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EQUITABLE RINGS DOMINATION IN GRAPHS

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Abstract. A dominating set S of G is an equitable dominating set of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in V(G)$ and $|\deg(u) - \deg(v)| \leq 1$. A dominating set S of G is a rings dominating set of G if every vertex $v \in V(G) \setminus S$ is adjacent to atleast two vertices $V(G) \setminus S$. In this paper, we examine the conditions at which the equitable dominating set and the rings dominating set coincide, and thus naming the dominating set as equitable rings dominating set. The minimum cardinality of an equitable rings dominating set of a graph G is called the equitable rings domination number of G, and is denoted by $\gamma_{eri}(G)$. Moreover, We examine and determine the equitable domination numbers of many graphs, as well as graphs formed by some binary operations.

1. Introduction

The concept of domination was introduced in 1962 by Berge [3], and because it offers a wide range of research ideas, many mathematicians have introduced many variants of dominations up until today, and it is evident due to the number of researches in domination such as in the paper of [7] and [8]. One of the variants of domination is the Equitable Domination in Graphs, which was presented in the paper of Anita, Arumugan and Chellali [2], and in the paper of Deepak, Sooner and Alwardi [6]. This study has caught the attention of many researchers, and thus, in 2017, Caay and Arugay [4] introduced the notion of Perfect Equitable Domination, which explained that every vertex outside the equitable domination set is dominated only once by a vertex in the equitable dominating set. Furthermore, in 2021, Caay and Durog [5] studied the Independent Equitable Domination, which they explained that an equitable dominating set must also be an independent set to achieve the goal of the research. One of the newest developed variants of domination is the concept of Rings Domination in Graphs which was introduced in 2022 by Abed and Al-Harere [1].

Interestingly, this study introduces the concept of the Equitable Rings Domination in Graphs. This means that the dominating set S of a graph

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G is an equitable and rings dominating set. The layout of this paper is as follows: Section 2 contains some basic definitions and preliminaries that are used in the study of this paper; Section 3 discusses the notions of equitable dominations and rings dominations, and this is where we show the conditions when these two dominating sets coincide to present the notion of equitable rings dominations, which is the main concept of this study; Section 4 presents the results of equitable rings dominations in some graphs, and examine the number; Section 5 presents the results and discussions on the equitable rings dominations and the numbers on binary operations such join, corona, and cartesian product of graphs; and lastly, we present the conclusion and recommendation in 6.

2. Basic Definitions

This study only considers simple connected graphs. That is, the graphs do not have loops and multiple edges. A pair $G = (V(G), E(G))$ is called a *graph* (on V). The elements of $V(G)$ are called the *vertices* of G, and the elements of $E(G)$ are called the *edges* of G.

We define the neighborhood of $v \in (G)$, denoted by $N_G(v)$, as the set

$$N_G(v) := \{u \in V(G) \ : \ uv \in E(G)\}.$$

Given a subset $S \subseteq V(G)$, the set $N_G(S) = N(S) = \bigcup_{v \in S} N_G(v)$ and the set $N_G[S] = N[S] = S \cup N(S)$ are the *open neighborhood* and the *closed neighborhood* of S, respectively.

Given a vertex $v \in V(G)$, we define the *degree of* v, denoted by $\deg(v)$, to be the number of edges incident to v. The *maximum degree of* G, denoted by $\Delta(G)$, is the degree of the vertex in G having the maximum degree, and the *minimum degree of* G, denoted by $\delta(G)$, is the degree of the vertex in G having the minimum degree.

The *join* $G + H$ of the two graphs G and H is the graph with vertex set

$$V(G + H) = V(G) + V(H),$$

and the edge set

$$E(G + H) = E(G) \cup (H) \cup \{uv \ : \ u \in V(G), v \in V(H)\}.$$

The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H.

The *cartesian product* $G \times H$ of two graphs G and H is the graph with vertex set $V(G \times H) = V(G) \times V(H)$, and $e \in E(G \times H)$ if and only if $(u_i, v_j)\,(u_k, v_l)$ where either

i. $i = k$ and $v_j v_l \in E(H)$; or
ii. $j = l$ and $u_i u_k \in E(G)$.

