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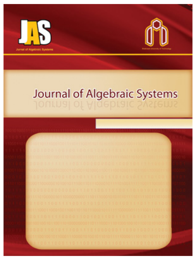

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A NOTE ON NONLINEAR MIXED $*$-JORDAN TYPE DERIVATIONS ON $*$-ALGEBRAS

M. A. SIDDEEQUE*, R. A. BHAT AND A. H. SHIKEH

Abstract. Let $\mathcal{S}$ be a $*$-algebra containing the unity and a nontrivial projection. In the present paper, we show that under certain restrictions if a map $\Psi : \mathcal{S} \to \mathcal{S}$ satisfies $\Psi(L \diamond N \bullet D) = \Psi(L) \diamond N \bullet D + L \diamond \Psi(N) \bullet D + L \diamond N \bullet \Psi(D)$ for all $L, N, D \in \mathcal{S}$, then Ψ is an additive $*$-derivation.

1. Introduction

Let $\mathcal{S}$ be a $*$-algebra with unity over the field $\mathbb{C}$ of complex numbers. For $L, N \in \mathcal{S}$, let $L \circ N = LN + NL$, $L \bullet N = LN + NL^*$, $L \triangleleft_\lambda N = LN + \lambda NL^*$ and $L \diamond N = L^*N + N^*L$ denote Jordan product, Jordan $*$-product, skew λ-Jordan product and bi-skew Jordan product of L and N respectively, where $\lambda \in \mathbb{C}$. An additive map $\Psi : \mathcal{S} \to \mathcal{S}$ is said to be an additive derivation if $\Psi(LN) = \Psi(L)N + L\Psi(N)$ for all $L, N \in \mathcal{S}$. Moreover, if $\Psi(L^*) = \Psi(L)^*$ holds for all $L \in \mathcal{S}$, then Ψ is called an additive $*$-derivation. Let $\Psi : \mathcal{S} \to \mathcal{S}$ be a mapping (not necessarily additive). Then Ψ is called a nonlinear Jordan $*$-derivation if

$$\Psi(L \bullet N) = \Psi(L) \bullet N + L \bullet \Psi(N)$$

holds for all $L, N \in \mathcal{S}$. A map (not necessarily additive) $\Psi : \mathcal{S} \to \mathcal{S}$ is said to be a nonlinear Jordan triple $*$-derivation if

$$\Psi(L \bullet N \bullet D) = \Psi(L) \bullet N \bullet D + L \bullet \Psi(N) \bullet D + L \bullet N \bullet \Psi(D)$$

holds for all $L, N, D \in \mathcal{S}$. Also a map (not necessarily additive) $\Psi : \mathcal{S} \to \mathcal{S}$ is said to be a nonlinear mixed Jordan triple derivation if

$$\Psi(E \circ K \bullet D) = \Psi(E) \circ K \bullet D + E \circ \Psi(K) \bullet D + E \circ K \bullet \Psi(D)$$

holds for all $E, K, D \in \mathcal{S}$, (for more details see [13]).

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Throughout the text, a map (not necessarily additive) $\Psi : \mathcal{S} \to \mathcal{S}$ is called a nonlinear mixed $(\diamond, \bullet)$-Jordan triple derivation if

$$\Psi(L \diamond N \bullet D) = \Psi(L) \diamond N \bullet D + L \diamond \Psi(N) \bullet D + L \diamond N \bullet \Psi(D)$$

holds for all $L, N, D \in \mathcal{S}$.

From the past few years, the evaluation of Jordan product, Jordan $*$-product, skew Jordan product, bi-skew Jordan product, mixed Lie and Jordan triple products have attracted the attention of many algebraists (see references [1, 2], [4]-[11], [15]-[19]). Recently, Darvish et al. [3] showed that if $\mathcal{S}$ is a prime $*$-algebra and $\Psi : \mathcal{S} \to \mathcal{S}$ is a map such that

$$\Psi(L \diamond N \diamond D) = \Psi(L) \diamond N \diamond D + L \diamond \Psi(N) \diamond D + L \diamond N \diamond \Psi(D)$$

holds for all $L, N, D \in \mathcal{S}$, then Ψ is an additive $*$-derivation. Taghavi et al. [14] showed that if $\mathcal{S}$ is a prime $*$-algebra and $\Psi : \mathcal{S} \to \mathcal{S}$ is a map satisfying

$$\Psi(L \triangleleft_\lambda N \triangleleft_\lambda D) = \Psi(L) \triangleleft_\lambda N \triangleleft_\lambda D + L \triangleleft_\lambda \Psi(N) \triangleleft_\lambda D + L \triangleleft_\lambda N \triangleleft_\lambda \Psi(D)$$

for all $L, N, D \in \mathcal{S}$, where $|\lambda| \neq 0, 1$, then Ψ is additive. Moreover, if $\Psi(I)$ is self-adjoint, then Ψ is an additive $*$-derivation. Liang et al. [12] studied the structure of nonlinear mixed Lie triple derivable mappings on factor von Neumann algebras and proved that every nonlinear mixed Lie triple derivation on factor von Neumann algebra is an additive $*$-derivation. This result was extended by Zhou et al. [20] to prime $*$-algebras and they obtained the same conclusion. Very recently, Rehman et al. [13] showed that every nonlinear mixed Jordan triple derivation on $*$-algebra is an additive $*$-derivation.

