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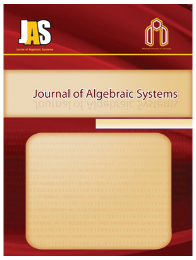

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HEMI-COMPLEMENTED LATTICES

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Abstract. The notion of hemi-complemented lattices is introduced and some of the properties of these algebras are studied. Some characterization theorems of hemi-complemented lattices are derived with the help of minimal prime D-filters, ideals, and congruences. The notion of D-Stone lattices is introduced and then derived a set of equivalent conditions for a hemi-complemented lattice to become a D-Stone lattice. Hemi-complemented lattices and D-Stone lattices are characterized in topological terms.

Introduction

In 1970, the theory of relative annihilators was introduced in lattices by Mark Mandelker [9] and he characterized distributive lattices in terms of their relative annihilators. Later, many authors introduced the concept of annihilators in the structures of rings as well as lattices and characterized several algebraic structures in terms of annihilators. T.P. Speed [13] and W.H. Cornish [3] made an extensive study of annihilators in distributive lattices. In [4], Cornish introduced and characterized the special class of distributive lattices, known as normal lattices, with the help of annihilator ideals and minimal prime ideals. In [5], some properties of dense elements of distributive lattice were studied in order to characterize another class of distributive lattices called quasi-complemented lattices.

In the year 2015, M. Sambasiva Rao [10] introduced the class of disjunctive ideals in terms of annihilator ideals of distributive lattices. He characterized the class of normal lattices with the help of disjunctive ideals and annihilator ideals. In 2016, Rao and Badawy [12] introduced the notion of co-annihilator filters of distributive lattices. In this paper, the authors introduced the concept of μ-filters of distributive lattices and studied some topological properties of the space of all prime μ-filters. In the year 2013, M. Sambasiva Rao [11] studied the properties of dense elements and D-filters of MS-algebras.

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Recently in 2020, M.S. Rao et.all studied the properties of D-filters of distributive lattices. The authors derived a set of equivalent conditions, in terms of D-filters, for a quasi-complemented lattice to become a Boolean algebra. Later, they investigated the properties of prime D-filters[8] of distributive lattices and derived some topological properties of the space of all prime D-filters of distributive lattices.

In this paper, the notion of condensed elements is introduced in distributive lattices and investigated certain properties of these condensed elements. The notion of hemi-complemented lattices is introduced, and a set of equivalent conditions is derived for a hemi-complemented lattice to become a quasi-complemented lattice. A necessary and sufficient condition is derived for a hemi-complemented lattice to become a Boolean algebra. A characterization theorem of hemi-complemented lattices is proved with the help of D-filters and minimal prime D-filters of distributive lattices. This class of hemi-complemented lattices is also characterized with the help of ideals and congruences.

The notion of D-Stone lattices is introduced and its properties are investigated in lattices. It is observed that every D-Stone lattice is hemi-complemented but not the converse. A set of equivalent conditions is established for every hemi-complemented lattice to become a D-Stone lattice. A necessary and sufficient condition is derived for the space of all minimal prime D-filters to become a compact space. Some topological characterizations of hemi-complemented lattices and D-Stone lattices are given in terms of the space of all minimal prime D-filters and the prime spectrum of D-filters of distributive lattices.

1. Preliminaries

The reader is referred to [1] and [2] for the elementary notions and notations of distributive lattices. However some of the preliminary definitions and results of [4], [3], [5], [11], [7], [8] and [13] are presented for the ready reference of the reader.

Definition 1.1. [1] An algebra $(L, \wedge, \vee)$ of type $(2, 2)$ is called a distributive lattice if for all $x, y, z \in L$, it satisfies the following properties $(1), (2), (3), (4)$ along with (5) or $(5')$,

(1) $x \wedge x = x$, $x \vee x = x$,
(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$,
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$,

(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$,
(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(5′) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Throughout this article, all lattices are bounded distributive lattices unless otherwise mentioned. A non-empty subset A of L is called an ideal(filter) of L if $a \vee b \in A(a \wedge b \in A)$ and $a \wedge x \in A(a \vee x \in A)$ whenever $a, b \in A$ and $x \in L$. The set $\mathcal{I}(L)$ of all ideals of $(L, \vee, \wedge, 0)$ forms a complete distributive lattice as well as the set $\mathcal{F}(L)$ of all filters of $(L, \vee, \wedge, 1)$ forms a complete distributive lattice. A proper ideal (filter) M of a lattice is called maximal if there exists no proper ideal(filter) N such that $M \subset N$.

Definition 1.2. [2] Let $(L, \wedge, \vee)$ be a lattice. A partial ordering relation $\leq$ is defined on L by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$. In this case, the pair $(L, \leq)$ is called a partially ordered set.

The set $(a] = \{x \in L \mid x \leq a\}$ is called *principal ideal* generated by a and the set of all principal ideals is a sublattice of $\mathcal{I}(L)$. Dually, the set $[a) = \{x \in L \mid a \leq x\}$ is called *principal filter* generated by a and the set of all principal filters is a sublattice of $\mathcal{F}(L)$. A proper ideal (proper filter) P of a lattice L is called prime if for all $a, b \in L$, $a \wedge b \in P$ $(a \vee b \in P)$ then $a \in P$ or $b \in P$. Every maximal ideal (maximal filter) is prime. A prime ideal P of L is called *minimal* [6] if there exists no prime ideal Q of L such that $Q \subset P$.

Theorem 1.3. [1] *Let F be a filter and I an ideal of a distributive lattice L such that $F \cap I = \emptyset$, then there exists a prime filter P of L such that $F \subseteq P$ and $P \cap I = \emptyset$.*

For any element a of a distributive lattice $(L, \vee, \wedge, 0)$, the *annihilator* of a is defined as $(a)^* = \{x \in L \mid x \wedge a = 0\}$. Properties of these annihilators are extensively studied by Cornish [3, 4], and T. P. Speed [13]. Normal lattices are characterized in terms of annihilators in [4].

Lemma 1.4. [13] *For any two elements a, b of a distributive lattice L with 0, we have*

(1) $a \leq b$ *implies* $(b)^* \subseteq (a)^*$,
(2) $(a \vee b)^* = (a)^* \cap (b)^*$,
(3) $(a \wedge b)^{**} = (a)^{**} \cap (b)^{**}$,
(4) $(a)^* = L$ *if and only if* $a = 0$.

An element a of a lattice L is called *dense* if $(a)^* = \{0\}$. The set D of all dense elements of a lattice L forms a filter of L. A lattice L with 0 is called quasi-complemented [5] if for each $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y \in D$. A lattice L is called *dense* if every non-zero element of L is dense.

Definition 1.5. [7] A filter F of a lattice L is called a D-filter if $D \subseteq F$.

Clearly D is the smallest D-filter of the distributive lattice. For any $a \in L$, the set $\langle a \rangle_D = [a) \vee D$ is the smallest D-filter containing a which is called the principal D-filter [7]. For any $\emptyset \neq A \subseteq L$, define

$$A^\circ = \{x \in L \mid a \vee x \in D \text{ for all } a \in A\}.$$

Then A° is a filter of L. Clearly $L^\circ = D$, $D^\circ = L$ and $D \subseteq A^\circ$ for any subset A of L. For any $a \in L$, we simply denote $(\{a\})^\circ$ by $(a)^\circ$.

Lemma 1.6. [8] *For any two subsets A, B of a lattice L, we have:*

(1) $A \subseteq B$ *implies* $B^\circ \subseteq A^\circ$,
(2) $A \subseteq A^{\circ\circ}$,
(3) $A^{\circ\circ\circ} = A^\circ$,
(4) $A^\circ = L$ *if and only if* $A \subseteq D$.

In case of filters, we have the following result.

Proposition 1.7. [8] *For any filters F, G and H of a distributive lattice L, we have:*

(1) $F^\circ \cap F^{\circ\circ} = D$,
(2) $F \cap G \subseteq D$ *implies* $F \subseteq G^\circ$,
(3) $(F \vee G)^\circ = F^\circ \cap G^\circ$,
(4) $(F \cap G)^{\circ\circ} = F^{\circ\circ} \cap G^{\circ\circ}$.

For any element x of a distributive lattice, it is clear that $([x))^\circ = (x)^\circ$. Then clearly $(0)^\circ = D$. The following corollary is a direct consequence of the above results.

Corollary 1.8. [8] *Let L be a distributive lattice and $a, b, c \in L$. Then*

(1) $a \leq b$ *implies* $(a)^\circ \subseteq (b)^\circ$,
(2) $(a \wedge b)^\circ = (a)^\circ \cap (b)^\circ$,
(3) $(a \vee b)^{\circ\circ} = (a)^{\circ\circ} \cap (b)^{\circ\circ}$,
(4) $(a)^\circ = L$ *if and only if* $a \in D$,
(5) $(a)^\circ = (b)^\circ$ *implies* $(a \wedge c)^\circ = (b \wedge c)^\circ$,
(6) $(a)^\circ = (b)^\circ$ *implies* $(a \vee c)^\circ = (b \vee c)^\circ$.

A D-filter P of a distributive lattice L is called prime [8] if $a \vee b \in P$ then either $a \in P$ or $b \in P$ for all $a, b \in L$. The intersection of all prime D-filters of a lattice is equal to D. Let I be an ideal and F a D-filter of L such that $I \cap F = \emptyset$, then there exists a prime D-filter P such that $F \subseteq P$ and $P \cap I = \emptyset$. A prime D-filter is minimal prime D-filter if and only if for each $x \in P$, there exists $y \notin P$ such that $x \vee y \in D$.