We now introduce the concept of domination in graphs.

Definition 2.1. [3] A subset $S \in V(G)$ is a *dominating set* of G if for every $v \in V(G) \backslash S$, there exists $u \in S$ such that $uv \in E(G)$. That is, $N[S] = V(G)$. The minimum cardinality of a dominating set S of G is called the *domination number* of G, and is denoted by $\gamma(G)$. In this case, we call a dominating set S of G to be a γ-set of G.

Definition 2.2. [2] A dominating set S of G is an *equitable dominating set* of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in V(G)$ and $|\deg(u) - \deg(v)| \leq 1$. The minimum cardinality of an equitable dominating set S of G is called the *equitable domination number* of G, and is denoted by $\gamma_e(G)$. In this case, we call an equitable dominating set S of G to be a γ_e-set of G.

Definition 2.3. [1] A dominating set S of G is a *rings dominating set* of G if every vertex $v \in V(G) \setminus S$ is adjacent to atleast two vertices $V(G) \setminus S$. The minimum cardinality of a rings dominating set S of G is called the *rings domination number* of G, and is denoted by $\gamma_{ri}(G)$. In this case, we call a rings dominating set S of G to be a γ_{ri}-set of G.

Remark 2.4. [1] For a γ_{ri}-set of G of order n, we have

i. the order of G is $n \geq 4$.
ii. for each $v \in V(G) \setminus S$, the $\deg(v) \geq 3$.
iii. $1 \leq |S| \leq n - 3$.
iv. $3 \leq |V(G) \setminus S| \leq n - 1$.
v. $1 \leq \gamma_{ri}(G) \leq |S| \leq n - 3$.

3. The Equitable Dominations and the Rings Dominations in Graphs

We present some important results of the equitable dominations and the rings dominations in graphs, and with this, we present a case where the equitable dominating set and the rings dominating set coincide to formally introduce the main definition of the study, which is the equitable rings domination in graphs.

Lemma 3.1. *Let G be any nontrivial graph. Then $\gamma_e(G) = 1$ if and only if $\Delta(G) - \delta(G) \leq 1$.*

Lemma 4.1 has a similar proof with the Theorem 2.2 of [5], but the study is on the independent equitable domination in graphs, and thus the concept of the proof is beyond the scope of this paper. However, Lemma 4.1 can be proven using the Definition 2.2, and is easy to follow.

Proposition 3.2. [1] *Given a complete graph K_n and a wheel graph W_{n+1} with $n \geq 4$, we have $\gamma_{ri}(K_n) = \gamma_{ri}(W_{n+1}) = 1$.*

Proposition 3.3. [1] *A path graph P_n has no γ_{ri}-set. In general, tree graphs T_n have no γ_{ri}-set.*

Proposition 3.4. [1] *A cycle graph C_n has no γ_{ri}-set.*

It is natural to ask what are the conditions for a graphs to have equal equitable domination number and rings domination number. That is, what are the conditions of a graph G such that if S is an equitable dominating set of G, then it is also a rings dominating set of G.

Lemma 3.5. *Let G be any graph of order $n \geq 5$. Then G has a γ_e-set and a γ_{ri}-set, and they coincide if and only if for such dominating set S,*

i. *for every vertex $v \in V(G) \setminus S$, there exists $u \in S$ with $uv \in S$ such that $|\deg(u) - \deg(v)| \leq 1$, and*
ii. *$V(G) \setminus S$ is a union of i number of cycles $C_{k_j}^{(i)}$ for some i, with $k_j \geq 3$.*

Proof. Let G be any graph such that its γ_e-set and γ_{ri}-set coincide. That is, $\gamma_e(G) = \gamma_{ri}(G) = |S|$. Then by Definition 2.2, (i) follows. Moreover, by Definition 2.3, it follows that every vertex in $V(G) \setminus S$ is adjacent to atleast two vertices in $V(G) \setminus S$.

If $|V(G) \setminus S| = 3$, then $V(G) \setminus S$ forms a cycle, and so we are done.

