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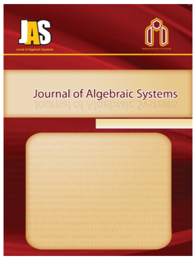

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FAULT-TOLERANT METRIC DIMENSION OF ANNIHILATOR GRAPHS OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with unity. The annihilator graph $AG(R)$ is a simple graph with vertex set as the set of all non-zero zero-divisors of R, and two distinct vertices a and b are adjacent if and only if $ann_R(a) \cup ann_R(b) \neq ann_R(a \cdot b)$. We depicted the relationship between the fault-tolerant metric dimension of $AG(R)$ and some graph parameters. Furthermore, we computed the fault-tolerant metric dimension of the annihilator graph of reduced and non-reduced rings.

1. Introduction

The study of graphs associated with different algebraic structures is one of the good approaches to studying the properties of algebraic structures. One of the most important and active research area in graphs associated with algebraic structures is the study of graphs from rings. There are many papers on graphs associated with rings, for more information, see [1, 3, 4].

Throughout this article, we assume that all rings are commutative with unity. The set of all non-zero zero-divisors and the set of all nilpotent elements is denoted as $Z(R)^*$ and $Nil(R)$, respectively. The annihilator of an element $a \in R$ is defined as $ann_R(a) = \{r \in R : a \cdot r = 0, a \in R\}$. A ring R is reduced if it does not contain non-zero nilpotent elements. For any undefined terminology or notation in commutative algebra, we refer the reader to [2].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The distance between two vertices a and b in G is denoted as $d(a,b)$ and defined as the length of the shortest path in G. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered subset of $V(G)$. The metric representation of a with respect to W is $r(a|W) = (d(a,w_1), d(a,w_2), \ldots, d(a,w_k))$. If vertices have distinct metric representations, then W is a resolving set. A resolving set with the minimum number of vertices is called a metric dimension and is denoted as $dim(G)$. If we remove an element in a resolving set, then the resulting

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set is also a resolving set, called the fault-tolerant resolving set. A fault-tolerant resolving set with the minimum number of elements is known as a fault-tolerant metric basis. The number of elements in a fault-tolerant metric basis is fault-tolerant metric dimension and, is denoted as $f_t dim\,(G)$. Let $F = \{f_1, f_2, \ldots, f_n\} \subseteq V\,(G)$. The absolute difference representation consists of $n-$vector

$$AD\left((a,b)\,|F\right) = \left(|d\,(a, f_1) - d\,(b, f_1)\,|, \ldots, |d\,(a, f_n) - d\,(b, f_n)\,|\right)$$

for any $a, b \in V\,(G)$ with respect to F, and denoted by $AD\left((a,b)\,|F\right)$. If $AD\left((a,b)\,|F\right)$ has at least two non-zero elements in its $n-vector$ for every $a \neq b \in V\,(G)$, then F is called the fault-tolerant resolving set. Refer to the book [19] for any undefined terminology or notation in graph theory.

Ayman Badawi [3] proposed the idea of an annihilator graph for a commutative ring. The annihilator graph $AG\,(R)$ is a simple graph with vertex set as the set of all non-zero zero-divisors of R and two distinct vertices, a and b are adjacent if and only if $ann_R\,(a) \cup ann_R\,(b) \neq ann_R\,(a \cdot b)$. Several authors studied the annihilator graphs of commutative rings; see [12, 14].

Motivated by the problem of uniquely recognizing the location of fault in a network, calculating the metric dimension of a graph was proposed by Harary and Meter [8]. The fault-tolerant metric dimension, which is a more powerful invariant than the metric dimension, was introduced by Hernando et al. [9] and studied in several articles. For more information, see [5, 10]. Recently, computing the metric dimension and strong metric dimension of graphs associated with algebraic structures has been started. For more details in this direction, see [6, 7, 13, 15, 16, 17, 18]. These papers inspired us to determine the fault-tolerant metric dimension of $AG\,(R)$.

In this paper, we study the relationship between certain graph characteristics and the fault-tolerant metric dimension of $AG\,(R)$. Then the fault-tolerant metric dimension of the annihilator graph of reduced and non-reduced rings is calculated. Finally, the fault-tolerant metric dimension of $AG\,(\mathbb{Z}_n)$ and $AG\,(\mathbb{Z}_n\,[i])$ are computed.

2. Fault-Tolerant Metric Dimension of $AG\,(R)$

In this section, we prove that the fault-tolerant metric dimension of $AG\,(R)$ is finite if and only if R is finite. Moreover, we find a relation between the fault-tolerant metric dimension of $AG\,(R)$ and certain graph characteristics.

Theorem 2.1. *Let R be a ring. Then $f_t dim\,(AG\,(R))$ is finite if and only if R is finite.*

Proof. Assume that $f_t dim\left(AG\left(R\right)\right)$ is finite. Let $F = \{f_1, \ldots, f_n\}$ be the fault-tolerant metric basis for $AG\left(R\right)$, and $|F| = n$, and $n > 0$. By [3, Theorem 2.2], $diam\left(AG\left(R\right)\right) \leq 2$ and so $d\left(a, b\right) \leq 2$, for arbitrary $a, b \in V\left(AG\left(R\right)\right)$. The fault-tolerant metric representation of a with respect to F is $D_F\left(a|F\right) = \left(d\left(a, f_1\right), \ldots, d\left(a, f_n\right)\right)$. Then $d\left(a, f_i\right) \in \{0, 1, 2\}$, for $1 \leq i \leq n$. The number of possibilities for $D_F\left(a|F\right)$ is at most 3^n and $D_F\left(a|F\right)$ is unique for each $a \in V\left(AG\left(R\right)\right)$. Therefore, $|V\left(AG\left(R\right)\right)| \leq 3^n$ and hence, R is finite. The converse part is trivial. $\square$

Theorem 2.2. *Let R be a ring. Then $f_t dim\left(AG\left(R\right)\right)$ is undefined if R is an integral domain.*

Proof. Let R be an integral domain. Then $V\left(AG\left(R\right)\right)$ is empty. Thus $f_t dim\left(AG\left(R\right)\right)$ is undefined. $\square$

The following is a remark of the above theorem.

Remark 2.3. The converse part of Theorem 2.2 is not true. Consider $R = \mathbb{Z}_4$ then $Z\left(R\right) = \{0, 2\}$. From the definition of $AG\left(R\right)$, $AG\left(R\right) \cong K_1$. Therefore, $f_t dim\left(AG\left(R\right)\right)$ is undefined. But R is not an integral domain.