Inspired by the results mentioned above, in this paper we characterize the form of nonlinear mixed $(\diamond, \bullet)$-Jordan triple derivations on $*$-algebras. Precisely, we show that under certain conditions every nonlinear mixed $(\diamond, \bullet)$-Jordan triple derivation on $*$-algebras is an additive $*$-derivation.

2. Main result

The following is our main result.

Theorem 2.1. *Let $\mathcal{S}$ be an unital $*$-algebra with a non trivial projection P_1 satisfying*

$$N\mathcal{S}P_1 = 0 \textit{ implies } N = 0 \tag{2.1}$$

and

$$N\mathcal{S}(I - P_1) = 0 \textit{ implies } N = 0, \tag{2.2}$$

where $N \in \mathcal{S}$. Suppose that a map $\Psi : \mathcal{S} \to \mathcal{S}$ satisfies the condition

$$\Psi(L \diamond N \bullet D) = \Psi(L) \diamond N \bullet D + L \diamond \Psi(N) \bullet D + L \diamond N \bullet \Psi(D)$$

for all $L, N, D \in \mathcal{S}$. Then Ψ is additive. Moreover, if $\Psi(P_1)$ is self-adjoint, then Ψ is a $*$-derivation.

Proof. Let $P_2 = I - P_1$ and $\mathcal{S}_{ij} = P_i \mathcal{S} P_j$ for $i, j = 1, 2$. By Peirce decomposition of $\mathcal{S}$, we have $\mathcal{S} = \mathcal{S}_{11} \oplus \mathcal{S}_{12} \oplus \mathcal{S}_{21} \oplus \mathcal{S}_{22}$. Note that any $L \in \mathcal{S}$ can be written as $L = L_{11} + L_{12} + L_{21} + L_{22}$, where $L_{ij} \in \mathcal{S}_{ij}$ for $i, j = 1, 2$. Now to show the additivity of Ψ on $\mathcal{S}$, we use the above partition on $\mathcal{S}$ and establish some claims that will show that Ψ is additive on each $\mathcal{S}_{ij}$ for $i, j = 1, 2$. Also the following multiplicative relations are satisfied:

(i) $\mathcal{S}_{ij}\mathcal{S}_{jl} \subseteq \mathcal{S}_{il}$ $(i, j, l = 1, 2)$.
(ii) $\mathcal{S}_{ij}\mathcal{S}_{kl} = 0$ if $j \neq k, (k = 1, 2)$.

□

We begin with the following lemmas, which may be used in the proof of the above theorem.

Lemma 2.2. $\Psi(0) = 0$.

Proof. It is clear that
$\Psi(0) = \Psi(0 \diamond 0 \bullet 0) = \Psi(0) \diamond 0 \bullet 0 + 0 \diamond \Psi(0) \bullet 0 + 0 \diamond 0 \bullet \Psi(0) = 0.$ □

Lemma 2.3. *Let $L_{12} \in \mathcal{S}_{12}$ and $L_{21} \in \mathcal{S}_{21}$. Then*

$$\Psi(L_{12} + L_{21}) = \Psi(L_{12}) + \Psi(L_{21}).$$

Proof. Let $K = \Psi(L_{12} + L_{21}) - \Psi(L_{12}) - \Psi(L_{21})$. Since $L_{12} \diamond P_2 \bullet P_1 = 0$ and invoking Lemma 2.2, we have

$$\begin{aligned} \Psi((L_{12} + L_{21}) \diamond P_2 \bullet P_1) &= \Psi(L_{12} \diamond P_2 \bullet P_1) + \Psi(L_{21} \diamond P_2 \bullet P_1) \\ &= \Psi(L_{12}) \diamond P_2 \bullet P_1 + L_{12} \diamond \Psi(P_2) \bullet P_1 \\ &\quad + L_{12} \diamond P_2 \bullet \Psi(P_1) + \Psi(L_{21}) \diamond P_2 \bullet P_1 \\ &\quad + L_{21} \diamond \Psi(P_2) \bullet P_1 + L_{21} \diamond P_2 \bullet \Psi(P_1). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \Psi((L_{12} + L_{21}) \diamond P_2 \bullet P_1) &= \Psi(L_{12} + L_{21}) \diamond P_2 \bullet P_1 \\ &\quad + (L_{12} + L_{21}) \diamond \Psi(P_2) \bullet P_1 \\ &\quad + (L_{12} + L_{21}) \diamond P_2 \bullet \Psi(P_1). \end{aligned}$$

From the last two relations, we infer that $K \diamond P_2 \bullet P_1 = 0$. It follows that $P_2 K P_1 + P_1 K^* P_2 = 0$. Multiplying the previous relation by P_1 from right,

we get $P_2KP_1 = 0$. Analogously, we can show that $P_1KP_2 = 0$.
Now, again $(P_1 - P_2) \diamond I \bullet L_{21} = 0$ and invoking Lemma 2.2, we have

$$\begin{aligned}\Psi((P_1 - P_2) \diamond I \bullet (L_{12} + L_{21})) &= \Psi((P_1 - P_2) \diamond I \bullet L_{12}) \\ &\quad + \Psi((P_1 - P_2) \diamond I \bullet L_{21}) \\ &= \Psi(P_1 - P_2) \diamond I \bullet L_{12} \\ &\quad + (P_1 - P_2) \diamond \Psi(I) \bullet L_{12} \\ &\quad + (P_1 - P_2) \diamond I \bullet \Psi(L_{12}) \\ &\quad + \Psi(P_1 - P_2) \diamond I \bullet L_{21} \\ &\quad + (P_1 - P_2) \diamond \Psi(I) \bullet L_{21} \\ &\quad + (P_1 - P_2) \diamond I \bullet \Psi(L_{21}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Psi((P_1 - P_2) \diamond I \bullet (L_{12} + L_{21})) &= \Psi(P_1 - P_2) \diamond I \bullet (L_{12} + L_{21}) \\ &\quad + (P_1 - P_2) \diamond \Psi(I) \bullet (L_{12} + L_{21}) \\ &\quad + (P_1 - P_2) \diamond I \bullet \Psi(L_{12} + L_{21}).\end{aligned}$$

