}$. If $f(t) > 0$ is a real-valued continuous strictly concave twice differentiable function on $[m, M]$, then

$$K_1(m, M, f)P_f\big(\langle x, Tx\rangle, \langle x, x\rangle\big) \leq \langle x, f(T)x\rangle, \tag{3.4}$$

where
$K_1(m, M, f) = \min_{m\leq t\leq M} \frac{a_f t + b_f}{f(t)}$, *$a_f = \frac{f(M)-f(m)}{M-m}$, and $b_f = \frac{Mf(m)-mf(M)}{M-m}$. Moreover, if f^{-1} exists and f^{-1} is monotone increasing or monotone decreasing, then*

$$K_1(f^{-1}(m), f^{-1}(M), f)P_f\big(\langle x, f^{-1}(T)x\rangle, \langle x, x\rangle\big) \leq \langle x, Tx\rangle. \tag{3.5}$$

Proof. For a nonsingular element x of $\mathcal{X}$, put

$$\phi_x(X) = \big\langle x\langle x, x\rangle^{-\frac{1}{2}}, Xx\langle x, x\rangle^{-\frac{1}{2}}\big\rangle, \quad X \in L(\mathcal{X}).$$

So, ϕ_x is a unital positive linear map from $L(\mathcal{X})$ to $\mathcal{A}$. According to [12, Corollary 2.4], we find that

$$K_1(m, M, f)f\big(\phi_x(T)\big) \leq \phi_x\big(f(T)\big).$$

Hence,

$$K_1(m, M, f)f\big(\langle x, \langle x, x\rangle^{-\frac{1}{2}}, Tx\langle x, x\rangle^{-\frac{1}{2}}\rangle\big) \leq \big\langle x\langle x, x\rangle^{-\frac{1}{2}}, f(T)x\langle x, x\rangle^{-\frac{1}{2}}\big\rangle$$

and so

$$K_1(m, M, f)f\big(\langle x, x\rangle^{-\frac{1}{2}}\langle x, Tx\rangle\langle x, x\rangle^{-\frac{1}{2}}\big) \leq \langle x, x\rangle^{-\frac{1}{2}}\langle x, f(T)x\rangle\langle x, x\rangle^{-\frac{1}{2}}.$$

This means

$$\begin{aligned}&K_1(m,M,f)P_f\big(\langle x,Tx\rangle,\langle x,x\rangle\big)\\&=K_1(m,M,f)\langle x,x\rangle^{\frac{1}{2}}f\big(\langle x,x\rangle^{-\frac{1}{2}}\langle x,Tx\rangle\langle x,x\rangle^{-\frac{1}{2}}\big)\langle x,x\rangle^{\frac{1}{2}}\\&\leq\langle x,f(T)x\rangle.\end{aligned}$$

Moreover, the inequality (3.5) follows from (3.4) by replacing $f^{-1}(T)$ with T. In this situation, if f^{-1} is monotone increasing, then

$$f^{-1}(m)I\leq f^{-1}(T)\leq f^{-1}(M)I$$

and so

$$K_1(f^{-1}(m),f^{-1}(M),f)=\min_{f^{-1}(m)\leq t\leq f^{-1}(M)}\tfrac{a_ft+b_f}{f(t)},$$

where $a_f=\frac{M-m}{f^{-1}(M)-f^{-1}(m)}$ and $b_f=\frac{mf^{-1}(M)-Mf^{-1}(m)}{f^{-1}(M)-f^{-1}(m)}$. Note that if f^{-1} is monotone decreasing, then

$$f^{-1}(M)I\leq f^{-1}(T)\leq f^{-1}(m)I$$

and we have $K_1(f^{-1}(M),f^{-1}(m),f)=K_1(f^{-1}(m),f^{-1}(M),f)$. □

By a similar approach and using [12, Corollary 2.5], one can deduce the following theorem for a continuous strictly convex twice differentiable function on the closed interval $[m,M]$.

Theorem 3.9. *Let T be a positive operator in $L(\mathcal{X})$ with*

$$mI\leq T\leq MI,$$

$0<m<M$ and let x be a nonsingular element of $\mathcal{X}$. If $f(t)>0$ is a real-valued continuous strictly convex twice differentiable function on $[m,M]$, then

$$\langle x,f(T)x\rangle\leq K_2(m,M,f)P_f\big(\langle x,Tx\rangle,\langle x,x\rangle\big),\tag{3.6}$$

where

$$K_2(m,M,f)=\max_{m\leq t\leq M}\frac{a_ft+b_f}{f(t)}.$$

Moreover, if f^{-1} exists and f^{-1} is monotone increasing or monotone decreasing, then

$$\langle x,Tx\rangle\leq K_2(f^{-1}(m),f^{-1}(M),f)P_f\big(\langle x,f^{-1}(T)x\rangle,\langle x,x\rangle\big).\tag{3.7}$$

We can obtain some complementary inequalities for the Hilbert C^*-module version of some inequalities which we proved in Corollaries 3.5, 3.6, and 3.7 associated with the operator perspective of a suitable function. Seo proved some complementary inequalities for the Hölder-McCarthy inequality associated with an operator perspective of the power function in [22, Theorem 3.3, Theorem 3.5]. We show that by our perspective approach, the proofs are short and straightforward.