A collection τ of subsets of a set X is called a *topology* on X if (1) both the empty set and X are elements of τ; (2) any union of elements of τ is an element of τ; (3) any intersection of finitely many elements of τ is an element of τ. If τ is a topology on X, then the pair (X, τ) is called a topological space. The members of τ are called open sets in X. A base or a basis for the topology τ of a topological space (X, τ) is a family $\mathcal{B}$ of open subsets of X such that every open set of the topology is equal to the union of some sub-family of $\mathcal{B}$. The *closure* of a subset S of points in a topological space consists of all points in S together with all limit points of S. The closure of S may equivalently be defined as the union of S and its boundary, and also as the intersection of all closed sets containing S. A topological space (X, τ) is said to be compact if every open cover of X has a finite subcover. A topological space X is called a *Hausdorff space* if for each $x, y \in X$ with $x \neq y$ there are disjoint open subsets A, B such that $x \in A$ and $y \in B$.

2. Hemi-complemented lattices

In this section, the notion of hemi-complemented lattices is introduced and these special classes of lattices are then characterized in terms of ideals, D-filters, congruences, and minimal prime D-filters. A set of equivalent conditions is derived for a hemi-complemented lattice to become a quasi-complemented lattice.

Definition 2.1. An elements x of a lattice L is called *condensed* if $(x)^\circ = D$.

Clearly 0 is a condensed element in the lattice L. Let us denote the set of all condensed elements of L by D^∞.

Proposition 2.2. *The following properties hold in a lattice L;*

(1) $D \cap D^\infty = \emptyset$,
(2) D^∞ *is an ideal of* L.

Proof. (1) Let $x \in D \cap D^\infty$. Then, we get $x \in D$ and $(x)^\circ = D$. Since $x \in D$, we get that $L = (x)^\circ = D$. Hence $0 \in D$, which is a contradiction. Therefore $D \cap D^\infty = \emptyset$.

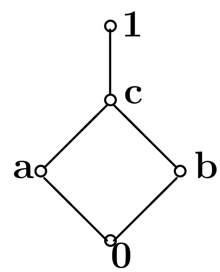
(2) Clearly $0 \in D^\infty$. Let $x, y \in D^\infty$. Then $(x)^{\circ\circ} = (y)^{\circ\circ} = D^\circ = L$. Hence $(x \vee y)^{\circ\circ} = (x)^{\circ\circ} \cap (y)^{\circ\circ} = L$. Thus $(x \vee y)^\circ = L^\circ = D$. Hence $x \vee y \in D^\infty$. Again, let $x \in D^\infty$ and $y \leq x$. Then $(y)^\circ \subseteq (x)^\circ = D$. Since $D \subseteq (y)^\circ$, we get that $(y)^\circ = D$. Hence $y \in D^\infty$. Therefore D^∞ is an ideal of L. □

Proposition 2.3. *Every dense lattice contains a unique condensed element, precisely 0.*

Proof. Let L be dense lattice. Then every non-zero element of L is a dense element. Let x be a condensed element of L. Then $(x)^\circ = D$. Suppose $x \neq 0$. Since L is dense, we get $x \in D$. By Corollary 1.8(4), we get $(x)^\circ = L$ which is a contradiction. Hence $x = 0$ is the unique condensed element of L, which is equilently to say that $D^\infty = \{0\}$. □

The converse of Proposition 2.3 is not true. That is a lattice containing unique condensed element 0 need not be a dense lattice. It can be seen the following example:

Example 2.4. Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given by:



For the elements a and b of the lattice, it can be easily observed that $(a)^\circ = \{b, c, 1\}$ and $(b)^\circ = \{a, c, 1\}$. It is easy to see that $D = \{c, 1\}$, therefore, $D = \{c, 1\} = (a)^\circ \cap (b)^\circ = (a \wedge b)^\circ = \{0\}^\circ$. Hence 0 is the only condensed element in L, but the lattice is not dense since a, b are not dense elements.

Proposition 2.5. *Define a binary relation θ on a lattice L as follows:*

$$(a, b) \in \theta \text{ if and only if } (a)^\circ = (b)^\circ$$

for all $a, b \in L$. Then θ is a congruence on L where D^∞ is the smallest congruence class with respect to θ and the unit congruence class is D.

Proof. Clearly θ is an equivalence relation on L. From (5) and (6) of Corollary 1.8, θ is a congruence on L. Let $x, y \in D^\infty$. Clearly $(x, y) \in \theta$. Hence D^∞ is a congruence class modulo θ. Let $a \in D^\infty$. Since D^∞ is an ideal, we get that $a \wedge x \in D^\infty$ for all $x \in L$. Hence $[a]_\theta \cap [y]_\theta = [a \wedge y]_\theta = [a]_\theta$ because of

$a, x \wedge a \in D^{\infty}$. Therefore $[a]_\theta = D^{\infty}$ is the smallest congruence class modulo θ. Since D is a filter, dually we get that D is the unit congruence class modulo θ. □

Definition 2.6. A lattice L is called a *hemi-complemented lattice* if to each $x \in L$, there exists $y \in L$ such that $x \wedge y \in D^{\infty}$ and $x \vee y \in D$.

Every pseudo-complemented lattice is a hemi-complemented lattice. For, consider a pseudo-complemented lattice L. Then, for each $x \in L$, there exists $x^* \in L$ such that $x \wedge x^* = 0 \in D^{\infty}$. It is obvious that $x \vee x^* \in D$. Hence L is a hemi-complemented lattice. Similarly, we can see that every quasi-complemented lattice is hemi-complemented. However, in the following theorem, we establish a set of equivalent conditions for a hemi-complemented lattice to become a quasi-complemented lattice.

Theorem 2.7. *Let L be a hemi-complemented lattice. Then the following assertions are equivalent:*

(1) *L is quasi-complemented;*
(2) *for any $x, y \in L - D$, $(x)^\circ = (y)^\circ$ implies $x = y$;*
(3) *L has a unique condensed element.*

Proof. (1) ⇒ (2): Assume that L is quasi-complemented. Let $x, y \in L - D$ be such that $(x)^\circ = (y)^\circ$. Suppose $x \neq y$. Clearly, either $(x] \cap \langle y \rangle_D = \emptyset$ or $(y] \cap \langle x \rangle_D = \emptyset$. Then there exists a prime D-filter P such that $\langle x \rangle_D \subseteq P$ and $(y] \cap P = \emptyset$. Hence $x \in \langle x \rangle_D \subseteq P$ and $y \notin P$. Since L is quasi-complemented, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' \in D$. Hence $x' \in (x)^\circ = (y)^\circ$. Hence $x' \vee y \in D \subseteq P$. Since $y \notin P$ and P is prime, we must have $x' \in P$, i.e., $0 = x \wedge x' \in P$, which is a contradiction. Therefore $x = y$.

(2) ⇒ (3): Assume condition (2). Suppose L has two distinct condensed elements, say x and y. Then $(x)^\circ = D = (y)^\circ$. By condition (2), we get $x = y$. Hence L possesses a unique condensed element.

(3) ⇒ (1): Assume that L has a unique condensed element, precisely 0. Let $x \in L$. Since L is hemi-complemented, there exists $x' \in L$ such that $x \wedge x' \in D^{\infty}$ and $x \vee x' \in D$. Since $D^{\infty} = \{0\}$, we get $x \wedge x' = 0$. Therefore L is quasi-complemented. □

Corollary 2.8. *A hemi-complemented lattice L is a Boolean algebra if and only if it contains a unique condensed element as well as a unique dense element.*

In the presence of Corollary 1.8, it can be easily verified that the class $\mathcal{D}^\circ(L)$ of all filters of the form $(x)^\circ,\ x \in L$ forms a distributive lattice with respect to the operations

$$(x)^\circ \cap (y)^\circ = (x \wedge y)^\circ \text{ and } (x)^\circ \sqcup (y)^\circ = (x \vee y)^\circ$$

for all $x, y \in L$. In the following theorem, a set of equivalent conditions is derived for the lattice $(\mathcal{D}^\circ(L), \sqcup, \cap)$ to become a Boolean algebra which leads to a characterization of hemi-complemented lattices.

Theorem 2.9. *The following assertions are equivalent in a lattice L:*

(1) *L is hemi-complemented;*
(2) *$(\mathcal{D}^\circ(L), \sqcup, \cap)$ is a Boolean algebra;*
(3) *$L_{/\theta}$ is a Boolean algebra;*
(4) *for each $x \in L$, there exists $y \in L$ such that $(x)^{\circ\circ} = (y)^\circ$;*
(5) *For any D-filter F of L with $F \cap D^\infty = \emptyset$, there exists a minimal prime D-filter P of L such that $F \subseteq P$.*

Proof. (1) $\Rightarrow$ (2): Assume that L is hemi-complemented. Let $(x)^\circ \in \mathcal{D}^\circ(L)$. Then there exists $y \in L$ such that $x \wedge y \in D^\infty$ and $x \vee y \in D$. Hence $(x)^\circ \cap (y)^\circ = (x \wedge y)^\circ = D$ and $(x)^\circ \sqcup (y)^\circ = (x \vee y)^\circ = L$. Therefore $(\mathcal{D}^\circ(L), \sqcup, \cap)$ is a Boolean algebra.