From the last two relations, we find that $(P_1 - P_2) \diamond I \bullet K = 0$, i.e., $2P_1K - 2P_2K + 2KP_1 - 2KP_2 = 0$. Multiplying the previous relation by P_1 from both left and right, we get $P_1KP_1 = 0$. Analogously, multiplying the previous relation by P_2 from both left and right, we get $P_2KP_2 = 0$. Hence, $K = 0$, i.e., $\Psi(L_{12} + L_{21}) = \Psi(L_{12}) + \Psi(L_{21})$. □

Lemma 2.4. *For every $L_{11} \in \mathcal{S}_{11}, L_{12} \in \mathcal{S}_{12}$, $L_{21} \in \mathcal{S}_{21}$ and $L_{22} \in \mathcal{S}_{22}$, we have*

(i) $\Psi(L_{11} + L_{12} + L_{21}) = \Psi(L_{11}) + \Psi(L_{12}) + \Psi(L_{21})$.
(ii) $\Psi(L_{12} + L_{21} + L_{22}) = \Psi(L_{12}) + \Psi(L_{21}) + \Psi(L_{22})$.

Proof. Let $K = \Psi(L_{11} + L_{12} + L_{21}) - \Psi(L_{11}) - \Psi(L_{12}) - \Psi(L_{21})$. On one hand, we have

$$\begin{aligned}\Psi((L_{11} + L_{12} + L_{21}) \diamond P_1 \bullet P_2) &= \Psi(L_{11} + L_{12} + L_{21}) \diamond P_1 \bullet P_2 \\ &\quad + (L_{11} + L_{12} + L_{21}) \diamond \Psi(P_1) \bullet P_2 \\ &\quad + (L_{11} + L_{12} + L_{21}) \diamond P_1 \bullet \Psi(P_2).\end{aligned}$$

On the other hand, invoking Lemma 2.3 and since $L_{11} \diamond P_1 \bullet P_2 = 0$, we have

$$\begin{aligned}\Psi((L_{11} + L_{12} + L_{21}) \diamond P_1 \bullet P_2) &= \Psi(L_{11} \diamond P_1 \bullet P_2) + \Psi(L_{12} \diamond P_1 \bullet P_2) \\ &\quad +\Psi(L_{21} \diamond P_1 \bullet P_2) \\ &= \Psi(L_{11}) \diamond P_1 \bullet P_2 + L_{11} \diamond \Psi(P_1) \bullet P_2 \\ &\quad +L_{11} \diamond P_1 \bullet \Psi(P_2) + \Psi(L_{12}) \diamond P_1 \bullet P_2 \\ &\quad +L_{12} \diamond \Psi(P_1) \bullet P_2 + L_{12} \diamond P_1 \bullet \Psi(P_2) \\ &\quad +\Psi(L_{21}) \diamond P_1 \bullet P_2 + L_{21} \diamond \Psi(P_1) \bullet P_2 \\ &\quad +L_{21} \diamond P_1 \bullet \Psi(P_2).\end{aligned}$$

From the last two relations, we find that $K \diamond P_1 \bullet P_2 = 0$. This gives us $P_1KP_2 + P_2K^*P_1 = 0$. Multiplying the last relation by P_2 from right, we find that $P_1KP_2 = 0$. Analogously, we can show that $P_2KP_1 = 0$.
Since

$$\frac{I}{2} \diamond (P_1 - P_2) \bullet L_{12} = \frac{I}{2} \diamond (P_1 - P_2) \bullet L_{21} = 0.$$

Now invoking Lemma 2.2, we get

$$\begin{aligned}&\Psi(\frac{I}{2} \diamond (P_1 - P_2) \bullet (L_{11} + L_{12} + L_{21})) \\ &= \Psi(\frac{I}{2} \diamond (P_1 - P_2) \bullet L_{11}) + \Psi(\frac{I}{2} \diamond (P_1 - P_2) \bullet L_{12}) + \Psi(\frac{I}{2} \diamond (P_1 - P_2) \bullet L_{21}) \\ &= \Psi(\frac{I}{2}) \diamond (P_1 - P_2) \bullet L_{11} + \frac{I}{2} \diamond \Psi(P_1 - P_2) \bullet L_{11} \\ &\quad +\frac{I}{2} \diamond (P_1 - P_2) \bullet \Psi(L_{11}) + \Psi(\frac{I}{2}) \diamond (P_1 - P_2) \bullet L_{12} \\ &\quad +\frac{I}{2} \diamond \Psi(P_1 - P_2) \bullet L_{12} + \frac{I}{2} \diamond (P_1 - P_2) \bullet \Psi(L_{12}) \\ &\quad +\Psi(\frac{I}{2}) \diamond (P_1 - P_2) \bullet L_{21} + \frac{I}{2} \diamond \Psi(P_1 - P_2) \bullet L_{21} \\ &\quad +\frac{I}{2} \diamond (P_1 - P_2) \bullet \Psi(L_{21}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\Psi(\frac{I}{2}\diamond(P_1-P_2)\bullet(L_{11}+L_{12}+L_{21}))\\&=\Psi(\frac{I}{2})\diamond(P_1-P_2)\bullet(L_{11}+L_{12}+L_{21})\\&\quad+\frac{I}{2}\diamond\Psi(P_1-P_2)\bullet(L_{11}+L_{12}+L_{21})\\&\quad+\frac{I}{2}\diamond(P_1-P_2)\bullet\Psi(L_{11}+L_{12}+L_{21}).\end{aligned}$$