Theorem 2.4. *Let R be a ring. Then, we have*

(1) *$f_t dim\left(AG\left(R\right)\right)$ is undefined if $diam\left(AG\left(R\right)\right) = 0$.*
(2) *$f_t dim\left(AG\left(R\right)\right) = |Z\left(R\right)^*|$ if and only if $diam\left(AG\left(R\right)\right) = 1$.*

Proof. (1) If $diam\left(AG\left(R\right)\right) = 0$, then $V\left(AG\left(R\right)\right)$ contains a single element. It is clear that the fault-tolerant metric basis is not defined in $(AG\left(R\right)$. Therefore, $f_t dim\left(AG\left(R\right)\right)$ is undefined.

(2) $diam\left(AG\left(R\right)\right) = 1 \iff AG\left(R\right) \cong K_n \iff f_t dim\left(AG\left(R\right)\right) = n \iff f_t dim\left(AG\left(R\right)\right) = |Z\left(R\right)^*|$

$\square$

2.1. **Fault-Tolerant Metric Dimension of Annihilator Graph of Reduced Rings.**

We find a formula for the fault-tolerant metric dimension of the annihilator graph of reduced rings in this section.

Proposition 2.5. *Let R be a reduced ring and, P_1 and P_2 be two minimal prime ideals such that $P_1 \cap P_2 = \{0\}$, and $Z\left(R\right) = P_1 \cup P_2$. Then $f_t dim\left(AG\left(R\right)\right) = |P_1| + |P_2| - 2$.*

Proof. $AG\left(R\right) \cong K_{|P_1|-1, |P_2|-1}$ follows from [3, Theorem 3.6]. By [5, Proposition 1], $f_t dim\left(AG\left(R\right)\right) = |P_1| + |P_2| - 2$. $\square$

Theorem 2.6. *Let n be a non-negative integer and $n \geq 2$. Assume that $R = \prod_{i=1}^{n} \mathbb{Z}_2$. Then*

$$f_t dim\left(AG\left(R\right)\right) = \begin{cases} n & if \quad 2 \leq n \leq 3, \\ n+1 & if \quad n \geq 4. \end{cases}$$

Proof. If $n = 2$, then by Proposition 2.5, $AG\left(R\right) \cong P_2$ and thus

$$f_t dim\left(AG\left(\mathbb{Z}_2 \times \mathbb{Z}_2\right)\right) = 2.$$

If $n = 3$, then $dim\left(AG\left(R\right)\right) = 2$ follows from [18, Theorem 2.1].
By [10, Corollary 1], $f_t dim\left(AG\left(R\right)\right) \geq 3$. Choose

$$F = \{(1,0,0), (0,1,0), (0,0,1)\}.$$

It is easy to show that F is a resolving set. If an element from F is removed, the set is a resolving set. Therefore, F is a fault-tolerant resolving set. Thus $f_t dim\left(AG\left(R\right)\right) \leq 3$. Hence, $f_t dim\left(AG\left(R\right)\right) = 3$. If $n \geq 4$, then by [18, Theorem 2.1], $dim\left(AG\left(\prod_{i=1}^{n} \mathbb{Z}_2\right)\right) = n$. Also, by [10, Corollary 1], we have $f_t dim\left(AG\left(\prod_{i=1}^{n} \mathbb{Z}_2\right)\right) \geq n+1$. We state the following claim:

Claim: $f_t dim\left(AG\left(\prod_{i=1}^{n} \mathbb{Z}_2\right)\right) \leq n+1$, for $n \geq 4$.

Let $F = \{f_1, \ldots, f_n, f_{n+1}\}$, where $f_i = (0, \ldots, 1, \ldots, 0)$, whose i^{th} component is 1 and $f_{n+1} = (1, \ldots, 1, 0)$, whose n^{th} component is 0. We have to prove that F is a fault-tolerant resolving set of $AG\left(R\right)$. Let $a \in V\left(AG\left(R\right)\right) \setminus F$, then the metric representation of a is,

$$D_F\left(a|F\right) = \left(d\left(a, f_1\right), \ldots, d\left(a, f_n\right), d\left(a, f_{n+1}\right)\right)$$

Let $a, b \in V\left(AG\left(R\right)\right), a \neq b$. Consider the product $a.f_i$, for $1 \leq i \leq n$,

$$a \cdot f_i = \begin{cases} (0, \ldots, 0) & if \quad i^{th} \quad component \quad of \quad a \quad is \quad 0, \\ f_i & if \quad i^{th} \quad component \quad of \quad a \quad is \quad 1. \end{cases}$$

$$a \cdot f_{n+1} = \begin{cases} a & if \quad n^{th} \quad component \quad of \quad a \quad is \quad 0, \\ f_i & if \quad i^{th} \quad and \quad n^{th} \quad component \quad of \quad a \quad is \quad 1, \\ f & if \quad n^{th} \quad component \quad of \quad a \quad is \quad 1, \quad and \\ & i^{th} \quad component \quad of \quad a \quad is \quad 0, \end{cases}$$

where $f \in V(AG(R)) \setminus F$. We have to show that $D_F(a|F) \neq D_F(b|F)$. So we have the following cases:

Case 1: Let $a \cdot f_i = (0, \ldots, 0)$ and $b \cdot f_i = (0, \ldots, 0)$, where $1 \leq i \leq n$. Then we can choose a fixed $j \in \{1, \ldots, n\}$ such that $a \cdot f_j = (0, \ldots, 0)$ and $b \cdot f_j \neq (0, \ldots, 0)$. It is clear that $b \cdot f_j = f_j \Rightarrow d(b, f_j) = 2$. Since $a \cdot f_j = (0, \ldots, 0)$ which implies that $d(a, f_j) = 1$. Therefore, $d(a, f_j) \neq d(b, f_j)$, and hence, $D_F(a|F) \neq D_F(b|F)$.

Case 2: Let $a \cdot f_{n+1} = a$ and $b \cdot f_{n+1} = b$ or $a \cdot f_{n+1} = f$ and $b \cdot f_{n+1} = f$, where $f \in V(AG(R)) \setminus F$. As in the proof of case 1, we can select a fixed $j \in \{1, \ldots, n\}$ such that $d(b, f_j) \neq d(a, f_j)$. Thus $D_F(a|F) \neq D_F(b|F)$.