Suppose $3 < |V(G) \setminus S| < \infty$. We choose $v_1 v_2 \in V(G) \setminus S$ with $v_1 v_2 \in E(G)$ and $v_1 \neq v_2$. Take a vertex $v_s \in V(G) \setminus S$. Since $\deg(v_i) \geq 2$, for all i, we can choose $v_{s+1} \notin \{v_s, v_{s-1}\}$. Continue the process until we have the sequence of vertices $v_1, \cdots, v_{k+1}$. By Pigeonhole Principle, there exists i, j with $i \neq j$ such that $v_i = v_j$. Thus, $V(G) \setminus S$ contains a cycle $C_{k_i}^{(1)}$. Now, since every vertex $v_t \in (V(G) \setminus S) \setminus C_{k_i}^1$ has degree of at least 2, we do the same process, and so $(V(G) \setminus S) \setminus C_{k_i}^1$ contains a cycle $C_{k_j}^{(2)}$. Continue the process inductively, it follows that $V(G) \setminus S$ contains cycles $C_k^{(i)}$ or is a union of i number of cycles for some i.

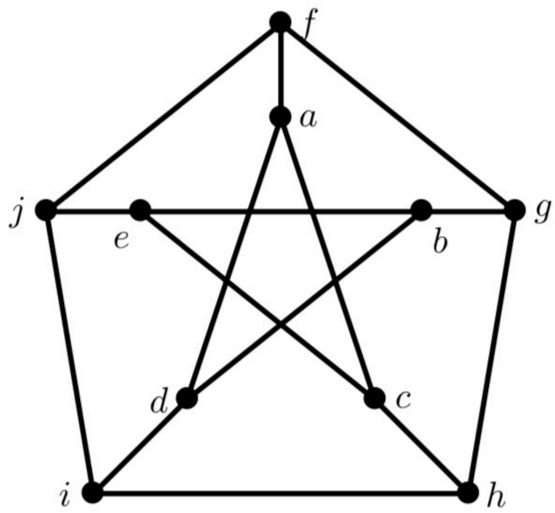

FIGURE 1. Graph G

Conversely, suppose (i) and (ii) hold. Let $S \subseteq V(G)$. Since (i) holds, by Definition 2.2, S is γ_e. Since (ii) holds, it follows that every vertex $V(G) \setminus S$ is adjacent to at least two vertices $V(G) \setminus S$. Thus, this follows the Definition 2.3.

Therefore, the γ_e-set of G and the γ_{ri}-set of G coincide. □

Lemma 3.5 serves as the motivation of this paper. This introduces the notion of equitable rings domination in graphs. To explore on this idea, we now formally introduce the formal definition.

Definition 3.6. A dominating set $S \subseteq V(G)$ is said to be an *equitable rings dominating set* of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ with $uv \in E(G)$ such that $|\deg(u) - \deg(v)| \leq 1$, and v is adjacent to at least two vertices in $V(G) \setminus S$. The minimum cardinality of an equitable rings dominating set of G is called *equitable rings domination number* of G, and is denoted by $\gamma_{eri}(G)$. An equitable rings dominating set S of G with $|S| = \gamma_{eri}(G)$ is said to be γ_{eri}-set of G.

Example 3.7. Consider the graph in Figure 3.7. Then we have γ_{eri}-sets of G as follows:

$$\{a, b, c, d, e\}, \{a, g, c, i, e\}, \{a, b, h, d, j\}, \text{ and } \{f, g, h, i, j\}.$$

Therefore, $\gamma_{eri}(G) = 5$.

4. The Equitable Rings Dominations in Graphs

The first two lemmas below follow from Definition 3.6.

Lemma 4.1. *Let G be any graph of order n with $\Delta(G) - \delta(G) \leq 1$. Then $\gamma_{eri} = 1$ if and only if $\Delta(G) = n - 1$.*

Lemma 4.2. *Let G be any graph. If $u \in S \subseteq V(G)$ with $\deg(u) = \Delta(G)$, then S is a γ_{eri}-set of G.*

Proposition 4.3. *There does not exist a γ_{eri}-set in a tree graph T_n. In particular, there does not exist a γ_{eri}-set in a path graph P_n.*

Proposition 4.4. *There does not exist a γ_{eri}-set in a cycle graph C_n.*

The proofs of Propositions 4.3 and 4.4 follows directly from Propositions 3.2 and 3.3, respectively.