Corollary 3.10. *Let T be a positive invertible operator in $L(\mathcal{X})$ such that $mI \le T \le MI$ for some scalars $0 < m < M$, and let x be a nonsingular element of X.*

(i) *If $p \ge 1$, then*

$$\langle x, x\rangle \#_{\frac{1}{p}} \langle x, T^p x\rangle \le K(m, M, p)^{\frac{1}{p}} \langle x, Tx\rangle.$$

(ii) *If $p \le -1$ or $\frac{1}{2} \le p \le 1$, then*

$$\langle x, Tx\rangle \le K(m^p, M^p, \frac{1}{p}) \langle x, x\rangle \#_{\frac{1}{p}} \langle x, T^p x\rangle,$$

(iii) *If for $0 < p < 1$, then*

$$K(m, M, p)\langle x, x\rangle \#_p \langle x, Tx\rangle \le \langle x, T^p x\rangle \le \langle x, x\rangle \#_p \langle x, Tx\rangle,$$

where the generalized Kantorovich constant $K(m, M, p)$ is defined by

$$K(m, M, p) = \frac{mM^p - Mm^p}{(p-1)(M-m)} \Big(\frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p} \Big)^p \tag{3.8}$$

for $0 < m < M$ and $p \ge 0$.

Proof. (i) Consider $f(t) = t^{\frac{1}{p}}$. According to Theorem 3.8 and the inequality (3.5) with $f^{-1}(t) = t^p$ one can deduce

$$K_1(m^p, M^p, f)\langle x, x\rangle \#_{\frac{1}{p}} \langle x, T^p x\rangle \le \langle x, Tx\rangle. \tag{3.9}$$

Note that in this situation, we have

$$K_1(m^p, M^p, f) = \frac{1}{K(m, M, p)^{\frac{1}{p}}}.$$

So, the result follows.

(ii) Let $f(t) = t^{\frac{1}{p}}$. Then, in view of Theorem 3.9 and the inequality (3.7) with $f^{-1}(t) = t^p$ we have

$$\langle x, Tx\rangle \le K_2(m^p, M^p, f)\langle x, x\rangle \#_{\frac{1}{p}} \langle x, T^p x\rangle. \tag{3.10}$$

On the other hand, a simple calculation indicates that

$$K_2(m^p, M^p, f) = K(m^p, M^p, \frac{1}{p})$$

(iii) Consider $f(t) = t^p$. The first inequality follows from Theorem 3.8 with $K_1(m, M, f) = K(m, M, p)$. One can deduce the second inequality from Theorem 3.1. □

4. HÖLDER INEQUALITY

In this section, we obtain Hölder type inequalities on a Hilbert C^*-module and its reverse one. For any continuous function $f : (0, \infty) \longrightarrow \mathbb{R}$ the transpose $\tilde{f}$ of f is defined by

$$\tilde{f}(x) = xf(x^{-1}), \quad x > 0.$$

We use the following lemma in our main results; for the proof, see [20, Lemma 2.1].

Lemma 4.1. *Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function and let $\tilde{f}$ be the transpose of f. Then, $P_{\tilde{f}}(A, B) = P_f(B, A)$ for every $A, B \in \mathcal{B}(\mathcal{H})^+$.*

Theorem 4.2. *Let A and B be positive invertible operators in $L(\mathcal{X})$ and let x be a nonsingular element of $\mathcal{X}$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f is operator concave, then*

(i) $\langle x, P_f(A^p, B^q)x\rangle \leq P_f\big(\langle x, A^p x\rangle, \langle x, B^q x\rangle\big)$,
(ii) $\langle x, P_{\tilde{f}}(B^q, A^p)x\rangle \leq P_{\tilde{f}}\big(\langle x, B^q x\rangle, \langle x, A^p x\rangle\big)$.

Proof. (i) Replacing x and T by $B^{\frac{q}{2}}x$ and $B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}}$ in Theorem 3.1, respectively we have

$$\begin{aligned}&\langle B^{\frac{q}{2}}x, f(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}})B^{\frac{q}{2}}x\rangle \\ &\leq P_f\big(\langle B^{\frac{q}{2}}x, (B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}})B^{\frac{q}{2}}x\rangle, \langle B^{\frac{q}{2}}x, B^{\frac{q}{2}}x\rangle\big).\end{aligned}$$

Hence,

$$\begin{aligned}&\langle x, B^{\frac{q}{2}}f(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}})B^{\frac{q}{2}}x\rangle \\ &\leq P_f\big(\langle x, B^{\frac{q}{2}}(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}})B^{\frac{q}{2}}x\rangle, \langle x, B^{\frac{q}{2}}B^{\frac{q}{2}}x\rangle\big)\end{aligned}$$

from which part (i) follows.