From the last two relations, we obtain $\frac{I}{2}\diamond(P_1-P_2)\bullet K=0$, which yields $P_1KP_1=P_2KP_2=0$. Hence, $K=0$ i.e.,

$$\Psi(L_{11}+L_{12}+L_{21})=\Psi(L_{11})+\Psi(L_{12})+\Psi(L_{21}).$$

Analogously, we can show

$$\Psi(L_{12}+L_{21}+L_{22})=\Psi(L_{12})+\Psi(L_{21})+\Psi(L_{22}).$$

□

Lemma 2.5. *For any $L_{ij}\in\mathcal{S}_{ij}, 1\leq i,j\leq 2$, we have*

$$\Psi(\sum_{i,j=1}^{2}L_{ij})=\sum_{i,j=1}^{2}\Psi(L_{ij}).$$

Proof. Let $K=\Psi(L_{11}+L_{12}+L_{21}+L_{22})-\Psi(L_{11})-\Psi(L_{12})-\Psi(L_{21})-\Psi(L_{22})$. On one hand, we have

$$\begin{aligned}\Psi(I\diamond P_1\bullet(L_{11}+L_{12}+L_{21}+L_{22}))&=\Psi(I)\diamond P_1\bullet(L_{11}+L_{12}+L_{21}+L_{22})\\&\quad+I\diamond\Psi(P_1)\bullet(L_{11}+L_{12}+L_{21}+L_{22})\\&\quad+I\diamond P_1\bullet\Psi(L_{11}+L_{12}+L_{21}+L_{22}).\end{aligned}$$

On the other hand, since $I \diamond P_1 \bullet L_{22} = 0$ and invoking Lemmas 2.2 and 2.4, we have

$$\begin{aligned}
&\Psi(I \diamond P_1 \bullet (L_{11} + L_{12} + L_{21} + L_{22}))\\
&= \Psi(I \diamond P_1 \bullet L_{11}) + \Psi(I \diamond P_1 \bullet L_{12})\\
&\quad +\Psi(I \diamond P_1 \bullet L_{21}) + \Psi(I \diamond P_1 \bullet L_{22})\\
&= \Psi(I) \diamond P_1 \bullet L_{11} + I \diamond \Psi(P_1) \bullet L_{11}\\
&\quad +I \diamond P_1 \bullet \Psi(L_{11}) + \Psi(I) \diamond P_1 \bullet L_{12}\\
&\quad +I \diamond \Psi(P_1) \bullet L_{12} + I \diamond P_1 \bullet \Psi(L_{12})\\
&\quad +\Psi(I) \diamond P_1 \bullet L_{21} + I \diamond \Psi(P_1) \bullet L_{21}\\
&\quad +I \diamond P_1 \bullet \Psi(L_{21}) + \Psi(I) \diamond P_1 \bullet L_{22}\\
&\quad +I \diamond \Psi(P_1) \bullet L_{22} + I \diamond P_1 \bullet \Psi(L_{22}).
\end{aligned}$$

From the last two relations, we infer that $I \diamond P_1 \bullet K = 0$. Hence $P_1KP_2 = P_2KP_1 = P_1KP_1 = 0$. Analogously, we can show that $P_2KP_2 = 0$. Thus $K = 0$, i.e.,

$$\Psi(L_{11} + L_{12} + L_{21} + L_{22}) = \Psi(L_{11}) + \Psi(L_{12}) + \Psi(L_{21}) + \Psi(L_{22}).$$

□

Lemma 2.6. *For any $L_{ij}, N_{ij} \in \mathcal{S}_{ij}$ with $i \neq j$, $\Psi(L_{ij}+N_{ij}) = \Psi(L_{ij})+\Psi(N_{ij})$.*

Proof. Let $N = \Psi(L_{ij} + N_{ij}) - \Psi(L_{ij}) - \Psi(N_{ij})$. Since $P_i \diamond \frac{I}{2} \bullet 2P_j = 0$, we get

$$\begin{aligned}
\Psi(P_i \diamond \frac{I}{2} \bullet (2P_j + L_{ij} + N_{ij})) &= \Psi(P_i \diamond \frac{I}{2} \bullet 2P_j) + \Psi(P_i \diamond \frac{I}{2} \bullet (L_{ij} + N_{ij}))\\
&= \Psi(P_i) \diamond \frac{I}{2} \bullet 2P_j + P_i \diamond \Psi(\frac{I}{2}) \bullet 2P_j\\
&\quad +P_i \diamond \frac{I}{2} \bullet \Psi(2P_j) + \Psi(P_i) \diamond \frac{I}{2} \bullet (L_{ij} + N_{ij})\\
&\quad +P_i \diamond \Psi(\frac{I}{2}) \bullet (L_{ij} + N_{ij})\\
&\quad +P_i \diamond \frac{I}{2} \bullet \Psi(L_{ij} + N_{ij}).
\end{aligned}$$