Theorem 4.5. *There exists a γ_{eri}-set in a complete k-partite graph $G = K_{n_1,n_2,\cdots,n_k}$, $k \geq 3$, if there exists a vertex partition P_i of cardinality at least 2, such that $||P_i| - |P_j|| \leq 1$, for all $i \neq j$. Moreover,*

$$\gamma_{eri}\left(K_{n_1,n_2,\cdots,n_k}\right) = \left|P_i\right|.$$

Proof. Suppose there exists a partition P_i such that $||P_i| - |P_j|| \leq 1$ for all $i \neq j$, and let u_{ik} be the kth vertex in the ith partition. Let $|P_i| = r$. Then the rest of the partitions are either P_j, P'_j or P''_j such that $|P_j| = r - 1$, $|P'_j| = r$ and $|P''_j| = r + 1$. Suppose there are k_1 copies P_j, k_2 copies of P'_j and k_3 copies of P''_j. Then $\deg(u_{ik}) = k_1(r-1) + k_2 r + k_3(k+1)$.

If v_s is one of the vertices of P_j's, then

$$\deg(v_s) = (k_1 - 1)(r - 1) + k_2 r + k_3(r + 1) + r.$$

Thus,

$$\begin{aligned} |\deg(u_{ij}) - \deg(v_s)| &= |(k_1(r-1) + k_2 r + k_3(k+1)) - ((k_1 - 1)(r-1) \\ &\quad + k_2 r + k_3(r+1) + r)| \\ &= 1. \end{aligned}$$

If v_t is one of the vertices of P'_j's, then

$$\deg(v_t) = k_1(r - 1) + (k_2 - 1)r + k_3(r + 1) + r.$$

Thus,

$$\begin{aligned} |\deg(u_{ij}) - \deg(v_t)| &= |(k_1(r-1) + k_2 r + k_3(k+1)) - (k_1(r-1) \\ &\quad + (k_2 - 1)r + k_3(r+1) + r)| \\ &= 0. \end{aligned}$$

Lastly, if v_z is one of the vertices of P''_j's, then

$$\deg(v_z) = k_1(r-1) + k_2 r + (k_3 - 1)(r+1) + r.$$

Thus,

$$\begin{aligned}|\deg(u_{ij}) - \deg(v_z)| &= |(k_1(r-1) + k_2 r + k_3(k+1)) - (k_1(r-1) \\ &\quad + k_2 r + (k_3-1)(r+1) + r)| \\ &= 1.\end{aligned}$$

Since this holds for all u_{ij} in the partition P_i, $j = 1, \cdots, r$, it follows that we can choose P_i to be the γ_{eri}-set. Therefore, γ_{eri}-set exists, and $\gamma_{eri}(K_{n_1,n_2,\cdots,n_k}) = |P_i|$. □

Corollary 4.6. *There does not exist a γ_{eri}-set of $G = K_{n_1,n_2,\cdots,n_k}$, $k \geq 3$, if there exists a vertex partition P_i such that $||P_i| - |P_j|| \geq 2$, for all $i \neq j$.*

Theorem 4.7. *There exists a γ_{eri}-set in a bipartite graph $G = K_{n_1,n_2}$ if $|n_1 - n_2| \leq 1$ for $n_1 \geq 3, n_2 \geq 3$. Moreover, $\gamma_{eri}(K_{n_1,n_2}) = 2$.*

Proof. Let P_1 and P_2 be the vertex partitions of K_{n_1,n_2}. Let u_i and v_j be the ith vertex and the jth vertex of P_1 and P_2, respectively. If $|P_1| = r$, then $|P_2|$ is either $r-1$ or $r+1$. Without loss generality, take $|P_2| = r-1$. Then $\deg(u_i) = r-1$ and $\deg(v_j) = r$. Thus, $|\deg(u_i) - \deg(v_j)| \leq 1$. This means that γ_{eri}-set exists. In particular, we may $u_1 \in P_1$ and $v_1 \in P_2$, and u_1 is adjacent to all $v_j \in P_2$, for all $j = 1, \cdots, r-1$, and v_1 is adjacent to all u_i's, for all $i = 1, \cdots, r$. Hence, we may take $S = \{u_1, v_1\}$ as a γ_{eri}-set of K_{n_1,n_2}. Therefore, $\gamma_{eri}(K_{n_1,n_2}) = 2$. □