Case 3: Assume that $a \cdot f_i = f_i$ and $b \cdot f_i = f_i$, where $i \in \{1, \ldots, n\}$. Then $a \cdot f_{n+1} = f$ and $b \cdot f_{n+1} \neq f$. It is obvious that, $b \cdot f_{n+1} = b$ and $d(b, f_{n+1}) = 2$. Therefore, $d(b, f_{n+1}) \neq d(a, f_{n+1})$ and thus $D_F(a|F) \neq D_F(b|F)$. Similarly, if $a \cdot f_{n+1} = a$ and $b \cdot f_{n+1} \neq b$. Then $b \cdot f_{n+1} \neq f_{n+1} \Rightarrow b \cdot f_{n+1} = f_i$ or f, where $i \in \{1, \ldots, n\}$. In both cases, $d(b, f_j) \neq d(a, f_j)$. Therefore, $D_F(a|F) \neq D_F(b|F)$. Assume that $a \cdot f_{n+1} = f_{n+1}$ and $b \cdot f_{n+1} = f_{n+1}$. Subsequently, we can opt for a fixed $j \in \{1, \ldots, n\}$ such that $a \cdot f_j = f_j$ and $b \cdot f_j \neq f_j$. Then $b \cdot f_j = (0, \ldots, 0) \Rightarrow d(b, f_j) = 1$. As a result, it is clear that $D_F(a|F) \neq D_F(b|F)$. Therefore, F is a resolving set. We can also examine that, for every $a, b \in V(AG(R))$, at least two elements in the $n+1-vector\ \ AD((a,b)|F)$ are non zero. Therefore, F is a fault-tolerant resolving set. Hence, $f_t dim\left(AG\left(\prod_{i=1}^{n} \mathbb{Z}_2\right)\right) \leq n+1$, for $n \geq 4$. □

Remark 2.7. Let G be a connected graph and $P_1, P_2, \ldots P_k$ be a partition of $V(G)$ such that for every $1 \leq i \leq k$, if $x, y \in P_i$, then $N(x) = N(y)$. Then $f_t dim(G) \geq |V(G)| - m$, where $m = |A(G)|$ and

$$A(G) = \{P_i : |P_i| = 1, 1 \leq i \leq k\}.$$

Theorem 2.8. *Suppose that* $R = \prod_{i=1}^{n} \mathbb{F}_i$, *where* $n \geq 2$ *is an integer, each* $\mathbb{F}_i$ *is a finite field and* $\mathbb{F}_i \ncong \mathbb{Z}_2$, *for* $1 \leq i \leq n$. *Then* $f_t dim(AG(R)) = |Z(R)^*|$.

Proof. Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ be vertices of $AG(R)$, where $a_i, b_i \in \mathbb{F}_i$, for every $1 \leq i \leq n$. Consider the relation $\sim$ defined on $V(AG(R))$ by $A \sim B$, whenever $a_i = 0$ if and only if $b_i = 0$, for every $1 \leq i \leq n$. Then $\sim$ is an equivalence relation on $V(AG(R))$. The equivalence class of A is denoted as $[A]$. Let $A_1, A_2 \in [A]$. Since $A_1 \sim A_2$, this implies that $ann_R(A_1) = ann_R(A_2)$, and by [18, Lemma 2.1], we infer that $N(A_1) = N(A_2)$.

Consider $A(G) = \{[A] : |[A]| = 1\}$. Let $A_1 = (a_1, \ldots, a_n) \in A(G)$. Without loss of generality, assume that $a_i \neq 0$, for any i. For an arbitrary $a_1 \neq 0$ in A_1. Since $\mathbb{F}_1 \ncong \mathbb{Z}_2$, there exists a vertex $B \in V(AG(R))$ such that $ann_R(A_1) = ann_R(B)$. Then $B \in [A]$, which is not possible. Therefore, $a_i = 0$, for all i, and $A(G) = \phi$. Thus $f_t dim(AG(R)) \geq |Z(R)^*|$ follows from Remark 2.7. Trivially $f_t dim(AG(R)) \leq |Z(R)^*|$. Hence,

$$f_t dim(AG(R)) = |Z(R)^*|.$$

□

2.2. **Fault-Tolerant Metric Dimension of Annihilator Graph of Non-Reduced Rings.** In this section, we compute the fault-tolerant metric dimension of the annihilator graph of non-reduced rings. We begin with the following proposition.

Proposition 2.9. *Let R be a quasi-local ring with maximal ideal m and $m^2 = (0)$, then $f_t dim(AG(R)) = |m| - 1$.*

Proof. Since R is a quasi-local ring with maximal ideal m, $Z(R) = m$. Thus $Z(R) = Nil(R)$, since $m^2 = (0)$ and R is not a field. It is clear that $ann_R(a) = Z(R)$, for all $a \in Z(R)$. From [3, Theorem 3.10], $AG(R) \cong K_{|m|-1}$. From [5, Proposition 1] , $f_t dim(AG(R)) = |m| - 1$.

□

The following remark is due to V. Soleymanivarniab et al. [18].

Remark 2.10. Suppose that $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ are vertices of $AG(R)$, where $a_i, b_i \in R_i$, for every $1 \leq i \leq n$. The relation $\sim$ defined on $V(AG(R))$ by, $A \sim B$ whenever for every $1 \leq i \leq n$, the following conditions hold:

- $a_i = 0$ if and only if $b_i = 0$, for every $1 \leq i \leq n$.
- $a_i \in Nil(R_i)^*$ if and only if $b_i \in Nil(R_i)^*$, for every $1 \leq i \leq n$.
- $a_i \in U(R_i)$ if and only if $b_i \in U(R_i)$, for every $1 \leq i \leq n$.

Then $\sim$ is an equivalence relation on $V(AG(R))$. The equivalence class of A is denoted as $[A]$.