Also by Lemma 2.4, we have

$$\begin{aligned}
\Psi(P_i \diamond \frac{I}{2} \bullet (2P_j + L_{ij} + N_{ij})) &= \Psi(P_i) \diamond \frac{I}{2} \bullet (2P_j + L_{ij} + N_{ij}) \\
&\quad + P_i \diamond \Psi(\frac{I}{2}) \bullet (2P_j + L_{ij} + N_{ij}) \\
&\quad + P_i \diamond \frac{I}{2} \bullet \Psi(2P_j + L_{ij} + N_{ij}) \\
&= \Psi(P_i) \diamond \frac{I}{2} \bullet (2P_j + L_{ij} + N_{ij}) \\
&\quad + P_i \diamond \Psi(\frac{I}{2}) \bullet (2P_j + L_{ij} + N_{ij}) \\
&\quad + P_i \diamond \frac{I}{2} \bullet (\Psi(P_j + L_{ij}) + \Psi(P_j + N_{ij})) \\
&= \Psi(P_i) \diamond \frac{I}{2} \bullet (2P_j + L_{ij} + N_{ij}) \\
&\quad + P_i \diamond \Psi(\frac{I}{2}) \bullet (2P_j + L_{ij} + N_{ij}) \\
&\quad + P_i \diamond \frac{I}{2} \bullet (\Psi(2P_j) + \Psi(L_{ij}) + \Psi(N_{ij})).
\end{aligned}$$

From the last two relations, we infer that $P_i \diamond \frac{I}{2} \bullet N = 0$. Hence $P_i N P_i = P_i N P_j = 0$. Thus $N = 0$. ☐

Lemma 2.7. *For any $L_{ii}, N_{ii} \in \mathcal{S}_{ii}, 1 \leq i \leq 2$, we have*

$$\Psi(L_{ii} + N_{ii}) = \Psi(L_{ii}) + \Psi(N_{ii}).$$

Proof. Let $Q = \Psi(L_{ii} + N_{ii}) - \Psi(L_{ii}) - \Psi(N_{ii})$. Since $I \diamond P_j \bullet L_{ii} = 0$ for $i \neq j$. Invoking Lemma 2.2, we have

$$\begin{aligned}
\Psi(I \diamond P_j \bullet (L_{ii} + N_{ii})) &= \Psi(I \diamond P_j \bullet L_{ii}) + \Psi(I \diamond P_j \bullet N_{ii}) \\
&= \Psi(I) \diamond P_j \bullet L_{ii} + I \diamond \Psi(P_j) \bullet L_{ii} \\
&\quad + I \diamond P_j \bullet \Psi(L_{ii}) + \Psi(I) \diamond P_j \bullet N_{ii} \\
&\quad + I \diamond \Psi(P_j) \bullet N_{ii} + I \diamond P_j \bullet \Psi(N_{ii}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\Psi(I \diamond P_j \bullet (L_{ii} + N_{ii})) &= \Psi(I) \diamond P_j \bullet (L_{ii} + N_{ii}) \\
&\quad + I \diamond \Psi(P_j) \bullet (L_{ii} + N_{ii}) \\
&\quad + I \diamond P_j \bullet \Psi(L_{ii} + N_{ii}).
\end{aligned}$$

From the last two relations, we infer that $I \diamond P_j \bullet Q = 0$. It follows that $P_jQP_j = P_jQP_i = P_iQP_j = 0$.
Next, for any $X_{ij} \in \mathcal{S}_{ij}$ with $i \neq j$, we have

$$\begin{aligned}\Psi(P_i \diamond X_{ij} \bullet (L_{ii} + N_{ii})) &= \Psi(P_i) \diamond X_{ij} \bullet (L_{ii} + N_{ii}) \\ &+ P_i \diamond \Psi(X_{ij}) \bullet (L_{ii} + N_{ii}) \\ &+ P_i \diamond X_{ij} \bullet \Psi(L_{ii} + N_{ii}).\end{aligned}$$

On the other hand, using Lemma 2.6, we have

$$\begin{aligned}\Psi(P_i \diamond X_{ij} \bullet (L_{ii} + N_{ii})) &= \Psi(X_{ij}^*L_{ii} + X_{ij}^*N_{ii} + L_{ii}X_{ij} + N_{ii}X_{ij}) \\ &= \Psi(X_{ij}^*L_{ii} + L_{ii}X_{ij}) + \Psi(X_{ij}^*N_{ii} + N_{ii}X_{ij}) \\ &= \Psi(X_{ij}^*L_{ii}) + \Psi(L_{ii}X_{ij}) + \Psi(X_{ij}^*N_{ii}) + \Psi(N_{ii}X_{ij}) \\ &= \Psi(X_{ij}^*L_{ii} + L_{ii}X_{ij}) + \Psi(X_{ij}^*N_{ii} + N_{ii}X_{ij}) \\ &= \Psi(P_i \diamond X_{ij} \bullet L_{ii}) + \Psi(P_i \diamond X_{ij} \bullet N_{ii}) \\ &= \Psi(P_i) \diamond X_{ij} \bullet L_{ii} + P_i \diamond \Psi(X_{ij}) \bullet L_{ii} \\ &\quad + P_i \diamond X_{ij} \bullet \Psi(L_{ii}) + \Psi(P_i) \diamond X_{ij} \bullet N_{ii} \\ &\quad + P_i \diamond \Psi(X_{ij}) \bullet N_{ii} + P_i \diamond X_{ij} \bullet \Psi(N_{ii}).\end{aligned}$$