(2) $\Rightarrow$ (3): Assume that $(\mathcal{D}^\circ(L), \sqcup, \cap)$ is a Boolean algebra. Let $[x]_\theta \in L_{/\theta}$. Since $\mathcal{D}^\circ(L)$ is a Boolean algebra, there exists $y \in L$ such that $(x \wedge y)^\circ = (x)^\circ \cap (y)^\circ = D$ and $(x \vee y)^\circ = (x)^\circ \sqcup (y)^\circ = L$. Hence $x \wedge y \in D^\infty$ and $x \vee y \in D$. Thus $[x]_\theta \cap [y]_\theta = [x \wedge y]_\theta = D^\infty$ and $[x]_\theta \vee [y]_\theta = [x \vee y]_\theta = D$. Therefore $L_{/\theta}$ is a Boolean algebra.

(3) $\Rightarrow$ (4): Assume that $L_{/\theta}$ is a Boolean algebra. Let $x \in L$. Then there exists $[y]_\theta \in L_{/\theta}$ such that $[x \wedge y]_\theta = [x]_\theta \cap [y]_\theta = D^\infty$ and

$$[x \vee y]_\theta = [x]_\theta \vee [y]_\theta = D.$$

Hence $x \wedge y \in D^\infty$ and $x \vee y \in D$. Now

$$\begin{aligned} x \wedge y \in D^\infty &\Rightarrow (x \wedge y)^\circ = D \\ &\Rightarrow (x)^\circ \cap (y)^\circ = D \qquad \text{(by Proposition 1.7(2))} \\ &\Rightarrow (y)^\circ \subseteq (x)^{\circ\circ} \end{aligned}$$

$$\begin{aligned} x \vee y \in D &\Rightarrow x \in (y)^\circ \\ &\Rightarrow [x) \subseteq (y)^\circ \qquad \text{(since } (y)^\circ \text{ is a filter)} \\ &\Rightarrow (x)^{\circ\circ} \subseteq (y)^\circ \end{aligned}$$

Hence, to each $x \in L$, there exists $y \in L$ such that $(x)^{\circ\circ} = (y)^{\circ}$.

(4) $\Rightarrow$ (5): Assume condition (4). Let F be a D-filter of L such that $F \cap D^{\infty} = \emptyset$. Then there exists a prime filter P such that $F \subseteq P$ and $P \cap D^{\infty} = \emptyset$. Clearly P is a D-filter. We now show that P is minimal. Let $x \in P$. By the condition (4), there exists $y \in L$ such that $(x)^{\circ\circ} = (y)^{\circ}$. Hence $x \in (x)^{\circ\circ} = (y)^{\circ}$, which implies that $x \vee y \in D$. Again, we get $(x \wedge y)^{\circ} = (x)^{\circ} \cap (y)^{\circ} = (x)^{\circ} \cap (x)^{\circ\circ} = D$. Hence $x \wedge y \in D^{\infty}$. Since $P \cap D^{\infty} = \emptyset$, we get $x \wedge y \notin P$. Suppose $y \in P$. Then $x \wedge y \in P$, which is a contradiction. Therefore P is the minimal prime D-filter.

(5) $\Rightarrow$ (1): Assume condition (5). Let $x \in L$. Then clearly $(x)^{\circ} \vee (x)^{\circ\circ}$ is a D-filter in L. Suppose there exists a minimal prime D-filter P such that $(x)^{\circ} \vee (x)^{\circ\circ} \subseteq P$. Then $x \in (x)^{\circ\circ} \subseteq P$. Since $(x)^{\circ} \subseteq P$ and P is minimal, we get $x \notin P$, which is a contradiction. Hence $(x)^{\circ} \vee (x)^{\circ\circ}$ is not contained in any minimal prime D-filter. Thus by hypothesis (5), we get $\{(x)^{\circ} \vee (x)^{\circ\circ}\} \cap D^{\infty} \neq \emptyset$. Choose $c \in \{(x)^{\circ} \vee (x)^{\circ\circ}\} \cap D^{\infty}$. Then $(c)^{\circ} = D$ and $c = a \wedge b$ for some $a \in (x)^{\circ}$ and $b \in (x)^{\circ\circ}$. Since $a \in (x)^{\circ}$, it is clear that $a \vee x \in D$. Since $b \in (x)^{\circ\circ}$, we get $(x)^{\circ} \subseteq (b)^{\circ}$. Now we get the following consequence:

$$\begin{aligned} c = a \wedge b &\Rightarrow (a \wedge b)^{\circ} = (c)^{\circ} = D \\ &\Rightarrow (a)^{\circ} \cap (b)^{\circ} = D \\ &\Rightarrow (a)^{\circ} \cap (x)^{\circ} = D \qquad \text{(since } (x)^{\circ} \subseteq (b)^{\circ}) \\ &\Rightarrow (a \wedge x)^{\circ} = D \\ &\Rightarrow a \wedge x \in D^{\infty} \end{aligned}$$

Therefore L is hemi-complemented. Hence the proof is completed. ☐

In the following, another characterization is given for the congruence θ in hemicomplemented lattice. For this, we observe another congruence defined in terms of D^{∞}.

Definition 2.10. Let L be a lattice. For any $a \in L$, defined

$$\langle a, D^{\infty} \rangle = \{x \in L \mid x \wedge a \in D^{\infty}\}.$$

Lemma 2.11. *Let L be a lattice. For any $a, b \in L$, the following properties hold:*

1. $\langle a, D^{\infty} \rangle$ *is an ideal in L,*
2. $D^{\infty} \subseteq \langle a, D^{\infty} \rangle$,
3. $a \leq b$ *implies* $\langle b, D^{\infty} \rangle \subseteq \langle a, D^{\infty} \rangle$,
4. $\langle a \vee b, D^{\infty} \rangle = \langle a, D^{\infty} \rangle \cap \langle b, D^{\infty} \rangle$,

(5) *$a \in D^{\infty}$ if and only if $\langle a, D^{\infty}\rangle = L$.*

Proof. (1) Clearly $0 \in \langle a, D^{\infty}\rangle$. Let $x, y \in \langle a, D^{\infty}\rangle$. Then $a \wedge x \in D^{\infty}$ and $a \wedge y \in D^{\infty}$. Hence

$$(a \wedge (x \vee y))^{\circ\circ} = ((a \wedge x) \vee (a \wedge y))^{\circ\circ} = (a \wedge x)^{\circ\circ} \cap (a \wedge y)^{\circ\circ} = L \cap L = L.$$

Hence $(a \wedge (x \vee y))^{\circ} = L^{\circ} = D$, which implies $a \wedge (x \vee y) \in D^{\infty}$. Thus $x \vee y \in \langle a, D^{\infty}\rangle$. Again, let $x \in \langle a, D^{\infty}\rangle$ and $y \leq x$. Then $y \wedge a \leq x \wedge a \in D^{\infty}$. Hence $y \in \langle a, D^{\infty}\rangle$, which yields that $\langle a, D^{\infty}\rangle$ is an ideal in L.

(2) and (3) are trivial.

(4) Clearly $\langle a \vee b, D^{\infty}\rangle \subseteq \langle a, D^{\infty}\rangle \cap \langle b, D^{\infty}\rangle$. Conversely, let

$$x \in \langle a, D^{\infty}\rangle \cap \langle b, D^{\infty}\rangle.$$

Then, we get $a \wedge x \in D^{\infty}$ and $b \wedge x \in D^{\infty}$. Since D^{∞} is an ideal, we get $(a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x) \in D^{\infty}$. Therefore $x \in \langle a \vee b, D^{\infty}\rangle$.

(5) Assume that $a \in D^{\infty}$. Since D^{∞} is an ideal, we get $a \wedge x \in D^{\infty}$ for all $x \in L$. Hence $\langle a, D^{\infty}\rangle = L$. Conversely, assume that $\langle a, D^{\infty}\rangle = L$. Then, we get $1 \in \langle a, D^{\infty}\rangle$. Hence $a = 1 \wedge a \in D^{\infty}$. $\square$

Proposition 2.12. *Let L be a lattice. Define a relation $\Psi_{D^{\infty}}$ on L as*

$$(a, b) \in \Psi_{D^{\infty}} \text{ if and only if } \langle a, D^{\infty}\rangle = \langle b, D^{\infty}\rangle$$

for all $a, b \in L$. Then $\Psi_{D^{\infty}}$ is a congruence on L.

Proof. Clearly $\Psi_{D^{\infty}}$ is an equivalence relation on L. Let $(x, y) \in \Psi_{D^{\infty}}$. For any $c \in L$, we get

$$\langle x \vee c, D^{\infty}\rangle = \langle x, D^{\infty}\rangle \cap \langle c, D^{\infty}\rangle = \langle y, D^{\infty}\rangle \cap \langle c, D^{\infty}\rangle = \langle y \vee c, D^{\infty}\rangle.$$

Hence $(x \vee c, y \vee c) \in \Psi_{D^{\infty}}$. Again, for any $t \in L$, we get

$$\begin{aligned} t \in \langle x \wedge c, D^{\infty}\rangle &\Leftrightarrow t \wedge x \wedge c \in D^{\infty} \\ &\Leftrightarrow t \wedge c \in \langle x, D^{\infty}\rangle = \langle y, D^{\infty}\rangle \\ &\Leftrightarrow t \wedge c \wedge y \in D^{\infty} \\ &\Leftrightarrow t \in \langle y \wedge c, D^{\infty}\rangle. \end{aligned}$$

Thus $\langle x \wedge c, D^{\infty}\rangle = \langle y \wedge c, D^{\infty}\rangle$. Hence $(x \wedge c, y \wedge c) \in \Psi_{D^{\infty}}$. Thus $\Psi_{D^{\infty}}$ is a congruence. $\square$

Theorem 2.13. *In a hemi-complemented lattice L, $\theta = \Psi_{D^{\infty}}$.*

Proof. Let $(x,y) \in \theta$. Then $(x)^\circ = (y)^\circ$. For any $t \in L$, we get

$$\begin{aligned} t \in \langle x, D^\infty \rangle &\Leftrightarrow x \wedge t \in D^\infty \\ &\Leftrightarrow (x)^\circ \cap (t)^\circ = (x \wedge t)^\circ = D \\ &\Leftrightarrow (y \wedge t)^\circ = (y)^\circ \cap (t)^\circ = D \\ &\Leftrightarrow y \wedge t \in D^\infty \\ &\Leftrightarrow t \in \langle y, D^\infty \rangle \end{aligned}$$

Hence $\langle x, D^\infty \rangle = \langle y, D^\infty \rangle$. Thus $(x,y) \in \Psi_{D^\infty}$. Therefore $\theta \subseteq \Psi_{D^\infty}$.