Theorem 2.11. *Suppose that $R = \prod_{i=1}^{n} R_i$, $n \geq 2$ is an integer, for each $R_i \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x] / \langle x^2 \rangle$ for $1 \leq i \leq n$. Then $f_t dim(AG(R)) = 4^n - 2^{n+1} + n$.*

Proof. Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ be vertices of $AG(R)$, where $a_i, b_i \in R_i$, for every $1 \leq i \leq n$. Consider the equivalence relation defined in Remark 2.10. Let $A_1, A_2 \in [A]$. Since $A_1 \sim A_2$, by [18, Lemma 2.1], we can infer that $N(A_1) = N(A_2)$. We have to calculate the number of equivalence classes with cardinality 1. Consider $A(G) = \{[A] : |[A]| = 1\}$. Let $A \in A(G)$ and $A = (a_1, \ldots, a_n)$. Assume that $a_i \notin Nil(R_i)$, then $a_i \in U(R_i)$, for $1 \leq i \leq n$. Since $R_i \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/\langle x^2 \rangle$ for $1 \leq i \leq n$, $|U(R_i)| > 1$. Analogous to the proof of theorem 2.8, we get $a_i \in Nil(R_i)$, for $1 \leq i \leq n$. Thus $A = (a_1, \ldots, a_n)$, $a_i \in Nil(R_i) \Rightarrow a_i \in \{0, n_i\}, n_i \in Nil(R_i)^*, 1 \leq i \leq n$. Hence, $|A(G)| = 2^n - 1$. From Remark 2.7,

$$f_t dim(AG(R)) \geq |Z(R)^*| - 2^n + 1.$$

Consider an arbitrary element A in $[A]$, $[A]_k$ denotes the number of non-zero components of A in $[A]$, where $1 \leq k \leq n-1$. Let $[A]$ and $[B]$ be two equivalence classes in $V(AG(R))$. Next we investigate the cases when $N[A]$ and $N[B]$ are equal.

Case 1: Let $[A]_k \geq 2$ and $[B]_k \geq 2$. If $[A]_k < [B]_k$, then assume that $a_n = 0$ and $b_n \neq 0$. Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$. Choose $C = (1, \ldots, 1, a), a \in Nil(R)^*$ and $C \in [C]$. Since $|[C]| \geq 2$, we can assume that $C \neq B$. Then $C \in N[A]$ but $C \notin N[B]$. Therefore, $N[A] \neq N[B]$. Similarly if $[B]_k < [A]_k$, then $N[A] \neq N[B]$. If $[A]_k = [B]_k$, then we have to show that $N[A] \neq N[B]$. Assume that for some $1 \leq i \leq n$, $a_i = 0$ and $b_i \neq 0$. Without loss of generality, assume that $a_1 = 0$ and $b_1 \neq 0$. Then $b_1 \in Nil(R)^*$ or $b_1 \in U(R)$. Choose $A = (0, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$, where $b_1 \in Nil(R)^*$. Put $C = (a, 1, \ldots, 1, 1), a \in Nil(R)^*$. Then $C \in N[B]$ but $C \notin N[A]$. Therefore, $N[A] \neq N[B]$. Assume that $a_i = 0$ if and only if $b_i = 0$, for all $1 \leq i \leq n$. Since $[A]_k = [B]_k$ implies that $a_i \in Nil(R)^*$ and $b_i \in U(R)$ or $b_i \in Nil(R)^*$ and $a_i \in U(R)$. If $a_1 \in Nil(R_1)^*$ and $b_1 \in U(R_1)$, then we choose $C = (a, 0, \ldots, 0), a \in Nil(R_1)^*$. Therefore, $C \in N[A]$ but $C \notin N[B]$. Hence, $N[A] \neq N[B]$.

Case 2: Let $[A]_k = 1$ and $[B]_k \geq 2$ or $[B]_k = 1$ and $[A]_k \geq 2$. If $[A]_k = 1$ and $[B]_k \geq 2$, then we assume that $a_1 \neq 0$ and $b_2 \neq 0$. It is obvious that, $b_2 \in Nil(R_2)^*$ or $b_2 \in U(R_2)$. If $b_2 \in U(R_2)$, then we choose $A = (a, 0, \ldots, 0)$ and $B = (0, b_2, \ldots, b_n)$, where $a \in Nil(R_1)^*$. Put $C = (0, u_2, \ldots, 0, 0)$, $u_2 \in U(R_2)$. Then $C \in N[A]$ but $C \notin N[B] \Rightarrow N[A] \neq N[B]$. If $b_2 \in Nil(R_2)^*$, then put $C = (1, u_2, \ldots, 0, 0), u_2 \in U(R_2)$. Then $C \in N[A]$ but $C \notin N[B] \Rightarrow N[A] \neq N[B]$. Therefore, $N[A] \neq N[B]$. Similarly, if $[B]_k = 1$ and $[A]_k \geq 2$, then $N[A] \neq N[B]$.

Case 3: Assume that $[A]_k = 1$ and $[B]_k = 1$. Let

$$M_1 = \{[A] : [A] \in A(G) \quad and \quad [A]_k = 1\},$$

$M_2 = \{[A] : [A] \notin A(G) \ \ and \ \ [A]_k = 1\}$ and $[A], [B] \in M_1 \cup M_2$. We have to show that $N[A] = N[B]$ if and only if $a_i = 0$ if and only if $b_i = 0$, for every $1 \leq i \leq n$ and $[A] \in M_1$ if and only if $[B] \in M_2$. Assume that $N[A] = N[B]$ and $[A] \in M_1$. If $[B] \in M_1$, then there exists a $C \in [C]$ such that $C \in N[B]$ but $C \notin N[A]$, which is a contradiction. Thus, if $[A] \in M_1$, then $[B] \in M_2$. Similarly if $[A] \in M_2$, then $[B] \in M_1$. If $[A], [B] \in M_1$. Assume that $A = (a, 0, \ldots, 0)$, and $B = (0, a, \ldots, 0)$. Put $C = (u, 0, \ldots, 0, 0)$. Hence, $C \in N[B]$ but $C \notin N[A]$. Therefore, $N[A] \neq N[B]$, a contradiction. Thus $[A] \in M_1$ if and only if $[B] \in M_2$. Assume that $[A] \in M_1$ and $a_i = 0$. If $b_i \neq 0$ then there exists a $C \in [B] \setminus \{B\}$ such that $C \in N[B]$ but $C \notin N[A]$, which is a contradiction. Therefore, $a_i = 0$ if and only if $b_i = 0$, for every $1 \leq i \leq n$. Combining case 1 and 2, for random equivalence classes $[A]$ and $[B]$, then $N[A] \neq N[B]$. By case 3, there is one equivalence class of $[B]$ such that $N[A] = N[B]$. Since $|M_1| = n$, we get to the conclusion that $f_t dim(AG(R)) \geq |Z(R)^*| - (2^n - n - 1)$. We know that

$$\begin{aligned} |Z(R)^*| = 4^n - 2^n - 1 \Rightarrow f_t dim(AG(R)) &\geq 4^n - 2^n - 1 - 2^n + 1 + n \\ &= 4^n - 2^{n+1} + n. \end{aligned}$$

Hence, $f_t dim(AG(R)) \geq 4^n - 2^{n+1} + n$.