Corollary 4.8. *There does not exist a γ_{eri}-set of K_{n_1,n_2}, if $|n_1 - n_2| \geq 2$.*

5. The Equitable Rings Dominations in Graphs Under Some Binary Operations

In this section, we modify how we write the degree of a vertex to avoid confusion. Given two graphs G and H, we denote $\deg_G(u)$ to refer the degree of u in the graph G alone. Unless no binary operation is done yet, we use the convention $\deg(u)$. Also, we denote $\deg_{G*H}(u)$ to refer the degree of u in $G * H$, for any binary operations $*$ presented in this paper.

5.1. **Join of Graphs.**

Theorem 5.1. *Let G and H be any graphs of order n and m, respectively. Let S be γ_{eri}-set of G and $u \in S$ with $\deg_G(u) = n-1$. Then S is a γ_{eri}-set of $G + H$ if and only if every vertices in H have degree of atleast $m-2$.*

Proof. Let $u \in S \subset V(G+H)$, with S a γ_{eri}-set of $G+H$. Since $\deg_G(u) = n-1$, it follows that $\deg_{G+H}(u) = n-1+m$. If $v_i \in V(H)$, then $\deg_{G+H}(v_i) = n + \deg_H(v_i)$, for all $i = 1, \cdots, m$. Let $\deg_H(v_i) = x$. Then $\deg_{G+H}(v_i) = n + x$. Thus,

$$|\deg(u) - \deg(v_i)| = |(n-1+m-(n+x))| \leq 1.$$

This means that x must be at least $m-2$. Since v_i is arbitrary, this holds for all vertices in H.

Conversely, suppose every vertices in H have degree of atleast $m-2$, and let $v_i \in V(H)$. Then, $\deg_{G+H}(v_i) \geq m-2+n$. Since $u \in S$ and S is a γ_{eri}-set of G, $\deg_{G+H}(u) = n-1+m$. Hence, we have

$$\begin{aligned}|\deg_{G+H}(u) - \deg_{G+H}(v_i)| &= |(n-1+m)-(m-2+n)| \text{ or}\\ &\quad |(n-1+m)-(m-1+n)|,\end{aligned}$$

which in either case is less than or equal to 1. This proves the claim. □

Theorem 5.2. *Let G and H be any graphs of order at least 4. If $\gamma_{eri}(G) = \gamma_{eri}(H) = 1$, then $\gamma_{eri}(G+H) = 1$.*

Proof. Let G and H be groups of order n and m, respectively. If $\gamma_{eri}(G) = \gamma_{eri}(H) = 1$, then by Lemma 4.1, $\Delta(G) = n-1$, $\delta(G) = n-2$, $\Delta(H) = m-1$ and $\delta(H) = n-2$. By Lemma 4.2, if $u \in S$, S is a γ_{eri}-set of G, then $\deg_G(u) = \Delta(G) = n-1$. Thus, $\deg_{G+H}(u) = n-1+m$. Now if $u' \in V(G) \setminus S$, by Lemma 4.1, $\deg_G(u') = n-2$, and so

$$\deg_{G+H}(u') = n-2+m.$$

Also, if $h_1, h_2 \in V(H)$ with $\deg_H(h_1) = \Delta(H)$ and $\deg_H(h_2) = \delta(H)$. Thus, by Lemma 4.1, $\deg_H(h_1) = m-1$ and $\deg_H(h_2) = m-2$, and so

$$\deg_{G+H}(h_1) = m-1+n \text{ and } \deg_{G+H}(h_2) = m-2+n.$$

Hence, we have

$$\begin{aligned}|\deg_{G+H}(u) - \deg_{G+H}(v)| &= |(n-1+m)-(n-2+m)| \leq 1.\\ |\deg_{G+H}(u) - \deg_{G+H}(h_1)| &= |(n-1+m)-(m-1+n)| \leq 1.\\ |\deg_{G+H}(u) - \deg_{G+H}(h_2)| &= |(n-1+m)-(m-2+n)| \leq 1.\end{aligned}$$