Conversely, let $(x,y) \in \Psi_{D^\infty}$ for $x, y \in L$. Then $\langle x, D^\infty \rangle = \langle y, D^\infty \rangle$. Since L is hemi-complemented, there exists $x' \in L$ such that $x \wedge x' \in D^\infty$ and $x \vee x' \in D$. Hence $x' \in \langle x, D^\infty \rangle = \langle y, D^\infty \rangle$. Hence $x' \wedge y \in D^\infty$. Hence $(x')^\circ \cap (y)^\circ = (x' \wedge y)^\circ = D$. Let $t \in (y)^\circ$. Since $(y)^\circ$ is a filter, we get $t \vee x \in (y)^\circ$. Since $x \vee x' \in D$, we get $x \in (x')^\circ$. Hence $t \vee x \in (x')^\circ$. Thus $t \vee x \in (x')^\circ \cap (y)^\circ = D$. Hence $t \in (x)^\circ$. Therefore $(y)^\circ \subseteq (x)^\circ$. By a similar argument, we can obtain $(x)^\circ \subseteq (y)^\circ$. Hence $(x,y) \in \theta$. Therefore $\psi_{D^\infty} \subseteq \theta$. □

Lemma 2.14. *For any ideal J of a lattice L, the set*

$$\mathcal{D}(J) = \{x \in L \mid x \vee a \in D \text{ for some } a \in J\}$$

is a filter of L.

Proof. Clearly $D \subseteq \mathcal{D}(J)$. Let $x, y \in \mathcal{D}(J)$. Then $x \vee a \in D$ and $y \vee b \in D$ for some $a, b \in J$. Hence $(x \wedge y) \vee (a \vee b) = (x \vee a \vee b) \wedge (y \vee a \vee b) \in D$. Since J is an ideal, we get that $x \wedge y \in \mathcal{D}(J)$. Again, let $x \in \mathcal{D}(J)$ and $x \leq y$. Then $x \vee a \in D$ for some $a \in J$. Hence we get $y \vee a \in D$, which shows that $y \in \mathcal{D}(J)$. Therefore $\mathcal{D}(J)$ is a filter of L. □

Theorem 2.15. *The following assertions are equivalent in a lattice L:*

(1) *L is hemi-complemented;*
(2) *For each filter F, there exists an ideal J such that $F^{\circ\circ} = \mathcal{D}(J)$;*
(3) *For each filter F, $F^{\circ\circ} = \mathcal{D}(J_{(F^{\circ\circ})})$ where $J_{(F^{\circ\circ})}$ is the ideal as*

$$J_{(F^{\circ\circ})} = \{x \in L \mid (a)^\circ \subseteq (x)^{\circ\circ} \text{ for some } a \in F^{\circ\circ}\}.$$

Proof. (1) ⇒ (3): Assume that L is hemi-complemented. Consider the set $J_{(F^{\circ\circ})} = \{x \in L \mid (a)^\circ \subseteq (x)^{\circ\circ}$ for some $a \in F^{\circ\circ}\}$. Clearly $0 \in J_{(F^{\circ\circ})}$. Let $x, y \in J_{(F^{\circ\circ})}$. Then $(a)^\circ \subseteq (x)^{\circ\circ}$ and $(b)^\circ \subseteq (y)^{\circ\circ}$ for some $a, b \in F^{\circ\circ}$. Now $(a \wedge b)^\circ = (a)^\circ \cap (b)^\circ \subseteq (x)^{\circ\circ} \cap (y)^{\circ\circ} = (x \vee y)^{\circ\circ}$ and $a \wedge b \in F^{\circ\circ}$. Hence $x \vee y \in J_{(F^{\circ\circ})}$. Again let $x \in J_{(F^{\circ\circ})}$ and $r \in L$. Then $(a)^\circ \subseteq (x)^{\circ\circ}$ for some $a \in F^{\circ\circ}$. Now $(a)^\circ \subseteq (x)^{\circ\circ} \subseteq (x \wedge r)^{\circ\circ}$. Thus $x \wedge r \in J_{(F^{\circ\circ})}$. Therefore

$J_{(F^{\circ\circ})}$ is an ideal of L. We now prove that $F^{\circ\circ} = \mathcal{D}(J_{(F^{\circ\circ})})$. Let $x \in F^{\circ\circ}$. Since L is hemi-complemented, by Theorem 2.9, there exists $y \in L$ such that $(x)^{\circ\circ} = (y)^{\circ}$. Since $x \in F^{\circ\circ}$, we get that $y \in J_{(F^{\circ\circ})}$. Now $x \in (x)^{\circ\circ} = (y)^{\circ}$ implies that $x \vee y \in D$ for some $y \in J_{(F^{\circ\circ})}$. Therefore $F^{\circ\circ} \subseteq \mathcal{D}(J_{(F^{\circ\circ})})$. Conversely, let $x \in \mathcal{D}(J_{(F^{\circ\circ})})$. Then $x \vee f \in D$ for some $f \in J_{(F^{\circ\circ})}$. Hence $x \in (f)^{\circ}$ for some $f \in J_{(F^{\circ\circ})}$. Now, we get

$$\begin{aligned} f \in J_{(F^{\circ\circ})} &\Rightarrow (a)^{\circ} \subseteq (f)^{\circ\circ} \text{ for some } a \in F^{\circ\circ} \\ &\Rightarrow (f)^{\circ} \subseteq (a)^{\circ\circ} \text{ for some } a \in F^{\circ\circ} \\ &\Rightarrow x \in (a)^{\circ\circ} \subseteq F^{\circ\circ} \end{aligned}$$

Hence $\mathcal{D}(J_{(F^{\circ\circ})}) \subseteq F^{\circ\circ}$. Therefore $F^{\circ\circ} = \mathcal{D}(J_{(F^{\circ\circ})})$.

(3) $\Rightarrow$ (2): It is obvious.

(2) $\Rightarrow$ (1): Assume condition (2). Let $x \in L$. Hence by condition (2), there exists an ideal J such that $(x)^{\circ\circ} = \mathcal{D}(J)$. Therefore,

$$\begin{aligned} x \in \mathcal{D}(J) &\Rightarrow x \vee i \in D \text{ for some } i \in J \\ &\Rightarrow [x) \subseteq (i)^{\circ} \\ &\Rightarrow (x)^{\circ\circ} \subseteq (i)^{\circ\circ\circ} = (i)^{\circ} \end{aligned}$$

Again let $a \in (i)^{\circ}$. Then $a \vee i \in D$ and $i \in J$. Thus $a \in \mathcal{D}(J) = (x)^{\circ\circ}$. Hence $(i)^{\circ} \subseteq (x)^{\circ\circ}$. Thus $(x)^{\circ\circ} = (i)^{\circ}$. Therefore L is hemi-complemented. □

3. D-Stone lattices

In this section, the notion of D-Stone lattices is introduced. A set of equivalent conditions is derived for a hemi-complemented lattice to become a D-Stone lattice. It is proved that every hemi-complemented lattice is D-Stone if and only if $\mathcal{D}^{\circ}(L)$ is a Boolean algebra.

Lemma 3.1. *Let L be a lattice. For any $a, b \in L$, the following assertions are equivalent:*

(1) $a \vee b \in D$;
(2) $(a)^{\circ\circ} \cap [b) \subseteq D$;
(3) $(a)^{\circ\circ} \cap (b)^{\circ\circ} \subseteq D$.

Proof. (1) $\Rightarrow$ (2): Assume condition (1). Let $a, b \in L$. Suppose $a \vee b \in D$. Then $b \in (a)^{\circ}$, which gives $[b) \subseteq (a)^{\circ}$. Hence $(a)^{\circ\circ} \cap [b) \subseteq (a)^{\circ\circ} \cap (a)^{\circ} = D$.

(2) $\Rightarrow$ (3): Assume that $(a)^{\circ\circ} \cap [b) \subseteq D$ for any $a, b \in L$. By Proposition 1.7(2), we get $(a)^{\circ\circ} \subseteq (b)^{\circ}$. Therefore $(a)^{\circ\circ} \cap (b)^{\circ\circ} \subseteq (b)^{\circ} \cap (b)^{\circ\circ} \subseteq D$.