Claim: $f_t dim(AG(R)) \leq 4^n - 2^{n+1} + n$. Let

$$P = \{(a_1, \ldots, a_n) \in Z(R)^* : a_i \in \{0, n_i\}, n_i \in Nil(R_i)^*\},$$

$Q = \{(a_1, \ldots, a_n) \in Z(R)^* : a_i \in \{0, n_i\}, n_i \in Nil(R_i)^* \ \ and \ \ [A]_k = 1\}$, and $F = Z(R)^* \setminus \{P \setminus Q\}$. Let $A, B \notin F$ and $A \neq B$. We have to prove that F is a fault-tolerant resolving set. Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$. If $A, B \in P \setminus Q$, then $D_F(A|F) \neq D_F(B|F)$. If $A, B \in P$, then $a_i = 0$ and $b_i \in Nil(R_i)$ or $b_i = 0$ and $a_i \in Nil(R_i)$, for $1 \leq i \leq n$. Assume that $A = (0, a_2, \ldots, a_n)$ and $b = (n, b_2, \ldots, b_n)$, where $n \in Nil(R_1)$. Choose $C = (n, 1, \ldots, 1)$. Then $C \in N[B]$ but $C \notin N[A]$. Since $|[C]| > 1, D_F(A|F) \neq D_F(B|F)$. If $A, B \in Q$, then we assume that $a_1 \neq 0$ and $b_2 \neq 0$. $A = (a_1, \ldots, 0)$ and $B = (0, b_2, \ldots, 0)$, where $a_1 \in Nil(R_1)$. Choose $C = (u_1, 0, \ldots, 0)$, where $u_1 \in U(R_1)$. Then $C \in N[B]$ but $C \notin N[A]$. Since $|[C]| > 1, D_F(A|F) \neq D_F(B|F)$. Therefore, F is a resolving set. Consider

$$AD((A, B)|F) = (|d(A, f_1) - d(B, f_1)|, \ldots, |d(A, f_n) - d(B, f_n)|).$$

If $A, B \in V \setminus F$, then $|[A]| > 1$ and $|[B]| > 1$. If $|[A]| = 1$ whose distance is similar to equivalence classes with $|[A]| > 1$. Similarly, holds for B.

Therefore, $AD((A,B)|F)$ has at least two non-zero terms. Hence F is a fault-tolerant resolving set. Since $|P| = 2^n - 1, |Q| = n$ and

$$F = Z(R)^* - |P| + |Q| = 4^n - 2^{n+1} + n.$$

Thus $f_t dim(AG(R)) \leq 4^n - 2^{n+1} + n$. This completes the proof. □

We have the subsequent Corollary.

Corollary 2.12. *Let $n \geq 2$ be a positive integer and $R = \prod_{i=1}^{n} R_i$, where each R_i is a finite local ring with $|Z(R_i)| > 2$. Then $f_t dim(AG(R)) = |Z(R)^*|$.*

Proof. Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ be vertices of $AG(R)$, where $a_i, b_i \in R_i$, for every $1 \leq i \leq n$. Consider the equivalence relation defined in Remark 2.10. Let $A_1, A_2 \in [A]$. Since $A_1 \sim A_2$, by [18, Lemma 2.1], we deduce that $N(A_1) = N(A_2)$. Consider $A(G) = \{[A] : |[A]| = 1\}$. Let $A = (a_1, \ldots, a_n) \in A(G)$. If $a_i \in Nil(R_i)^*$ or $a_i \in U(R_i)$, then $|[A]| > 1$, since $|Z(R_i)| > 2$, for $1 \leq i \leq n$. Therefore, $a_i = 0$, for $1 \leq i \leq n$, and $A(G) = \phi$. Then by Remark 2.7, $f_t dim(AG(R)) \geq |Z(R)^*|$. Trivially $f_t dim(AG(R)) \geq |Z(R)^*|$. This completes the proof. □

The following remark is due to V. Soleymanivarniab et al. [18].

Remark 2.13. Let $A = (a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m})$ and $B = (b_1, \ldots, b_n, b_{n+1}, \ldots, b_{n+m})$ are vertices of $AG(R)$, where $a_i, b_i \in R_i$, for $1 \leq i \leq n$ and $a_j, b_j \in \mathbb{F}_j$, for $n+1 \leq j \leq n+m$. The relation $\sim$ defined on $V(AG(R))$ by, $A \sim B$ then the following hold:

- $a_i = 0$ if and only if $b_i = 0$, for every $1 \leq i \leq n+m$.
- $a_i \in Nil(R_i)^*$ if and only if $b_i \in Nil(R_i)^*$, for every $1 \leq i \leq n$.
- $a_i \in U(R_i)$ if and only if $b_i \in U(R_i)$, for every $1 \leq i \leq n+m$.

Then $\sim$ is an equivalence relation on $V(AG(R))$.

Theorem 2.14. *Suppose that $R = R_1 \times R_2 \times \ldots R_n \times F_1 \times F_2 \times \ldots F_m$, where each $R_i \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/\langle x^2 \rangle$ for $1 \leq i \leq n$ and F_j is a finite field and $F_j \ncong \mathbb{Z}_2$ for $1 \leq j \leq m$. Then $f_t dim(AG(R)) = |Z(R)^*| - 2^n + n + 1$.*

Proof. Let $A = (a_1, \ldots, a_n, \ldots, a_{n+m})$ and $B = (b_1, \ldots, b_n, \ldots, b_{n+m})$ be vertices of $AG(R)$, where $a_i, b_i \in R_i$, for $1 \leq i \leq n$ and $a_j, b_j \in \mathbb{F}_j$, for $n+1 \leq j \leq n+m$. Consider the equivalence relation $\sim$ on $V(AG(R))$ defined in Remark 2.13. Let $A_1, A_2 \in [A]$. Then $N(A_1) = N(A_2)$. Consider $A(G) = \{[A] : |[A]| = 1\}$. Let $A = (a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) \in A(G)$. Assume that $a_j \neq 0$, for $n+1 \leq j \leq n+m$. Then $a_j \in U(R_j)$, for

$n+1 \leq j \leq n+m$ and so $|\,[A]\,| > 1$, which is not possible. Therefore, $a_j = 0$ for $n+1 \leq j \leq n+m$. Assume that $a_i \notin Nil\,(R_i)$, for $1 \leq i \leq n$. Then $a_i \in U\,(R_i)$, for $1 \leq i \leq n \Rightarrow |\,[A]\,| > 1$, which is not possible. Therefore, $a_i \in Nil\,(R_i)$ for $1 \leq i \leq n$. Thus

$$\begin{aligned} A\,(G) = \{&(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) : a_i \in \{0, n_i\}, a_i \in Nil\,(R)^*, 1 \leq i \leq n, \\ &\text{and } a_j = 0, n+1 \leq j \leq n+m\}. \end{aligned}$$