Therefore, u is in a γ_{eri}-set of $G+H$. Consequently, $\gamma_{eri}(G+H) = 1$. □

Proposition 5.3. *There exists a γ_{eri}-set in $K_n + K_m$ if $|n-m| \leq 1$. Otherwise, γ_{eri}-set does not exist. Moreover, $\gamma_{eri}(K_n + K_m) = 1$.*

Proof. Let $u_i \in V(K_n)$ and $v_j \in V(K_m)$, for $i = 1, \cdots, n$, and $j = 1, \cdots, m$. Thus, $\deg_{G+H}(u_i) = n - 1 + m$ and $\deg_{G+H}(v_j) = m - 1 + n$, for all $i = 1, \cdots, n$, and for all $j = 1, \cdots, m$. Hence,

$$|\deg_{G+H}(u_i) - \deg_{G+H}(v_j)| \leq 1.$$

Since it holds for any i and j, we may take u_1 in particular. Thus, $S = \{u_1\}$ is a γ_{eri}-set of $K_n + K_m$, and so $\gamma_{eri}(K_n + K_m) = 1$. □

5.2. **Corona of Graphs.**

Theorem 5.4. *Let G and H be any graphs of order $n \geq 3$ and $m \geq 3$, respectively. Then $G \circ H$ does not have a γ_{eri}-set whenever every vertices of G has degree at least 2.*

Proof. Suppose on the contrary that there exists a γ_{eri}-set S in $G \circ H$. Take $u \in S$. If $S \subset V(G)$, then $\deg_{G \circ H}(u) \geq 3 + m$. Let $v_1 \in V(H)$. Then $\deg_{G \circ H}(v_1) \leq 1 + m$. Thus, we have

$$\begin{aligned} |\deg_{G \circ H}(u) - \deg_{G \circ H}(v_1)| &\geq |\deg_{G \circ H}(u)| - |\deg_{G \circ H}(v_1)| \\ &\geq (2 + m) - (1 + m) \\ &\geq 1. \end{aligned}$$

This is a contradiction. Now, if $S \subset V(H)$, then $\deg_{G \circ H}(u) \leq m + 1$. Let $v_2 \in V(G)$. Then $\deg_{G \circ H}(v_2) \geq m + 2$. Following the same argument, this leads to contradiction. □

Corollary 5.5. *There exists γ_{eri}-set in $P_2 \circ K_n$, $n \geq 3$, and*

$$\gamma_{eri}(P_2 \circ K_n) = |V(P_2)| = 2.$$

The following result is easy to follow.

Theorem 5.6. *Let G and H be any graphs. If $V(G) = \{u\}$ and there exists a γ_{eri}-set S_H of H such that for some $v \in S_H$, $\deg_H(v) = |V(H)| - 1$. Then*

i. *G is a γ_{eri}-set of $G \circ H$.*
ii. *S_H is a γ_{eri}-set of $G \circ H$.*

Consequently, $\gamma_{eri}(G \circ H) = 1$.

5.3. **Cartesian Product of Graphs.**

Theorem 5.7. *Let G and H be graphs of order at least 4 having the properties that $|\Delta(G) - \Delta(H)| \leq 1$ and $|\delta(G) - \delta(H)| \leq 1$. Suppose further that either of the graph G and H has the property that the difference of its maximum and minimum degree is at most 1. Then there exists a γ_{eri}-set in $G \times H$.*