(3) $\Rightarrow$ (1): Assume that $(a)^{\circ\circ} \cap (b)^{\circ\circ} \subseteq D$ for any $a, b \in L$. By Proposition 1.7(2), we get $(a)^{\circ\circ} \subseteq (b)^{\circ\circ\circ} = (b)^{\circ}$. Hence $a \in (a)^{\circ\circ} \cap (b)^{\circ}$, which means $a \vee b \in D$. $\square$

Proposition 3.2. *The intersection of all minimal prime D-filters is D.*

Proof. Clearly $D \subseteq \bigcap\{P \mid P \text{ is a minimal prime } D\text{-filter}\}$. Let $x \notin D$. Then there exists an ideal I such that $x \in I$ and I is maximal with respect to the property of not meeting D. Clearly $L - I$ is a minimal prime D-filter and $x \notin L - I$. Hence $x \notin \cap\{P \mid P \text{ is a minimal prime } D\text{-filter }\}$. Thus $\bigcap\{P \mid P \text{ is a minimal prime } D\text{-filter }\} \subseteq D$. Therefore

$$D = \bigcap\{P \mid P \text{ is a minimal prime } D\text{-filter }\}.$$

$\square$

Corollary 3.3. *Let L be a lattice Then for any $x \in L$, we have*

$$(x)^{\circ} = \bigcap\{P \mid P \textit{ is a minimal prime } D\textit{-filter such that } x \notin P\}$$

Proof. Let $a \in (x)^{\circ}$ and P a minimal prime D-filter such that $x \notin P$. Then $x \vee a \in D \subseteq P$. Since $x \notin P$, we get $a \in P$ for all minimal prime D-filters with $x \notin P$. Hence

$$(x)^{\circ} \subseteq \cap\{P \mid P \text{ is a minimal prime } D\text{-filter}, x \notin P\}. \tag{3.1}$$

Conversely, suppose that $t \notin (x)^{\circ}$. Then $t \vee x \notin D$. By Proposition 3.2, there exists a minimal prime D-filter P such that $t \vee x \notin P$. Hence $t \notin \bigcap\{P \mid P \text{ is a minimal prime } D\text{-filter such that } x \notin P\}$. Therefore

$$\bigcap\{P \mid P \text{ is a minimal prime } D\text{-filter such that } x \notin P\} \subseteq (x)^{\circ}.$$

$\square$

Definition 3.4. A lattice L is called a *D-Stone lattice* if $(x)^{\circ} \vee (x)^{\circ\circ} = L$ for all $x \in L$.

The bounded distributive lattice $L = \{0, a, b, c, 1\}$ given in Example 2.4 is a D-Stone lattice. Clearly $(a)^{\circ} \vee (a)^{\circ\circ} = \{1, c, b\} \vee \{1, c, a\} = L$ and $(b)^{\circ} \vee (b)^{\circ\circ} = \{1, c, a\} \vee \{1, c, b\} = L$. Also $(c)^{\circ} \vee (c)^{\circ\circ} = D \vee L = L$.

Proposition 3.5. *If every prime D-filter of a lattice L is minimal, then L is a D-Stone lattice.*

Proof. Assume that every prime D-filter of L is minimal. Let $x \in L$. Suppose $(x)^{\circ} \vee (x)^{\circ\circ} \neq L$. Then there exists a prime D-filter P such that $(x)^{\circ} \vee (x)^{\circ\circ} \subseteq P$. Hence $(x)^{\circ} \subseteq P$ and $(x)^{\circ\circ} \subseteq P$. Since $(x)^{\circ} \subseteq P$, we get that $x \notin P$ because of Corollary 3.3. Clearly $x \in (x)^{\circ\circ} \subseteq P$. Hence $x \in P$, which is a contradiction. Thus $(x)^{\circ} \vee (x)^{\circ\circ} = L$. $\square$

Proposition 3.6. *Every D-Stone lattice is hemi-complemented.*

Proof. Suppose L is a D-Stone lattice. Let $x \in L$. Then $(x)^{\circ} \vee (x)^{\circ\circ} = L$. Hence $0 \in (x)^{\circ} \vee (x)^{\circ\circ}$, which implies $0 = a \wedge b$ for some $a \in (x)^{\circ}$ and $b \in (x)^{\circ\circ}$. Thus $(a)^{\circ} \cap (b)^{\circ} = (a \wedge b)^{\circ} = (0)^{\circ} = D$. Hence $(a)^{\circ} \subseteq (b)^{\circ\circ} \subseteq (x)^{\circ\circ}$. Since $a \in (x)^{\circ}$, we get $(x)^{\circ\circ} \subseteq (a)^{\circ}$. Thus $(x)^{\circ\circ} = (a)^{\circ}$. Therefore L is hemi-complemented. $\square$

In the following, a set of equivalent conditions is derived for every hemi-complemented lattice to become a D-Stone lattice. Let us call that a D-filter F of a lattice L is called a D-factor of L if there exists a proper D-filter G such that $F \cap G = D$ and $F \vee G = L$.

Theorem 3.7. *Let L be a hemi-complemented lattice. Then the following assertions are equivalent;*

(1) *L is a D-Stone lattice;*
(2) *each $(x)^{\circ}$ is a D-factor of L;*
(3) *for each $x \in L$, there exists $x' \in L$ such that $(x)^{\circ} \vee (x')^{\circ} = L$;*
(4) *for $x, y \in L$, $(x)^{\circ} \vee (y)^{\circ} = (x \vee y)^{\circ}$;*
(5) *$\mathcal{D}^{\circ\circ}(L) = \{(x)^{\circ\circ} \mid x \in L\}$ is a sublattice of $\mathcal{F}(L)$, where $\mathcal{F}(L)$ is the lattice of all filters of L.*

Proof. (1) $\Rightarrow$ (2): Assume that L is a D-Stone lattice. Let $x \in L$. By Proposition 1.7(1), $(x)^{\circ} \cap (x)^{\circ\circ} = D$. By (1), we get $(x)^{\circ} \vee (x)^{\circ\circ} = L$. Therefore $(x)^{\circ}$ is a D-factor of L.

(2) $\Rightarrow$ (3): Assume condition (2). Let $x \in L$. Since L is hemi-complemented, there exists $x' \in L$ such that $(x)^{\circ\circ} = (x')^{\circ}$. Since $(x)^{\circ}$ is a D-factor of L, there exists a D-filter G such that $(x)^{\circ} \cap G = D$ and $(x)^{\circ} \vee G = L$. Since $(x)^{\circ} \cap G = D$, we get $G \subseteq (x)^{\circ\circ} = (x')^{\circ}$. Therefore $L = (x)^{\circ} \vee G \subseteq (x)^{\circ} \vee (x')^{\circ}$. Therefore $(x)^{\circ} \vee (x')^{\circ} = L$.

(3) $\Rightarrow$ (4): Assume condition (3). Let $x, y \in L$. By (3), there exists $x' \in L$ such that $(x)^{\circ} \vee (x')^{\circ} = L$. Clearly $(x)^{\circ} \vee (y)^{\circ} \subseteq (x \vee y)^{\circ}$. Conversely, let $a \in (x \vee y)^{\circ}$. Then $a \vee x \vee y \in D$, which gives $a \vee y \in (x)^{\circ}$. By Proposition 1.7(2) and Lemma 3.1, we get

$$\begin{aligned} a \vee y \in (x)^{\circ} &\Rightarrow (x)^{\circ\circ} \subseteq (a \vee y)^{\circ} \\ &\Rightarrow (x)^{\circ\circ} \cap [a \vee y) \subseteq D \\ &\Rightarrow (x)^{\circ\circ} \cap \{[a) \cap [y)\} \subseteq D \end{aligned}$$

$$\begin{aligned}&\Rightarrow \{(x)^{\circ\circ} \cap [a)\} \cap [y) \subseteq D\\&\Rightarrow \{(x)^{\circ\circ} \cap [a)\} \subseteq (y)^{\circ}\\&\Rightarrow \{(x')^{\circ} \cap [a)\} \subseteq (y)^{\circ}\end{aligned}$$

Clearly $(x)^{\circ} \cap [a) \subseteq (x)^{\circ}$. Hence

$$\begin{aligned}a \in [a) = L \cap [a) &= \{(x)^{\circ} \vee (x')^{\circ}\} \cap [a)\\&= \{(x)^{\circ} \cap [a)\} \cap \{(x')^{\circ} \cap [a)\}\\&\subseteq (x)^{\circ} \vee (y)^{\circ}.\end{aligned}$$

Hence $(x \vee y)^{\circ} \subseteq (x)^{\circ} \vee (y)^{\circ}$.

(4) $\Rightarrow$ (5): For any $x, y \in L$, by Corollary 1.8(3), $(x)^{\circ\circ} \cap (y)^{\circ\circ} = (x \vee y)^{\circ\circ}$. Since L is hemi-complemented, there exist $x', y' \in L$ such that $(x)^{\circ\circ} = (x')^{\circ}$ and $(y)^{\circ\circ} = (y')^{\circ}$. Hence $(x)^{\circ\circ} \vee (y)^{\circ\circ} = (x')^{\circ} \vee (y')^{\circ} = (x' \vee y')^{\circ} = (c)^{\circ\circ}$ for some $c \in L$, as L is hemi-complemented. Therefore $\mathcal{D}^{\circ\circ}(L)$ is a sublattice of $\mathcal{F}(L)$.