(ii) It is a straightforward consequence of part (i) and Lemma 4.1. □

Corollary 4.3. *Under the hypotheses of Theorem 4.2, if f is operator convex, then the reverse inequalities hold in parts (i) and (ii) of Theorem 4.2.*

As an application of Theorem 4.2, one may show Hölder type inequalities on a Hilbert C^*–module for the power function which is proved in [22, Theorem 4.1], but has a simple proof by our approach.

Corollary 4.4. *Let A and B be positive invertible operators in $L(\mathcal{X})$ and let x be a nonsingular element of $\mathcal{X}$ and $\frac{1}{p}+\frac{1}{q}=1$.*

(i) *If $p>1$, then*

$$\langle x, B^q \#_{\frac{1}{p}} A^p x\rangle \leq \langle x, B^q x\rangle \#_{\frac{1}{p}} \langle x, A^p x\rangle$$

or

$$\langle x, A^p \#_{\frac{1}{q}} B^q x\rangle \leq \langle x, A^p x\rangle \#_{\frac{1}{q}} \langle x, B^q x\rangle .$$

(ii) *If $p \leq -1$ or $\frac{1}{2} \leq p < 1$, then*

$$\langle x, B^q \#_{\frac{1}{p}} A^p x\rangle \geq \langle x, B^q x\rangle \#_{\frac{1}{p}} \langle x, A^p x\rangle$$

or

$$\langle x, A^p \#_{\frac{1}{q}} B^q x\rangle \geq \langle x, A^p x\rangle \#_{\frac{1}{q}} \langle x, B^q x\rangle .$$

Proof. Similar to that of Corollary 3.4 consider $f(t)=t^{\frac{1}{p}}$ and apply Theorem 4.2 in part (i) and Corollary 4.3 in part (ii), respectively. Note that in this situation,

$$\tilde{f}(t)=tf(t^{-1})=t^{1-\frac{1}{p}}=t^{\frac{1}{q}}.$$

□

As a consequence of Theorem 4.2, we give some Hölder type inequalities on a Hilbert C^*–module in some senses.

Corollary 4.5. *Let A and B be positive invertible operators in $L(\mathcal{X})$ and let x be a nonsingular element of $\mathcal{X}$ and $\frac{1}{p}+\frac{1}{q}=1$. Then,*

(i) $\langle x, S(B^q, A^p)x\rangle \leq S(\langle x, B^q x\rangle, \langle x, A^p x\rangle)$,
(ii) $\langle x, S_1(B^q, A^p)x\rangle \geq S_1(\langle x, B^q x\rangle, \langle x, A^p x\rangle)$.

Proof. (i) Consider $f(t)=\log t$ and apply Theorem 4.2 (i).

(ii) Consider $f(t)=\log t$ and note that $\tilde{f}(t)=-t\log t$. The result follows from Theorem 4.2 (ii). □

Corollary 4.6. *Let A and B be positive invertible operators in $L(\mathcal{X})$ and let x be a nonsingular element of $\mathcal{X}$ and $\frac{1}{p}+\frac{1}{q}=1$. Then,*

$$\langle x, T_\lambda(B^q, (A^p+B^q))x\rangle \leq T_\lambda(\langle x, B^q x\rangle, \langle x, (A^p+B^q)x\rangle)$$

for every $\lambda \in [-1,0)\cup(0,1]$.

Proof. Consider

$$f(t)=\frac{(t+1)^\lambda-1}{\lambda},\ t\geq 0.$$

This function is operator concave; see Corollary 3.5. By Theorem 4.2 (i), we get the result. □

Corollary 4.7. *Let A and B be positive invertible operators in $L(\mathcal{X})$ and let x be a nonsingular element of $\mathcal{X}$ and $\frac{1}{p}+\frac{1}{q}=1$. Then,*

$$\langle x, T_\lambda(B^q, A^p)x\rangle \geq T_\lambda(\langle x, B^q x\rangle, \langle x, A^p x\rangle)$$

for every $\lambda \in [1,2]$.

Proof. Consider $f(t)=\frac{t^\lambda-1}{\lambda}$ and apply Corollary 4.3 for every $\lambda\in[1,2]$. □

5. Conclusions

We discovered the Hölder-McCarthy and Hölder type inequalities and their complementary associated with an operator convex (resp. concave) function via the operator perspective approach. In particular, we recovered the results presented by Seo [22] for the function $f(t)=t^{\frac{1}{p}}$. The advantage of our work is that we can use the other elementary functions like $\log t$, $\frac{t^\lambda-1}{\lambda}$, or $\frac{(t+1)^\lambda-1}{\lambda}$ to generate some Hölder-McCarthy and Hölder type inequalities.

The open problem is now how one can extend the results for the generalized perspective [5, 20] and what are its applications.

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Ismail Nikoufar
Department of Mathematics, Payame Noor University, Tehran, Iran.
Email: `nikoufar@pnu.ac.ir`

Zahra Baghernezhad Shayan
Department of Mathematics, Payame Noor University, Tehran, Iran.
Email: `zahrashayan96@gmail.com`