Proof. Let u and v be vertices of G and H, respectively, such that $\deg(u) = \Delta(G)$ and $\deg(v) = \Delta(H)$. Then $|\deg(u) - \deg(v)| \leq 1$. Furthermore, let u' and v' be vertices of G and H, respectively, such that $\deg(u') = \delta(G)$ and $\deg(v') = \delta(H)$. Then $|\deg(u') - \deg(v')| \leq 1$. Without loss of generality, assume that G has the property that

$$|\Delta(G) - \delta(G)| \leq 1.$$

Let (u, v) denote the vertex of $G \times H$ with respect to u and v after the cartesian product operation. Then $\deg_{G\times H}((u, v)) = \deg_G(u) + \deg_H(v)$. Now we have

$$|\deg_{G\circ H}((u, v)) - \deg_{G\times H}((u, v'))| \\ |\deg_G(u) + \deg_H(v) - \deg_G(u) - \deg_H(v')| \leq 1.$$

$$|\deg_{G\circ H}((u, v)) - \deg_{G\times H}((u', v'))| \\ |\deg_G(u) + \deg_H(v) - \deg_G(u') - \deg_H(v')| \leq 2.$$

$$|\deg_{G\circ H}((u, v)) - \deg_{G\times H}((u', v))| \\ |\deg_G(u) + \deg_H(v) - \deg_G(u') - \deg_H(v)| \leq 1.$$

$$|\deg_{G\circ H}((u, v')) - \deg_{G\times H}((u', v))| \\ |\deg_G(u) + \deg_H(v') - \deg_G(u') - \deg_H(v)| \leq 1.$$

Observe that the only absolute difference bounded by 2 is when between vertices (u, v) and (u', v'), and the rest of the absolute difference is at most 1. This implies that we can choose a pairing of vertices as (u, v) and (u', v), or (u, v) and (u, v'), or (u, v') and (u', v) to have a "*dominant-dominee*" relationship. Now (u, v) can be paired with either (u, v') or $(u'v)$, and either of these is a dominating vertex to dominate (u, v). This is all possible since u and v are indexed by the orders of G and H, respectively. Therefore, we can always choose a dominating vertex for $G \times H$. That is, γ_{eri}-set in $G \times H$. □

Proposition 5.8. *Let G be any graph of order $n \geq 1$. If for every $u \in V(G)$, $\deg(u) \leq 2$, then $G \times K_m$ has a γ_{eri}-set, and $\gamma_{eri}(G \times K_m) = n$.*

Proof. Let $u_i v_j \in V(G \times H)$, where $u_i \in V(G)$, and $v_j \in V(K_m)$. We present the graph in Figure 5.3.

We arrange vertices in columns such that the entries of the first column are of the form $u_i v_1$, the entries of the second column are $u_i v_2$, and so on, so that $u_i v_m$ is the entry of the mth column. This is possible because there are m vertices for K_m. Moreover, there are also n rows since there are n vertices of G.