(5) $\Rightarrow$ (1): Assume condition (5). Let $x \in L$. Since L is hemi-complemented, by Theorem 2.9(4), there exists $y \in L$ such that $(x)^{\circ\circ} = (y)^{\circ}$. Since $\mathcal{D}^{\circ\circ}(L)$ is a sublattice of $\mathcal{F}(L)$, we get $(x)^{\circ\circ} \vee (y)^{\circ\circ} = (t)^{\circ\circ}$ for some $t \in L$. Thus $x \wedge y \in (x)^{\circ\circ} \vee (y)^{\circ\circ} = (t)^{\circ\circ}$. Therefore

$$(t)^{\circ} = (t)^{\circ\circ\circ} \subseteq (x \wedge y)^{\circ} = (x)^{\circ} \cap (y)^{\circ} = (x)^{\circ} \cap (x)^{\circ\circ} = D$$

which implies that $(t)^{\circ\circ} = D^{\circ} = L$. Hence

$$(x)^{\circ} \vee (x)^{\circ\circ} = (y)^{\circ\circ} \vee (x)^{\circ\circ} = (t)^{\circ\circ} = L.$$

Therefore L is a D-stone lattice. □

Corollary 3.8. *If L is a D-Stone lattice, then $\mathcal{D}^{\circ}(L)$ is a sublattice of $\mathcal{F}(L)$.*

The following corollary states a property of D-Stone lattices in terms of minimal prime D-filters. Two D-filters F and G of a lattice L are called comaximal if $F \vee G = L$.

Corollary 3.9. *If L is a D-Stone lattice, then any two distinct minimal prime D-filters of L are comaximal.*

Proof. Assume that L is a D-Stone lattice. By condition (2) of Theorem 3.7, each $(x)^{\circ}$ is a D-factor of L. Let P and Q be two distinct minimal prime D-filters of L. Choose $a \in P - Q$. Hence $(a)^{\circ} \subseteq Q$. Since P is minimal, we get that $(a)^{\circ\circ} \subseteq P$. Since $(a)^{\circ}$ is a D-factor of L, there exists a D-filter G such that $(a)^{\circ} \cap G = D$ and $(a)^{\circ} \vee G = L$. Hence $G \subseteq (a)^{\circ\circ} \subseteq P$. Thus $L = (a)^{\circ} \vee G \subseteq Q \vee P$. Therefore P and Q are comaximal. □

Theorem 3.10. *Let L be a hemi-complemented lattice. Then L is a D-Stone lattice if and only if $\mathcal{D}^{\circ}(L)$ is a Boolean algebra.*

Proof. Assume that L is a D-Stone lattice. Let $(x)^{\circ}, (y)^{\circ} \in \mathcal{D}^{\circ}(L)$. Clearly $(x)^{\circ} \cap (y)^{\circ} = (x \wedge y)^{\circ}$. Since L is D-Stone lattice, by Theorem 3.7(4), we get $(x)^{\circ} \vee (y)^{\circ} = (x \vee y)^{\circ}$. Hence $(\mathcal{D}^{\circ}(L), \vee, \cap)$ is a lattice. Since $(x)^{\circ} = D$ for all $x \in D^{\infty}$, we get that D is the smallest element of $\mathcal{D}^{\circ}(L)$. Since $(d)^{\circ} = L$ for each $d \in D$, we get that L is the greatest element in $\mathcal{D}^{\circ}(L)$. It can be routinely verified that $(\mathcal{D}^{\circ}(L), \vee, \cap, D, L)$ is a bounded distributive lattice. Now, let $(x)^{\circ} \in \mathcal{D}^{\circ}(L)$ where $x \in L$. Since L is hemi-complemented, there exists $x' \in L$ such that $(x)^{\circ\circ} = (x')^{\circ}$. Hence $(x)^{\circ} \cap (x')^{\circ} = (x)^{\circ} \cap (x)^{\circ\circ} = D$. Since L is D-Stone, we get $(x)^{\circ} \vee (x)^{\circ\circ} = L$. Hence $(x)^{\circ} \vee (x')^{\circ} = L$. Thus $(x')^{\circ}$ is the complement of $(x)^{\circ}$ in $\mathcal{D}^{\circ}(L)$. Therefore $\mathcal{D}^{\circ}(L)$ is a Boolean algebra.

Conversely, assume that $\mathcal{D}^{\circ}(L)$ is a Boolean algebra. Let $x \in L$. Then $(x)^{\circ} \in \mathcal{D}^{\circ}(L)$. Since $\mathcal{D}^{\circ}(L)$ is a Boolean algebra, there exists $(x')^{\circ} \in \mathcal{D}^{\circ}(L)$ such that $(x)^{\circ} \cap (x')^{\circ} = D$ and $(x)^{\circ} \vee (x')^{\circ} = L$. Since $(x)^{\circ} \cap (x')^{\circ} = D$, we get $(x')^{\circ} \subseteq (x)^{\circ\circ}$. Hence $L = (x)^{\circ} \vee (x')^{\circ} \subseteq (x)^{\circ} \vee (x)^{\circ\circ}$, which gives $(x)^{\circ} \vee (x)^{\circ\circ} = L$. Therefore L is a D-Stone lattice. □

4. Topological characterizations

In this section, the classes of hemi-complemented lattices and D-Stone lattices are characterized in topological terms. Let us denote the set of all minimal prime D-filters of L by $Spec^{D}_{MF}(L)$. For any subset S of L, define $\mathfrak{K}_m(S) = \{P \in Spec^{D}_{MF}(L) \mid S \nsubseteq P\}$. In case of $S = \{x\}$, we write $\mathfrak{K}_m(x)$ for $\mathfrak{K}_m(\{x\}) = \{P \in Spec^{D}_{MF}(L) \mid x \notin P\}$. Then it can be easily seen that $\mathfrak{K}_m(S) = \bigcup\limits_{x \in S} \mathfrak{K}_m(x)$.

Lemma 4.1. *For any x, y of a lattice L, the following properties hold:*

1. $\mathfrak{K}_m(x) \cap \mathfrak{K}_m(y) = \mathfrak{K}_m(x \vee y)$,
2. $\mathfrak{K}_m(x \wedge y) \subseteq \mathfrak{K}_m(x) \cup \mathfrak{K}_m(y)$,
3. $\mathfrak{K}_m(x) = \mathfrak{K}_m((x)^{\circ\circ})$,
4. $(x)^{\circ} \subseteq (y)^{\circ}$ *if and only if* $\mathfrak{K}_m(y) \subseteq \mathfrak{K}_m(x)$,
5. $\mathfrak{K}_m(x) = \emptyset$ *if and only if* $x \in D$,
6. $\mathfrak{K}_m(x) = Spec^{D}_{MF}(L)$ *if and only if x is condensed.*

Proof. The proofs of (1) and (2) are trivial.

(3) Let $P \in \mathfrak{K}_m(x)$. Then $x \notin P$. Since P is minimal, we get $(x)^{\circ\circ} \nsubseteq P$. Hence $P \in \mathfrak{K}_m((x)^{\circ\circ})$. Therefore $\mathfrak{K}_m(x) \subseteq \mathfrak{K}_m((x)^{\circ\circ})$. Conversely, let $P \in \mathfrak{K}_m((x)^{\circ\circ})$. Then $(x)^{\circ\circ} \nsubseteq P$. Suppose $x \in P$. Since P is

minimal, we get $(x)^{\circ\circ} \subseteq P$, which is a contradiction. Hence $P \in \mathfrak{K}_m(x)$. Thus $\mathfrak{K}_m((x)^{\circ\circ}) \subseteq \mathfrak{K}_m(x)$. Therefore $\mathfrak{K}_m(x) = \mathfrak{K}_m((x)^{\circ\circ})$.

(4) Assume that $(x)^\circ \subseteq (y)^\circ$. Let $P \in \mathfrak{K}_m(y)$. Then $y \notin P$. Hence $(x)^\circ \subseteq (y)^\circ \subseteq P$. Since P is minimal, we get $x \notin P$. Hence $P \in \mathfrak{K}_m(x)$. Therefore $\mathfrak{K}_m(y) \subseteq \mathfrak{K}_m(x)$. Conversely, assume that $\mathfrak{K}_m(y) \subseteq \mathfrak{K}_m(x)$. Let $a \notin (y)^\circ$. Then $a \vee y \notin D$. Then by Proposition 3.2, there exists a minimal prime D-filter P_0 such that $a \vee y \notin P_0$. Hence $a \notin P_0$ and $y \notin P_0$. Thus $a \notin P_0$ and $P_0 \in \mathfrak{K}_m(y) \subseteq \mathfrak{K}_m(x)$. Hence $a \notin P_0$ and $x \notin P_0$. Thus $a \vee x \notin P_0$, which implies that $a \vee x \notin D$. Hence $a \notin (x)^\circ$. Therefore $(x)^\circ \subseteq (y)^\circ$.

(5) It is obvious by Proposition 3.2.

(6) Assume that $\mathfrak{K}_m(x) = Spec^D_{MF}(L)$. Then

$$(x)^\circ = \bigcap_{P \in \mathfrak{K}_m(x)} P = \bigcap_{P \in Spec^D_{MF}(L)} P = D.$$

Hence x is condensed. Conversely, assume that x is condensed. Then $(x)^\circ = D \subseteq P$ for all $P \in Spec^D_{MF}(L)$. Hence $x \notin P$ for all $P \in Spec^D_{MF}(L)$. Therefore $\mathfrak{K}_m(x) = Spec^D_{MF}(L)$. □

From Lemma 4.1, we conclude that $\{\mathfrak{K}_m(x) \mid x \in L\}$ is the class of all subsets of $Spec^D_{MF}(L)$ which is closed under finite intersections. Also, since every minimal prime D-filter is proper, we get that $\bigcup_{x \in L} \mathfrak{K}_m(x) = Spec^D_{MF}(L)$. Therefore $\{\mathfrak{K}_m(x) \mid x \in L\}$ forms a base for a topology on $Spec^D_{MF}(L)$. For any subset S of a lattice L, define

$$H_m(S) = \{P \in Spec^D_{MF}(L) \mid S \subseteq P\}.$$

In case of $S = \{x\}$, we write $H_m(x)$ for

$$H_m(\{x\}) = \{P \in Spec^D_{MF}(L) \mid x \in P\}.$$

Keeping in view of the above facts, we can have the following:

Lemma 4.2. *For any two D-filters F and G of a lattice L, the following properties hold:*

(1) $H_m(F) \cap H_m(G) = H_m(F \vee G)$,

(2) $H_m(F) = Spec^D_{MF}(L)$ *if and only if* $F = D$,

(3) $\mathfrak{K}_m(x) = H_m((x)^\circ)$ *for all* $x \in L$,

(4) $H_m(x) = \mathfrak{K}_m((x)^\circ)$.