Hence, $|A\,(G)\,| = 2^n - 1$. By Remark 2.7,

$$f_t dim\,(AG\,(R)) \geq |Z\,(R)^*\,| - 2^n + 1.$$

Suppose $[A]$ and $[B]$ are two arbitrary equivalence classes. From the proof of Theorem 2.14, if $[A]_k \geq 2$ and $[B]_k \geq 2$, then $N\,[A] \neq N\,[B]$. Let $M_1 = \{[A] : [A] \in A\,(G)\ \ and\ \ [A]_k = 1\}$,

$$M_2 = \{[A] : [A] \notin A\,(G)\ \ and\ \ [A]_k = 1\}$$

and $[A]\,, [B] \in M_1 \cup M_2$. Then $N\,[A] = N\,[B]$ if and only if $a_i = 0$ if and only if $b_i = 0$, for every $1 \leq i \leq n, a_i = b_j = 0, n+1 \leq j \leq n+m$ and $[A] \in M_1$ if and only if $[B] \in M_2$. Since $|M_1| = n$. Thus

$$f_t dim\,(AG\,(R)) \geq |Z\,(R)^*\,| - 2^n + n + 1.$$

Claim: $f_t dim\,(AG\,(R)) \leq |Z\,(R)^*\,| - 2^n + n + 1$. Let

$$\begin{aligned} P = \{&(a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) : a_i \in \{0, n_i\}, n_i \in Nil\,(R)^*, \\ &1 \leq i \leq n, \quad and \quad a_j = 0, n+1 \leq j \leq n+m\}, \end{aligned}$$

$$\begin{aligned} Q = \{&(a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) : a_i \in \{0, n_i\}, n_i \in Nil\,(R)^*, \\ &1 \leq i \leq n, \quad and \quad a_j = 0, n+1 \leq j \leq n+m \quad and \quad [A]_k = 1\} \end{aligned}$$

and $F = Z\,(R)^* \setminus \{P \setminus Q\}$. We have to prove that F is a fault-tolerant resolving set. Let $A, B \notin F$ and $A \neq B$. Next, we have to show that $D_F\,(A|F) \neq D_F\,(B|F)$. For this, let $A = (a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m})$ and $B = (b_1, \ldots, b_n, b_{n+1}, \ldots, b_{n+m})$. If $A, B \in P \setminus Q$ then $N\,[A] \neq N\,[B]$. If $|\,[A]\,| = |\,[B]\,| = 1$. This implies that $a_i = 0$ and $b_i \in Nil\,(R_i)$ or $b_i = 0$ and $a_i \in Nil\,(R_i)$, for some $1 \leq i \leq n$. Without loss of generality, assume that $A = (0, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+m})$ and $B = (n, b_2, \ldots, b_n, b_{n+1}, \ldots, b_{n+m})$, where $n \in Nil\,(R_i)$. Put $C = (n, u_2, \ldots, u_n, u_{n+1}, \ldots, u_{n+m})$. Then $C \in N\,[B]$ but $C \notin N\,[A]$. Since $|\,[C]\,| > 1$, $D_F\,(A|F) \neq D_F\,(B|F)$. It is easy to show that $AD\,((A, B)\,|F)$ contains at least two non-zero terms. Therefore, F is a fault-tolerant resolving set. Hence,

$$f_t dim\,(AG\,(R)) \leq |Z\,(R)^*\,| - 2^n + n + 1.$$

□

Theorem 2.15. *Suppose that $R = R_1 \times R_2 \times \dots R_n \times F_1 \times F_2 \times \dots F_m$, where each R_i is a finite local ring with $|Z(R_i)| > 2$ for $1 \leq i \leq n$ and F_j is a finite field and $F_j \cong \mathbb{Z}_2$ for $1 \leq j \leq m$. Then $f_t dim(AG(R)) = |Z(R)^*| - 2^m + 1$.*

Proof. Let $A = (a_1, \dots, a_n, a_{n+1}, \dots, a_{n+m})$ and

$$B = (b_1, \dots, b_n, b_{n+1}, \dots, b_{n+m})$$

be vertices of $AG(R)$, where $a_i, b_i \in R_i$, for $1 \leq i \leq n$ and $a_j, b_j \in \mathbb{F}_j$, for $n+1 \leq j \leq n+m$. Consider the equivalence relation $\sim$ on $V(AG(R))$ defined in Remark 2.13. Let $A_1, A_2 \in [A]$. Then $ann_R(A_1) = ann_R(A_2)$. From [18, Lemma 2.1], $N(A_1) = N(A_2)$. Consider $A(G) = \{[A] : |[A]| = 1\}$. Let $A = (a_1, \dots, a_n, a_{n+1}, \dots, a_{n+m}) \in A(G)$. Assume that $a_i \in Nil(R_i)^*$ or $a_i \in U(R_i)$ for $1 \leq i \leq n$. Since $|Z(R_i)| > 2$ for $1 \leq i \leq n$, similar to the proof of Therem 2.8, we conclude that, $a_i = 0$, for $1 \leq i \leq n$. Thus $A = (a_1, \dots, a_n, a_{n+1}, \dots, a_{n+m}) \in A(G), a_i = 0$, for $1 \leq i \leq n$ and $a_j \in \{0,1\}$, for $n+1 \leq j \leq n+m$. Therefore, the number of elements in $A(G)$ is $2^m - 1$ and Remark 2.7 implies that $f_t dim(AG(R)) \geq |Z(R)^*| - 2^m + 1$. Let

$$P = \{(a_1, a_2, \dots, a_n, a_{n+1}, \dots, a_{n+m}) : a_i = 0, 1 \leq i \leq n, \\ \text{and } a_j \in \{0,1\}, n+1 \leq j \leq n+m\}$$

and $F = Z(R)^* \setminus P$. Let $A, B \notin F$ and $A \neq B$. We have to show that F is a fault-tolerant resolving set. Let $A = (a_1, \dots, a_n, a_{n+1}, \dots, a_{n+m})$ and $B = (b_1, \dots, b_{n+1}, \dots, b_{n+m})$. Next, we have to show that

$$D_F(A|F) \neq D_F(B|F).$$

If $A, B \in P$ then $N(A) \neq N(B)$. It is clear that $|[A]| = |[B]| = 1$. This implies that $a_i = 0$, for $1 \leq i \leq n$, $a_j \in \{0,1\}, n+1 \leq j \leq n+m$. Without loss of generality, assume that $A = (0, \dots, 0, 1, \dots, 0)$ and $B = (0, \dots, 0, 0, 1, \dots, 0)$. Put $C = (u_1, u_2, \dots, u_n, 1, \dots, 1)$. Then $C \in N[B]$ but $C \notin N[A]$. Since $|[C]| > 1$, $D_F(A|F) \neq D_F(B|F)$. We can easily verify that $AD((A,B)|F)$ contains at least two non-zero terms. Therefore, F is a fault-tolerant resolving set. Hence, $f_t dim(AG(R)) \leq |Z(R)^*| - 2^m + 1$. □