From the last two relations, we infer that $P_i \diamond X_{ij} \bullet Q = 0$. It follows that $P_i \diamond X_{ij} \bullet Q_{ii} = 0$. Now solving this, we get $X_{ij}^*Q_{ii} + Q_{ii}X_{ij} = 0$. Now multiply both sides on right by P_j, we get $Q_{ii}X_{ij} = 0$ and it follows from (2.1) and (2.2) that $Q_{ii} = 0$. Thus $Q = 0$ □

Lemma 2.8. Ψ *is additive.*

Proof. For any $L, N \in \mathcal{S}$, we write $L = L_{11} + L_{12} + L_{21} + L_{22}$ and $N = N_{11} + N_{12} + N_{21} + N_{22}$. Invoking Lemmas 2.5 - 2.7, we get

$$\begin{aligned}\Psi(L + N) &= \Psi(L_{11} + L_{12} + L_{21} + L_{22} + N_{11} + N_{12} + N_{21} + N_{22}) \\ &= \Psi(L_{11} + N_{11}) + \Psi(L_{12} + N_{12}) + \Psi(L_{21} + N_{21}) + \Psi(L_{22} + N_{22}) \\ &= \Psi(L_{11}) + \Psi(N_{11}) + \Psi(L_{12}) + \Psi(N_{12}) + \Psi(L_{21}) \\ &\quad + \Psi(N_{21}) + \Psi(L_{22}) + \Psi(N_{22}) \\ &= \Psi(L_{11} + L_{12} + L_{21} + L_{22}) + \Psi(N_{11} + N_{12} + N_{21} + N_{22}) \\ &= \Psi(L) + \Psi(N).\end{aligned}$$

Hence the additivity of Ψ follows from the above lemmas. □

Now in the rest of the paper, we show that Ψ is a $*$-derivation.

Lemma 2.9. *(i)* $P_1\Psi(P_1)P_2 = -P_1\Psi(P_2)P_2$.
(ii) $P_2\ \Psi(P_1)P_1 = -P_2\Psi(P_2)P_1$.
(iii) $P_1\Psi(P_2)P_1 = P_2\Psi(P_1)P_2 = 0$.

Proof. (i) It follows from $P_1 \diamond P_1 \bullet P_2 = 0$ and Lemma 2.2 that

$$\begin{aligned} 0 &= \Psi(P_1 \diamond P_1 \bullet P_2) \\ &= \Psi(P_1) \diamond P_1 \bullet P_2 + P_1 \diamond \Psi(P_1) \bullet P_2 + P_1 \diamond P_1 \bullet \Psi(P_2) \\ &= 2P_1\Psi(P_1)P_2 + 2P_2\Psi(P_1)^*P_1 + 2P_1\Psi(P_2) + 2\Psi(P_2)P_1. \end{aligned}$$

Multiplying the previous relation by P_1 from the left and by P_2 from the right, we get

$$P_1\Psi(P_1)P_2 = -P_1\Psi(P_2)P_2.$$

(ii) Since $P_2 \diamond P_2 \bullet P_1 = 0$, applying Lemma 2.2, we get

$$\begin{aligned} 0 &= \Psi(P_2 \diamond P_2 \bullet P_1) \\ &= \Psi(P_2) \diamond P_2 \bullet P_1 + P_2 \diamond \Psi(P_2) \bullet P_1 + P_2 \diamond P_2 \bullet \Psi(P_1) \\ &= 2P_2\Psi(P_2)P_1 + 2P_1\Psi(P_2)^*P_2 + 2P_2\Psi(P_1) + 2\Psi(P_1)P_2. \end{aligned}$$

Multiplying the previous relation by P_2 from left and by P_1 from right, we get $P_2\Psi(P_2)P_1 = -P_2\Psi(P_1)P_1$.

(iii) From (i), we have

$$0 = 2P_1\Psi(P_1)P_2 + 2P_2\Psi(P_1)^*P_1 + 2P_1\Psi(P_2) + 2\Psi(P_2)P_1. \tag{2.3}$$

Multiplying (2.3) by P_1 from both right and left, we get $P_1\Psi(P_2)P_1 = 0$. Analogously from (ii), we have

$$0 = 2P_2\Psi(P_2)P_1 + 2P_1\Psi(P_2)^*P_2 + 2P_2\Psi(P_1) + 2\Psi(P_1)P_2. \tag{2.4}$$

Multiplying (2.4) by P_2 from both right and left, we get $P_2\Psi(P_1)P_2 = 0$. □

Lemma 2.10. $P_1\Psi(P_1)P_1 = P_2\Psi(P_2)P_2 = 0$.