Proof. (1) It is obvious.

(2) Assume that $H_m(F) = Spec^D_{MF}(L)$. Then $F \subseteq P$ for all $P \in Spec^D_{MF}(L)$. Let $x \in F$. Suppose $x \notin D$. Then by Proposition 3.2, there exists $P \in Spec^D_{MF}(L)$ such that $x \notin P$. Hence we get $F \nsubseteq P$, which is

a contradiction. Thus $x \in D$ and hence we get $F \subseteq D$. Since F is a D-filter, we get $F = D$. Converse is clear.

(3) Let $P \in Spec^D_{MF}(L)$. Then

$$P \in \mathfrak{K}_m(x) \Leftrightarrow x \notin P \Leftrightarrow (x)^\circ \subseteq P \Leftrightarrow P \in H_m((x)^\circ).$$

(4) Similarly, it can be obtained. □

In the following two theorems, some topological properties of the space $Spec^D_{MF}(L)$ of all minimal prime D-filters of a lattice L can be observed.

Theorem 4.3. *For any lattice L, the space $Spec^D_{MF}(L)$ is compact if and only if $H_m(F) = \emptyset$ implies $F \cap D^\infty \neq \emptyset$ for any filter F of L.*

Proof. Assume that $Spec^D_{MF}(L)$ is compact. Let F be a D-filter of L such that $H_m(F) = \emptyset$. Then $F \nsubseteq P$ for all $P \in Spec^D_{MF}(L)$. Hence $Spec^D_{MF}(L) = \mathfrak{K}_m(F) = \bigcup_{x \in F} \mathfrak{K}_m(x)$. Since $Spec^D_{MF}(L)$ is compact, we get that $Spec^D_{MF}(L) = \bigcup_{i=1}^{n} \mathfrak{K}_m(x_i)$ for some $x_1, x_2, \ldots, x_n \in F$. But by Lemma 4.2, we get

$$\bigcup_{i=1}^{n} \mathfrak{K}_m(x_i) = \bigcup_{i=1}^{n} H_m(x_i)^\circ = H_m\Big(\bigcap_{i=1}^{n} (x_i)^\circ\Big) = H_m\Big(\bigwedge_{i=1}^{n} (x_i)^\circ\Big) = \mathfrak{K}_m\Big(\bigwedge_{i=1}^{n} (x_i)^\circ\Big).$$

Now by the fact that F is a filter and each $x_i \in F$, we conclude, $\bigwedge_{i=1}^{n} x_i \in F$. Therefore there exists $a \in \bigwedge_{i=1}^{n} x_i \in F$ such that $Spec^D_{MF}(L) = \mathfrak{K}_m(a)$. Hence by Lemma 4.1(6), we get $a \in D^\infty$. Therefore $F \cap D^\infty \neq \emptyset$.

Conversely, assume that $H_m(F) = \emptyset$ implies $F \cap D^\infty \neq \emptyset$ for any D-filter F of L. Let $S \subseteq L$ be such that $Spec^D_{MF}(L) = \bigcup_{a \in S} \mathfrak{K}_m(a) = \mathfrak{K}_m(S) = \mathfrak{K}_m(F)$, where F is the D-filter generated by S. Now choose $c \in F \cap D^\infty$. Then we can write $c = \bigwedge_{i=1}^{n} a_i$ for some $a_1, a_2, \ldots, a_n \in S$ and $n \in N$. Hence by Lemma 4.1(6), we get $Spec^D_{MF}(L) = \mathfrak{K}_m(c) = \mathfrak{K}_m\Big(\bigwedge_{i=1}^{n} a_i\Big) \subseteq \bigcup_{i=1}^{n} \mathfrak{K}_m(a_i)$. This shows that $Spec^D_{MF}(L)$ is compact. □

Theorem 4.4. *For any lattice L, $Spec^D_{MF}(L)$ is a Hausdorff space.*

Proof. Let P and Q be two distinct elements of $Spec^D_{MF}(L)$. Choose $x \in L$ such that $x \in P$ and $x \notin Q$. Then we get $Q \in \mathfrak{K}_m(x)$. Since $x \in P$ and P

is minimal, there exists $y \notin P$ such that $x \vee y \in D$. Hence $P \in \mathfrak{K}_m(y)$ and also $\mathfrak{K}_m(x) \cap \mathfrak{K}_m(y) = \mathfrak{K}_m(x \vee y) = \emptyset$ because of Lemma 4.1(5). Therefore $Spec^D_{MF}(L)$ is a Hausdorff space. □

In the following theorem, some equivalent conditions are derived for a lattice to become hemi-complemented, which leads to a topological characterization.

Theorem 4.5. *The following assertion are equivalent in a lattice L:*

(1) *L is hemi-complemented;*
(2) *for each $x \in L$, there exists $y \in L$ such that $H_m(x) = \mathfrak{K}_m(y)$;*
(3) *for each $x \in L$, there exists $y \in L$ such that $\mathfrak{K}_m(x) = \mathfrak{K}_m((y)^\circ)$.*

Proof. (1) ⇒ (2): Assume that L is hemi-complemented. Let $x \in L$. Then there exists $y \in L$ such that $(x)^\circ = (y)^{\circ\circ}$. Then by Lemma 4.1(6) and Lemma 4.2(4), we can obtain that $H_m(x) = \mathfrak{K}_m((x)^\circ) = \mathfrak{K}_m((y)^{\circ\circ}) = \mathfrak{K}_m(y)$.

(2) ⇒ (3): Assume condition (2). Let $x \in L$. Then there exists $y \in L$ such that $\mathfrak{K}_m(x) = H_m(y)$. Then by Lemma 3.2, we get $H_m(y) = \mathfrak{K}_m((y)^\circ)$. Therefore $\mathfrak{K}_m(x) = \mathfrak{K}_m((y)^\circ)$.

(3) ⇒ (1): Assume condition (3). Let $x \in L$. Then by (3), there exists $y \in L$ such that $\mathfrak{K}_m(x) = \mathfrak{K}_m((y)^\circ)$. Let $a \notin (y)^\circ$. Then $a \vee y \notin D$. This means that there exists a minimal prime D-filter P such that $a \vee y \notin P$. Hence $a \notin P$ and $y \notin P$. Since $y \notin P$, we get $(y)^\circ \subseteq P$ and hence $P \notin \mathfrak{K}_m((y)^\circ) = \mathfrak{K}_m(x)$. Thus $x \in P$. Since P is minimal, we get $(x)^{\circ\circ} \subseteq P$. Since $a \notin P$, we get $a \notin (x)^{\circ\circ}$. Hence $(x)^{\circ\circ} \subseteq (y)^\circ$. Similarly, we can obtain $(y)^\circ \subseteq (x)^{\circ\circ}$. Therefore L is hemi-complemented. □

It can be easily observed that the class $\mathcal{K}(L) = \{\mathfrak{K}_m(x) \mid x \in L\}$ forms a distributive lattice with respect to the set theoretic operations $\cap$ and $\cup$. In the following theorem, a necessary and sufficient condition is derived for the above set to become a Boolean algebra.

Theorem 4.6. *A lattice L is hemi-complemented if and only if $\mathcal{K}(L) = \langle \{\mathfrak{K}_m(x) \mid x \in L\}, \cap, \cup \rangle$ is a Boolean algebra.*

Proof. Assume that L is a hemi-complemented lattice. Let $\mathfrak{K}_m(x) \in \mathcal{K}(L)$. Then there exists $y \in L$ such that $x \vee y \in D$ and $(x)^\circ \cap (y)^\circ = (x \wedge y)^\circ = D$. Hence $\mathfrak{K}_m(x) \cap \mathfrak{K}_m(y) = \mathfrak{K}_m(x \vee y) = \emptyset$. Also $\mathfrak{K}_m(x) \cup \mathfrak{K}_m(y) = H_m((x)^\circ) \cup H_m((y)^\circ) = H_m((x)^\circ \cap (y)^\circ) = H_m(D) = Spec^D_{MF}(L)$. Hence $\mathfrak{K}_m(y)$ is the complement of $\mathfrak{K}_m(x)$ in $\mathcal{K}(L)$. Therefore $\mathcal{K}(L)$ is a Boolean algebra.