Theorem 2.16. *Suppose that $R = R_1 \times R_2 \times \dots R_n \times F_1 \times F_2 \times \dots F_m$, where each $R_i \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/\langle x^2 \rangle$ for $1 \leq i \leq n$ and F_j is a finite field and $F_j \cong \mathbb{Z}_2$ for $1 \leq j \leq m$. Then $f_t dim(AG(R)) = |Z(R)^*| - 2^{n+m} + n + 1$.*

Proof. Let $A = (a_1, \ldots, a_{n+1}, \ldots, a_{n+m})$ and $B = (b_1, \ldots, b_{n+1}, \ldots, b_{n+m})$ be vertices of $AG(R)$, where $a_i, b_i \in R_i$, for $1 \leq i \leq n$ and $a_j, b_j \in \mathbb{F}_j$, for $n+1 \leq j \leq n+m$. Consider the equivalence relation mentioned in Remark 2.13. Let $A_1, A_2 \in [A]$ then $ann_R(A_1) = ann_R(A_2)$. From [18, Lemma 2.1], $N(A_1) = N(A_2)$. Consider $A(G) = \{[A] : |[A]| = 1\}$. Let

$$A = (a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) \in A(G).$$

Assume that $a_i \notin Nil(R_i)$, $a_i \in U(R_i)$, for $1 \leq i \leq n$. Since $R_i \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/\langle x^2 \rangle$ for $1 \leq i \leq n$, $|U(R_i)| > 1 \Rightarrow |[A]| > 1$, which is not possible, since $|[A]| = 1$. Therefore, $a_i \in Nil(R_i)$ for $1 \leq i \leq n$, $a_j \in \{0, 1\}, n+1 \leq j \leq n+m$. Thus

$$\begin{aligned} A(G) = \{[A] : |[A]| = 1, a_i \in Nil(R_i), 1 \leq i \leq n \text{ and} \\ a_j \in \{0,1\}, n+1 \leq j \leq n+m\}. \end{aligned}$$

From Theorem 2.7, $f_t dim(AG(R)) \geq |Z(R)^*| - 2^{n+m} + 1$. Let

$$A = (a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) \in A(G).$$

Then $A = A_1 + A_2$. The number of non zero components in $[A_1]$ is denoted as $[A]_k$ and the number of non zero components in $[A_2]$ is denoted as $[A]_l$, $0 \leq k \leq n$ and $n+1 \leq l \leq n+m$. Consider two arbitrary equivalence classes $[A]$ and $[B]$. From the proof of Theorem 2.14, if $[A]_k \geq 2$ and $[B]_k \geq 2$, then $N[A] \neq N[B]$. Let $M_1 = \{[A] : [A] \in A(G) \quad and \quad [A]_k = 1\}$, $M_2 = \{[A] : [A] \notin A(G) \quad and \quad [A]_k = 1\}$. Then $N[A] = N[B]$ if and only if $a_i = 0$ if and only if $b_i = 0$, for every $1 \leq i \leq n$ and $[A] \in M_1$ if and only if $[B] \in M_2$. Hence, $f_t dim(AG(R)) \geq |Z(R)^*| - 2^{n+m} + n + 1$. We state the following claim:

Claim: $f_t dim(AG(R)) \leq |Z(R)^*| - 2^{n+m} + n + 1$. Let

$$P = \{(a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) \in Z(R)^* : a_i \in \{0, n_i\}, n_i \in Nil(R_i)^*\},$$

$$\begin{aligned} Q = \{&(a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) \in Z(R)^* : a_i \in \{0, n_i\}, n_i \in Nil(R_i)^* \\ &\text{and } [A]_k = 1\} \end{aligned}$$

and $F = Z(R)^* \setminus \{P \setminus Q\}$. Let $A, B \notin F$ and $A \neq B$. We have to show that F is a fault-tolerant resolving set. Let $A = (a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+m})$ and $B = (b_1, b_2, \ldots, b_n, b_{n+1}, \ldots, b_{n+m})$. We have to prove that

$$D_F(A|F) \neq D_F(B|F).$$

Since $A, B \in P$, it is clear that $a_i = 0$ and $b_i \in Nil(R_i)$ or $b_i = 0$ and $a_i \in Nil(R_i)$, for $1 \leq i \leq n$. Assume that $A = (0, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+m})$

and $B = (n, b_2, \ldots, b_n, b_{n+1}, \ldots, b_{n+m})$, where $n \in Nil(R_1)$. Choose $C = (n, 1, \ldots, 1)$, where $n \in Nil(R_1)$. Then $C \in N[B]$ but $C \notin N[A]$. Since $|[C]| > 1$ which implies that $D_F(A|F) \neq D_F(B|F)$. Similarly, if $A, B \in P \setminus Q$, then $N[A] \neq N[B]$. We can easily verify that two elements in $AD((A,B)|F)$ are non-zero. Therefore, F is a fault-tolerant resolving set. Since $|P| = 2^{n+m} - 1, |Q| = n$ and

$$F = |Z(R)^*| - |P| + |Q| = |Z(R)^*| - 2^{n+m} + n + 1.$$

Thus $f_t dim(AG(R)) \leq |Z(R)^*| - 2^{n+m} + n + 1$. This completes the proof. □

Remark 2.17. Let $A = (a_1, a_2 \ldots, a_{n+1})$ and $B = (b_1, b_2 \ldots, b_n, b_{n+1})$ be vertices of $AG(R)$, where $a_1, b_1 \in R_1$, for and $a_i, b_i \in \mathbb{F}_i$, for $2 \leq i \leq n+1$. The relation $\sim$ defined on $V(AG(R))$ by, $A \sim B$ then the following hold:

- $a_i = 0$ if and only if $b_i = 0$, for every $1 \leq i \leq n+1$.
- $a_i \in Nil(R_i)$ if and only if $b_i \in Nil(R_i)$, for every $i = 1$.
- $a_i \in U(R_i)$ if and only if $b_i \in U(R_i)$, for every $1 \leq i \leq n+1$.