Proof. For every $L_{12} \in \mathcal{S}_{12}$, applying Lemma 2.8, we get

$$\Psi(P_1 \diamond P_1 \bullet L_{12}) = 2\Psi(L_{12}).$$

On the other hand, we have

$$\begin{aligned} \Psi(P_1 \diamond P_1 \bullet L_{12}) &= \Psi(P_1) \diamond P_1 \bullet L_{12} + P_1 \diamond \Psi(P_1) \bullet L_{12} + P_1 \diamond P_1 \bullet \Psi(L_{12}) \\ &= 2\Psi(P_1)^*L_{12} + 2P_1\Psi(P_1)L_{12} + 2L_{12}\Psi(P_1)^*P_1 \\ &\quad +2P_1\Psi(L_{12}) + 2\Psi(L_{12})P_1. \end{aligned}$$

From the last two relations, we infer that

$$\Psi(P_1)^* L_{12} + P_1\Psi(P_1)L_{12} + L_{12}\Psi(P_1)^* P_1 + P_1\Psi(L_{12}) \\ +\Psi(L_{12})P_1 - \Psi(L_{12}) = 0.$$

By the given hypothesis $\Psi(P_1)$ is self-adjoint. Hence multiplying above relation by P_2 from the right and by P_1 from the left, we get $P_1\Psi(P_1)L_{12} = 0$, i.e., $P_1\Psi(P_1)P_1LP_2 = 0$ for all $L \in \mathcal{S}$. It follows from (2.1) and (2.2) that $P_1\Psi(P_1)P_1 = 0$. Analogously, we can prove that $P_2\Psi(P_2)P_2 = 0$. □

Lemma 2.11. *(i)* $\Psi(P_1) = P_1\Psi(P_1)P_2 + P_2\Psi(P_1)P_1$ *and*

$$\Psi(P_2) = P_1\Psi(P_2)P_2 + P_2\Psi(P_2)P_1.$$

(ii) $\Psi(I) = 0$.

Proof. (i) By Peirce decomposition, we have

$$\Psi(P_1) = P_1\Psi(P_1)P_1 + P_1\Psi(P_1)P_2 + P_2\Psi(P_1)P_1 + P_2\Psi(P_1)P_2.$$

In view of Lemmas 2.9 - 2.10, it follows that $\Psi(P_1) = P_1\Psi(P_1)P_2 + P_2\Psi(P_1)P_1$. Analogously, we can show that $\Psi(P_2) = P_1\Psi(P_2)P_2 + P_2\Psi(P_2)P_1$.

(ii) Invoking Lemmas 2.8 - 2.10, we have

$$\begin{aligned}\Psi(I) &= \Psi(P_1 + P_2)\\ &= \Psi(P_1) + \Psi(P_2)\\ &= P_1\Psi(P_1)P_2 + P_2\Psi(P_1)P_1 + P_1\Psi(P_2)P_2 + P_2\Psi(P_2)P_1\\ &= 0.\end{aligned}$$

□

Lemma 2.12. Ψ *preserves '$*$' i.e.,* $\Psi(L^*) = \Psi(L)^*$ *for all* $L \in \mathcal{S}$.

Proof. By Lemma 2.8, we have

$$\Psi(L \diamond I \bullet I) = \Psi(2L + 2L^*) = 2\Psi(L) + 2\Psi(L^*).$$

On the other hand, using Lemma 2.11, we have

$$\Psi(L \diamond I \bullet I) = \Psi(L) \diamond I \bullet I = 2\Psi(L) + 2\Psi(L)^*.$$

Comparing the above two relations, we arrive at
$\Psi(L^*) = \Psi(L)^*$ for all $L \in \mathcal{S}$.

□

Lemma 2.13. *(i)* $\Psi(iI) = 0$.
(ii) $\Psi(-iI) = 0$, *where* i *is the imaginary unit.*

Proof. (i) $\Psi(iI)=0$.
Applying Lemmas 2.8 and 2.11, we have

$$\Psi(iI \diamond iI \bullet I)=\Psi(iI)\diamond iI \bullet I + iI \diamond \Psi(iI)\bullet I.$$

Thus, $4i\Psi(iI)^{*}-4i\Psi(iI)=0$. Also, we have $\Psi(iI\diamond I\bullet I)=\Psi(iI)\diamond I\bullet I$. Thus, $2\Psi(iI)^{*}+2\Psi(iI)=0$. Multiplying the previous relation by $2i$, we get $4i\Psi(iI)^{*}+4i\Psi(iI)=0$. From the above two relations, we obtain $8i\Psi(iI)=0$, which implies that $\Psi(iI)=0$.
(ii) Analogously, we can show that $\Psi(-iI)=0$. □

Lemma 2.14. *(i)* $\Psi(-iL)=-i\Psi(L)$.
(ii) $\Psi(iL)=i\Psi(L)$, *where* i *is the imaginary unit.*

Proof. (i) $\Psi(-iL)=-i\Psi(L)$. Since $(-iL)\diamond I\bullet I = L\diamond iI\bullet I$, utilizing Lemma 2.2, we have $\Psi((-iL)\diamond I\bullet I)=\Psi(L\diamond iI\bullet I)$. Invoking Lemmas 2.11 and 2.13, we have $\Psi(-iL)^{*}+\Psi(-iL)=i\Psi(L)^{*}-i\Psi(L)$. Also, since $(-iL)\diamond iI\bullet I=(-I)\diamond L\bullet I$, utilizing Lemma 2.2, we have

$$\Psi((-iL)\diamond iI\bullet I)=\Psi(-I\diamond L\bullet I).$$

Now by Lemma 2.11 and 2.13, we have

$$i\Psi(-iL)^{*}-i\Psi(-iL)=-\Psi(L)-\Psi(L)^{*}.$$

Multiply bothsides of the above relation by iI, to get

$$-\Psi(-iL)^{*}+\Psi(-iL)=-i\Psi(L)-i\Psi(L)^{*}.$$

By adding the above two relations, we get, $\Psi(-iL)=-i\Psi(L)$ for all $L\in\mathcal{S}$.
(ii) Analogously, we can show that $\Psi(iL)=i\Psi(L)$. □