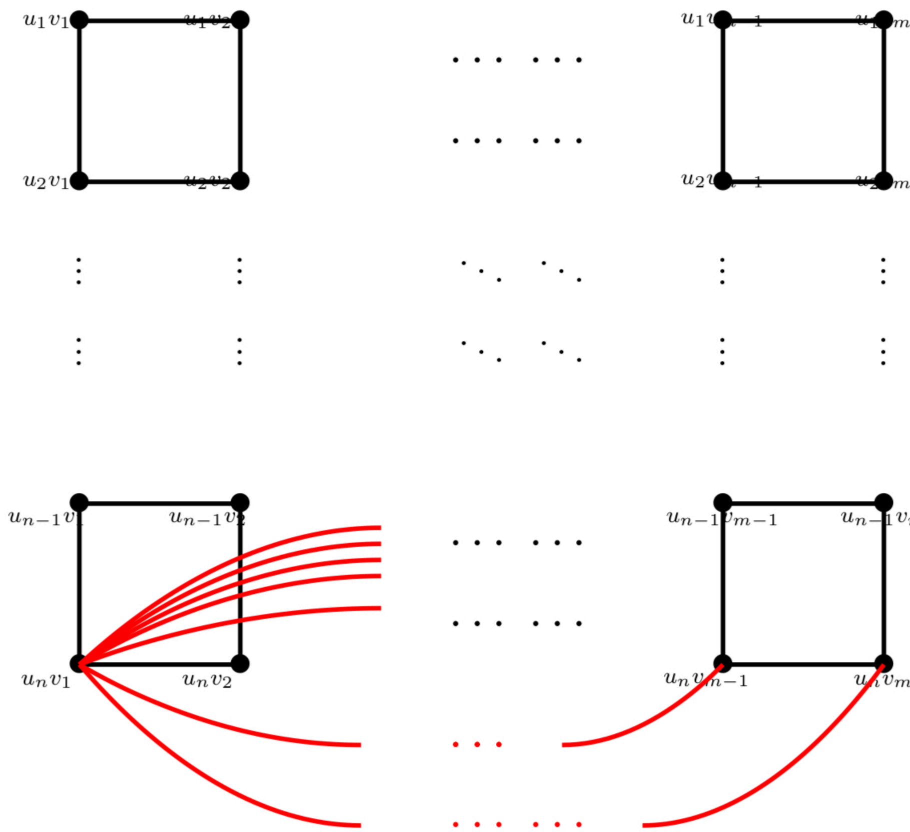

FIGURE 2. $G \times K_m$

Consider the vertex u_nv_1. Note that u_nv_1 is adjacent to all u_nv_j since all v_j's are adjacent to each other for they are vertices of K_m. This means that u_nv_1 is adjacent to $m+1$ vertices. Similarly, each vertices $u_nv_j's$, $j = 2, \cdots, m$, are adjacent to at most $m-1+2 = m+1$ vertices since u_i's are adjacent at most 2 vertices. This means that u_nv_1 is a equitable rings dominating vertex to all u_nv_j's since the difference of degree to u_nv_1 to u_nv_j's is at most 1, for all $j = 2, \cdots, m$. Since this holds for all u_iv_1's, $i = 1, \cdots, n$, all u_iv_1's is an equitable rings dominating vertices to other vertices. Therefore, we can pick γ_{eri}-set of $G \times H$ to the set of vertices of the first column. Hence, such γ_{eri}-set exists and $\gamma_{eri}(G \times H) = n$. □

6. Conlusions and Recommendations

This paper provides an interesting results on equitable rings dominations in graphs. We have shown the conditions when an equitable dominating set and a rings dominating set coincide, and that where our motivation is taken to present the notion of equitable rings domination. We also have shown results and conditions on the existence of equitable rings dominating sets in graphs, and examined the numbers up to the binary operations.

As this notion draws an interesting study in the are of domination in graphs, the researcher would like to recommend to the readers to study results and existence of this notion to other binary operations that are not covered in this study. Moreover, it would be a nice research if one could see the applications of this notion to the other disciplines in mathematics.

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EQUITABLE RINGS DOMINATION IN GRAPHS

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احاطه‌گر منصفانه حلقه‌ها در گراف‌ها

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یک مجموعه‌ی احاطه‌گری S از G را یک مجموعه‌ی احاطه‌گری منصفانه از G می‌نامند هرگاه برای هر $v \in V(G) \setminus S$، عنصر $u \in S$ وجود داشته باشد به طوری که $uv \in V(G)$ و $|\deg(u) - \deg(v)| \leq$ ۱. همچنین، یک مجموعه‌ی احاطه‌گری S از G یک مجموعه‌ی احاطه‌گری حلقه‌ها از G است هرگاه هر رأس $v \in V(G) \setminus S$ با حداقل دو رأس $V(G) \setminus S$ مجاور باشد. در این مقاله به بررسی شرایطی که تحت آن‌ها مجموعه‌ی احاطه‌گری منصفانه و مجموعه‌ی احاطه‌گری حلقه‌ها برابر هستند می‌پردازیم و در این حالت، آن را مجموعه‌ی احاطه‌گری منصفانه حلقه‌ها نام گذاری می‌کنیم. حداقل کاردینال مجموعه‌ی احاطه‌گری منصفانه حلقه‌ها از گراف G را عدد احاطه‌گری منصفانه حلقه‌ها G نامیده و با نماد $\gamma_{eri}(G)$ نشان می‌دهند. عدد احاطه‌گری منصفانه حلقه‌ها را برای بسیاری از گراف‌ها و گراف‌هایی که توسط برخی اعمال دوتایی ساخته شده‌اند، مشخص می‌کنیم.

کلمات کلیدی: احاطه‌گر، احاطه‌گر منصفانه، احاطه‌گر حلقه‌ها، احاطه‌گر منصفانه حلقه‌ها.