Then $\sim$ is an equivalence relation on $V(AG(R))$.

Corollary 2.18. *Suppose that $R = R_1 \times F_1 \times F_2 \times \ldots F_m$, where each $R_1 \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/\langle x^2 \rangle$ and $F_j \cong \mathbb{Z}_2$ for $1 \leq j \leq m$. Then*

$$f_t dim(AG(R)) = |Z(R)^*| - 2m.$$

2.3. **Fault-tolerant metric dimension of $AG(\mathbb{Z}_n)$ and $AG(\mathbb{Z}_n[i])$.** We calculate the fault-tolerant metric dimension of $AG(\mathbb{Z}_n)$ and
$AG(\mathbb{Z}_n[i])$ to conclude this paper.

The elements of the ring $\mathbb{Z}_n$ is notated as $0, 1, \ldots, n-1$. The number of zero-divisors of $\mathbb{Z}_n$ is $n - \phi(n) - 1$, where ϕ is the Euler's totient function. We calculate the $f_t dim(AG(\mathbb{Z}_n))$, where $n = p^m, pq$, in which p and q are distinct primes and $m > 1$.

Theorem 2.19. *Let $n > 1$ be an integer and consider the ring $\mathbb{Z}_n$. Then the following holds:*

(1) *$f_t dim(AG(\mathbb{Z}_n)) = p^{m-1} - 1$, where $n = p^m$, in which p is a prime number and $m > 1$.*
(2) *$f_t dim(AG(\mathbb{Z}_n)) = p + q - 2$, where $n = pq$, in which p and q are prime numbers.*
(3) *$f_t dim(AG(\mathbb{Z}_4))$ is undefined.*
(4) *$f_t dim(AG(\mathbb{Z}_p))$ is undefined*

Proof. (1) Suppose $n = p^m$, in which p is a prime number and $m > 1$. By Proposition 2.9, we derive that, $f_t dim\left(AG\left(\mathbb{Z}_n\right)\right) = p^{m-1} - 1$, where $n = p^m$, in which p is a prime number and $m > 1$.

(2) Suppose $n = pq$, in which p and q are prime numbers. By Proposition 2.5, we have $f_t dim\left(AG\left(\mathbb{Z}_n\right)\right) = p + q - 2$, where $n = pq$, in which p and q are prime numbers.

(3) Proof follows from Remark 2.3.

(4) Proof follows from Theorem 2.2.

□

The set of Gaussian integers $\mathbb{Z}[i] = \{\alpha = a + ib : a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{C}$. Gaussian norm is defined as $N(\alpha) = \alpha\overline{\alpha}$. If a is a prime integer, then it can be one of the form $a = 2$ or $a \equiv 1 \pmod 4$ or $a \equiv 3 \pmod 4$. Let p and q denote prime integers such that $q \equiv 3 \pmod 4$ and $p \equiv 1 \pmod 4$. Gaussian prime can be defined as $A = A_1 \cup A_2 \cup A_3$, where

(1) $A_1 = \{1 + i, 1 - i\}$.

(2) $A_2 = \{q : q \equiv 3 \pmod 4\}$.

(3)
$$\begin{aligned} A_3 &= \{a + ib : p = a^2 + b^2, \text{ for some } a, b \in \mathbb{Z} \text{ and } p \equiv 1 \pmod 4\} \\ &\cup \{a - ib : p = a^2 + b^2, \text{ for some } a, b \in \mathbb{Z} \; and \; p \equiv 1 \pmod 4\}. \end{aligned}$$

Theorem 2.20. *Consider the ring $\mathbb{Z}_n[i]$, where $n > 1$, an integer.*

(1) *$f_t dim\left(AG\left(\mathbb{Z}_2[i]\right)\right)$ is undefined.*

(2) *$f_t dim\left(AG\left(\mathbb{Z}_n[i]\right)\right) = q^2 - 1$, where $n = q^2, q \equiv 3 \pmod 4$.*

(3) *$f_t dim\left(AG\left(\mathbb{Z}_n[i]\right)\right) = q_1^2 + q_2^2 - 2$, where $n = q_1 q_2$, $q_1, q_2 \equiv 3 \pmod 4$.*

(4) *$f_t dim\left(AG\left(\mathbb{Z}_n[i]\right)\right) = 2(p - 1)$, where $n = p, p \equiv 1 \pmod 4$.*

Proof. (1) $Z\left(\mathbb{Z}_2[i]\right) = \{1 + i\}$. Therefore $f_t dim\left(AG\left(\mathbb{Z}_2[i]\right)\right)$ is undefined.

(2) If $n = q^2, q \equiv 3 \pmod 4$ then $\Gamma(R) \cong K_{q^2-1}$. Therefore,
$$AG(R) \cong K_{q^2-1}.$$
Hence $f_t dim\left(AG\left(\mathbb{Z}_n[i]\right)\right) = q^2 - 1$.

(3) If $n = q_1 q_2$, $q_1, q_2 \equiv 3 \pmod 4$ then $\Gamma(R) \cong K_{q_1^2-1, q_2^2-1}$. By [3, Theorem 3.6], $AG(R) \cong K_{q_1^2-1, q_2^2-1}$. Hence $f_t dim\left(AG\left(\mathbb{Z}_n[i]\right)\right) = q_1^2 + q_2^2 - 2$.

(4) If $n = p, p \equiv 1 \pmod 4$, then $\Gamma(R) \cong K_{p-1,p-1}$. By [3, Theorem 3.6], $AG(R) \cong K_{p-1,p-1}$. Hence
$$f_t dim\left(AG\left(\mathbb{Z}_n[i]\right)\right) = 2(p - 1).$$

□

3. CONCLUSION

The fault-tolerant metric dimension of $AG(R)$ was studied in this paper. We depicted the connection between the fault-tolerant metric dimension of $AG(R)$ and some graph parameters. Furthermore, we computed the fault-tolerant metric dimension of $AG(R)$ when R is a reduced ring. Then we derived the fault-tolerant metric dimension of the annihilator graph of non-reduced rings. Finally, we determined the fault-tolerant metric dimension of $AG(\mathbb{Z}_n)$ and $AG(\mathbb{Z}_n[i])$.

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