Conversely, assume that $\mathcal{K}(L)$ is a Boolean algebra. Let $x \in L$. Then $\mathfrak{K}_m(x) \in \mathcal{K}(L)$. Then there exists $\mathfrak{K}_m(y) \in \mathcal{K}(L)$ such that

$\mathfrak{K}_m(x \wedge y) = \mathfrak{K}_m(x) \cap \mathfrak{K}_m(y) = \emptyset$ and $\mathfrak{K}_m(x) \cup \mathfrak{K}_m(y) = Spec^D_{MF}(L)$. Hence $x \vee y \in D$. Also

$$\begin{aligned}\mathfrak{K}_m(x) \cup \mathfrak{K}_m(y) = Spec^D_{MF}(L) &\Rightarrow H_m((x)^\circ) \cup H_m((y)^\circ) = Spec^D_{MF}(L)\\ &\Rightarrow H_m((x)^\circ \cap (y)^\circ) = Spec^D_{MF}(L)\\ &\Rightarrow (x)^\circ \cap (y)^\circ = D \text{ by Lemma 4.2(2)}\end{aligned}$$

Therefore L is a hemi-complemented lattice. □

We now present a topological characterization of D-Stone lattices. Let $Spec^D_F(L)$ be the set of all prime D-filters of a lattice L. For any subset A of L, define $\mathfrak{K}(A) = \{P \in Spec^D_F(L) \mid A \nsubseteq P\}$ and

$$H(A) = \{P \in Spec^D_F(L) \mid A \subseteq P\}.$$

For $A = \{x\}$, we simply represent $\mathfrak{K}(x) = \{P \in Spec^D_F(L) \mid x \notin P\}$ and $H(x) = \{P \in Spec^D_F(L) \mid x \in P\}$. Then there are no hidden difficulties to prove the following properties hence the proof can be omitted.

Lemma 4.7. *Let L be a lattice and $x, y \in L$. Then*

(1) $\bigcup\limits_{x \in L} \mathfrak{K}(x) = Spec^D_F(L)$,
(2) $\mathfrak{K}(x) \cap \mathfrak{K}(y) = \mathfrak{K}(x \vee y)$,
(3) $\mathfrak{K}(x) \cup \mathfrak{K}(x) = \mathfrak{K}(x \wedge y)$,
(4) $\mathfrak{K}(x) = \emptyset$ *if and only if* $x \in D$,
(5) $\mathfrak{K}(0) = Spec^D_F(L)$.

From the above result, it can be easily observed that the collection $\{\mathfrak{K}(x) \mid x \in L\}$ forms a base for a topology on $Spec^D_F(L)$. For the topology on $Spec^D_{MF}(L)$, the open set corresponding to any $x \in L$ will become $Spec^D_{MF}(L) \cap \mathfrak{K}(x)$ and is denoted by $\mathfrak{K}_m(x)$. This topology on $Spec^D_F(L)$ is known as the hull-kernel topology for which $\{H(x) \mid x \in L\}$ is the hull which is the basis and the kernel is $\{\mathfrak{K}(x) \mid x \in L\}$.

Lemma 4.8. *Let F, G be two D-filters of a lattice L. Then*

(1) $H(F) = Spec^D_F(L)$ *if and only if* $F = D$,
(2) $H(F) = \emptyset$ *if and only if* $F = L$,
(3) $F \subseteq G$ *implies* $H(G) \subseteq H(F)$,
(4) $H(F) \cap H(G) = H(F \vee G)$,
(5) $H(F) \cup H(G) = H(F \cap G)$.

Proof. (1) Since F is a D-filter, we get $D \subseteq F$. Suppose $H(F) = Spec_F^D(L)$. Then $F \subseteq P$ for all $P \in Spec_F^D(L)$. Hence $F \subseteq \bigcap_{Spec_F^D(L)} P = D$. Hence $F = D$. Conversely, suppose that $F = D$. Then $F = D \subseteq P$ for all $Spec_F^D(L)$. Hence $H(F) = Spec_F^D(L)$.

(2) Assume that $H(F) = \emptyset$. Suppose $F \neq L$. Then there exists a prime D-filter P such that $F \subseteq P$. Hence $P \in H(F) = \emptyset$, which is a contradiction. Therefore $F = L$. Conversely, let $F = L$. Since there is no prime D-filter containing F, we get $H(F) = \emptyset$.

(3) It is clear.

(4) and (5) follow immediately due to P is a prime D-filter of L. □

From Lemma 4.8, we observe that the collection $\{\mathfrak{K}(F) \mid F \in \mathcal{F}^D(L)\}$ forms a base for a topology on $Spec_F^D(L)$. In this hull-kernel topology, open sets are of the form $\mathfrak{K}(F)$ where $\mathfrak{K}(F) = \{P \in Spec_F^D(L) \mid F \nsubseteq P\}$ and the closed sets are of the form

$$H(F) = Spec_F^D(L) - \mathfrak{K}(F)$$

For any subset A of $Spec_F^D(L)$, the closure $\bar{A}$ of A in the hull-kernel topology is given by

$$\overline{A} = \{Q \in Spec_F^D(L) \mid \bigcap_{P \in A} P \subseteq Q\}$$

Lemma 4.9. *Let L be a lattice and $x \in L$. Then $\overline{\mathfrak{K}(x)} = H((x)^\circ)$.*

Proof. Let $x \in L$. Then we have the following consequence:

$$\begin{aligned}\overline{\mathfrak{K}(x)} &= \{Q \in Spec_F^D(L) \mid \bigcap_{P \in \mathfrak{K}(x)} P \subseteq Q\} \\ &= \{Q \in Spec_F^D(L) \mid (x)^\circ \subseteq Q\} \\ &= H((x)^\circ).\end{aligned}$$

□

Theorem 4.10. *A lattice L is a D-Stone lattice if and only if for any $x \in L$, the closure $\overline{\mathfrak{K}(x)}$ is open in the hull-kernel topology on $Spec_F^D(L)$.*

Proof. Assume that L is a D-Stone lattice. Let $x \in L$. Then

$$\begin{aligned}\overline{\mathfrak{K}(x)} &= \left\{Q \in Spec_F^D(L) \mid \bigcap_{P \in \mathfrak{K}(x)} P \subseteq Q\right\} \\ &= \left\{Q \in Spec_F^D(L) \mid (x)^\circ \subseteq Q\right\}\end{aligned}$$

$$= \{Q \in Spec_F^D(L) \mid (x)^{\circ\circ} \nsubseteq Q\} \quad \text{since } L \text{ is } D\text{-Stone}$$
$$= \mathfrak{K}((x)^{\circ\circ})$$

Since $\mathfrak{K}((x)^{\circ\circ})$ is open in $Spec_F^D(L)$, we get $\overline{\mathfrak{K}(x)}$ is open in $Spec_F^D(L)$.

Conversely, assume that $\overline{\mathfrak{K}(x)}$ is open in the hull-kernel topology on $Spec_F^D(L)$. Then $Spec_F^D(L) - \overline{\mathfrak{K}(x)}$ is closed in $Spec_F^D(L)$. Then there exists a prime D-filter F of L such that $Spec_F^D(L) - \overline{\mathfrak{K}(x)} = H(F)$. By the above lemma, we have $\overline{\mathfrak{K}(x)} = H((x)^\circ)$. Hence $H(F)$ and $H((x)^\circ)$ are complements to each other in $Spec_F^D(L)$. Therefore $H((x)^\circ) \cup H(F) = Spec_F^D(L)$ and $H((x)^\circ) \cap H(F) = \emptyset$. By Lemma 4.8(4) and (5). we get

$$H((x)^\circ \cap F) = Spec_F^D(L) \text{ and } H((x)^\circ \vee F) = \emptyset$$

By Lema 4.8(1) and (2), we get $(x)^\circ \cap F = D$ and $(x)^\circ \vee F = L$. Since $(x)^\circ \cap F = D$, by Proposition 1.7(2), we get $F \subseteq (x)^{\circ\circ}$. Hence

$$L = (x)^\circ \vee F \subseteq (x)^\circ \vee (x)^{\circ\circ}.$$

Therefore L is a D-Stone lattice. □

5. Conclusion

In this article, an investigation is made to characterize the properties of hemi-complemented lattices and D-Stone lattices. Certain relations between hemi-complemented lattices and D-Stone lattices. A set of equivalent conditions is established for every hemi-complemented lattice to become a D-Stone lattices. Certain topological properties of hemi-complemented lattices and D-Stone lattices are also investigated. In the future work, it is proposed to instigate some properties of hemi-complemented lattices and D-Stone lattices with the help congruences which may characterize several structures of distributive lattices.

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References

1. G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. XXV, Providence, U.S.A., 1967.
2. S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Springer Verlag, 1981.
3. W. H. Cornish, Annulets and α-ideals in distributive lattices, *J. Aust. Math. Soc.*, **15** (1973), 70–77.
4. W. H. Cornish, Normal lattices, *J. Aust. Math. Soc.*, **14** (1972), 200–215.

5. W. H. Cornish, Quasicomplemented lattices, *Comment. Math. Univ. Carolin.*, **15**(3) (1974), 501–511.
6. J. Kist, Minimal prime ideals in commutative semigroups, *Proc. Lond. Math. Soc., Sec. B*, **13** (1963), 31–50.
7. A. P. P. Kumar, M. Sambasiva Rao and K. S. Babu, Filters of distributive lattices generated by dense elements, *Palest. J. Math.*, **11**(2) (2022), 45–54.
8. A. P. P. Kumar, M. Sambasiva Rao and K. S. Babu, Generalized prime D-filters of distributive lattices, *Arch. Math.*, **57**(3) (2021), 157–174.
9. M. Mandelker, Relative annihilators in lattices, *Duke Math. J.*, **37** (1970), 377–386.
10. M. Sambasiva Rao, Disjunctive ideals of distributive lattices, *Acta Math. Vietnam.*, **40**(4) (2015), 671–682.
11. M. Sambasiva Rao, e-filters of MS-algebras, *Acta Math. Sci.*, **33B**(3) (2013), 738–746.
12. M. Sambasiva Rao and A. Badawy, μ-filters of distributive lattices, *Southeast Asian Bull. Math.*, **40**(2) (2016), 251–264.
13. T. P. Speed, Some remarks on a class of distributive lattices, *J. Aust. Math. Soc.*, **9** (1969), 289–296.

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