Lemma 2.15. Ψ *is a derivation.*

Proof. For every L, N $\in\mathcal{S}$, we have $(L^{*}\diamond N\bullet I)=2(LN+N^{*}L^{*})$. Now, applying Lemmas 2.11 and 2.12, we have

$$\begin{aligned}2\ \Psi(LN+N^{*}L^{*}) &= \Psi(L^{*}\diamond N\bullet I)\\ &= \Psi(L)^{*}\diamond N\bullet I + L^{*}\diamond\Psi(N)\bullet I\\ &= 2\Psi(L)N+2N^{*}\Psi(L)^{*}+2L\Psi(N)+2\Psi(N)^{*}L^{*}.\end{aligned}$$

Therefore,

$$2\Psi(LN+N^{*}L^{*}) = 2\Psi(L)N+2N^{*}\Psi(L)^{*}+2L\Psi(N)+2\Psi(N)^{*}L^{*}.$$

Also, we have $(L^* \diamond iN \bullet I) = 2\ i\ (LN - N^*L^*)$. So, invoking Lemmas 2.11, 2.13 and 2.14, we have

$$\begin{aligned} 2\ \Psi(i(LN - N^*L^*)) &= \Psi(L^* \diamond iN \bullet I) \\ &= \Psi(L)^* \diamond iN \bullet I + L^* \diamond \Psi(iN) \bullet I \\ &= 2i\Psi(L)N + 2iL\Psi(N) - 2i\Psi(N)^*L^* - 2iN^*\Psi(L)^*. \end{aligned}$$

Therefore

$$2i\Psi(LN - N^*L^*) = 2i\Psi(L)N + 2iL\Psi(N) - 2i\Psi(N)^*L^* - 2iN^*\Psi(L)^*.$$

Multiply the above relation by iI to get

$$2\Psi(LN - N^*L^*) = 2\Psi(L)N + 2L\Psi(N) - 2\Psi(N)^*L^* - 2N^*\Psi(L)^*.$$

Adding the above two relations, we get $\Psi(LN) = \Psi(L)N + L\Psi(N)$. Hence Ψ is a derivation. This completes the proof of Theorem 2.1. □

3. Corollaries

An algebra $\mathcal{S}$ is called prime if $L\mathcal{S}N = 0$ for $L, N \in \mathcal{S}$ implies either $L = 0$ or $N = 0$. So, it is very simple to see that any prime $*$-algebra satisfies (2.1) and (2.2). So we have the following corollary.

Corollary 3.1. *Suppose $\mathcal{S}$ is a unital prime $*$-algebra with a non-trivial projection P_1. If $\Psi : \mathcal{S} \to \mathcal{S}$ satisfies*

$$\Psi(L \diamond N \bullet D) = \Psi(L) \diamond N \bullet D + L \diamond \Psi(N) \bullet D + L \diamond N \bullet \Psi(D)$$

for all $L, N, D \in \mathcal{S}$. Then Ψ is additive. Moreover, if $\Psi(P_1)$ is self-adjoint, then Ψ is a $*$-derivation.

Consider $\mathcal{H}$ as a complex Hilbert space. Assume $\mathcal{B}(\mathcal{H})$ and $\mathbb{T}(\mathcal{H})$ denote the algebra of all bounded linear operators and the subalgebra of bounded operators of finite rank respectively. It is well known that $\mathbb{T}(\mathcal{H})$ forms a $*$-closed ideal of $\mathcal{B}(\mathcal{H})$. An algebra $\mathcal{L} \subset \mathcal{B}(\mathcal{H})$ is called a standard operator algebra, if $\mathbb{T}(\mathcal{H}) \subset \mathcal{L}$. As a result, we have the following immediate corollary.

Corollary 3.2. *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{S}$ be a unital standard operator algebra on $\mathcal{H}$ such that $\mathcal{S}$ is closed under adjoint operation. Suppose that $\Psi : \mathcal{S} \to \mathcal{S}$ is a map satisfying*

$$\Psi(L \diamond N \bullet D) = \Psi(L) \diamond N \bullet D + L \diamond \Psi(N) \bullet D + L \diamond N \bullet \Psi(D)$$

for all $L, N, D \in \mathcal{S}$. Then Ψ is additive. Moreover, if $\Psi(P_1)$ is self-adjoint for some nontrivial projection P_1, then Ψ is a $*$-derivation.

A von Neumann algebra $\mathcal{Z}$ is a weakly closed self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator. Also it is well known that if a von Neumann algebra has no central summands of type I_1, then $\mathcal{Z}$ satisfies (2.1) and (2.2). As a result, we have the following immediate corollary.

Corollary 3.3. *Let $\mathcal{Z}$ be a von Neumann algebra with no central summands of type I_1 and consider the map $\Psi : \mathcal{Z} \to \mathcal{Z}$ satisfying*

$$\Psi(L \diamond N \bullet D) = \Psi(L) \diamond N \bullet D + L \diamond \Psi(N) \bullet D + L \diamond N \bullet \Psi(D)$$

for all $L, N, D \in \mathcal{Z}$. Then Ψ is additive. Moreover, if $\Psi(P_1)$ is self-adjoint for some nontrivial projection P_1, then Ψ is a $*$-derivation.

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