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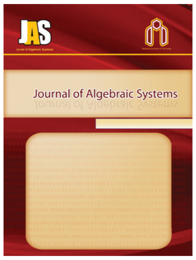

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A STUDY ON TRI REVERSIBLE RINGS

H. M. IMDADUL HOQUE* AND H. K. SAIKIA

Abstract. This article embodies a ring theoretic property which, preserves the reversibility of elements at non-zero tripotents. A ring R is defined as quasi tri reversible if any non-zero tripotent element ab of R implies ba is also a tripotent element in R for $a, b \in R$. We explore the quasi tri reversibility of 2 by 2 full and upper triangular matrix rings over various kinds of reversible rings, deducing that the quasi tri reversibility is a proper generalization of reversible rings. It is proved that the polynomial rings are not quasi tri reversible rings. The relation of symmetric rings, IFP and Abelian rings with reversibility and quasi tri reversibility are studied. It is also observed that the structure of weakly tri normal rings and quasi tri reversible rings are independent of each other.

1. Introduction

Throughout this paper, all rings are associative with identity unless otherwise stated. Let R be a ring, $T(R)$ denotes the set of all tripotents of R and $T(R)^{'} = \{t \in T(R) | t \neq 0\}$. Also $N^*(R)$, $J(R)$, and $N(R)$, represent the nilradical, the Jacobson radical and the set of all nilpotent elements of R, respectively. Further, $Mat_n(R)$ and $M_n(R)$ denote the n by n full matrix ring and the upper triangular matrix ring over R respectively. Also, $D_n(R) = \{(a_{ij}) \in M_n(R) | a_{11} = a_{22} = ... = a_{nn}\}$ and T_{ij} for the matrix with (i, j)-entry 1 and zeros elsewhere and I_n denotes the identity matrix in $Mat_n(R)$.

A ring is usually called reduced if it has no nilpotent elements other than zero. Following Lambek [13], a ring R is called symmetric if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$. Later on, Anderson and Camillo [1], used the term ZC_3 for symmetric. Clearly commutative rings are symmetric. Also reduced rings are symmetric by [1, Theorem I.3], but there are many types of non-reduced rings which are commutative (e.g., the ring of the type $\mathbb{Z}_{n^m}$ for n, $m \geq 2$). Cohn [3], in 1999 stated that a ring is said to be reversible on the condition that for any $a, b \in R$, $ab = 0$ implies $ba = 0$. Anderson and Camillo [1] used the term ZC_2 for the reversibility, they proved that a semigroup with

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no non-zero nilpotent elements satisfy ZC_2 and investigated rings that satisfy ZC_2. Clearly symmetric rings are reversible, but the converse is not true by [1, Example I.5] or G. Marks [15, Examples 5 and 7]. According to Bell [2], a ring R is said to satisfy the $Insertion-of-Factors-Property$ (simply, an IFP ring) on the condition that for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. It is proved that the reversible rings are IFP. A ring is called Abelian if every idempotents are central. By [16, Lemma 2.7], the IFP rings are Abelian rings. It is obvious that reversible rings are Abelian. A ring R is called directly finite if $ab = 1$ implies $ba = 1$ for $a, b \in R$. It is clear that Abelian rings are directly finite.

Following [9], a ring R is said to be quasi-reversible if for any $a, b \in R$, $0 \neq ab \in I(R)$ implies $ba \in I(R)$, where $I(R)$ is the set of all idempotent elements of R. They generalised the notion of reversibility into a quasi-reversibility and they investigated the quasi-reversibility of 2 by 2, the full and upper triangular matrix rings over various kinds of reversible rings.

An element t in a ring R is called tripotent if $t^3 = t$, the set of all tripotent elements are denoted by $T(R)$. Clearly, every idempotents are tripotents but the converse is not true. For example let $R = Mat_2(\mathbb{R})$, then $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \in R$ is tripotent but not idempotent.

In this study, we extend and generalize the structure of reversible rings using the concept of non-zero tripotent elements. Secondly, our main objective is to study and to define a new type of ring called quasi tri reversible ring using tripotent elements. A ring R is called quasi tri reversible ring if $0 \neq ab \in T(R)$ implies that $ba \in T(R)$. For example let $R = M_2(\mathbb{R})$ be an upper triangular matrix ring over a real number field $\mathbb{R}$. Then

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \in T(R)'$$

and

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \in T(R).$$

This exhibits that the quasi tri reversible ring need not be Abelian, while reversible rings are Abelian.

2. Reversibility on Tripotents

In this section we study the structure of reversible rings related to tripotent elements in a ring. We begin with the following equivalent conditions.

Theorem 2.1. *For a ring R the following conditions are equivalent:*

(1) *R is reversible*
(2) $ab \in T(R)$ *implies* $ba \in T(R)$ *for* $a, b \in R$.

Proof. (1) $\implies$ (2). Let R be a reversible ring. Let $ab \in T(R)$ for $a, b \in R$. So, $ab = (ab)^3 \implies ab(1-(ab)^2) = 0$. Since R is reversible, so

$$\begin{aligned} b(1-(ab)^2)a = 0 &\implies ba - b(ab)^2a = 0 \\ &\implies ba - b(abab)a = 0 \\ &\implies ba - (bababa) = 0 \\ &\implies ba = (ba)^3. \end{aligned}$$

Thus, $ba \in T(R)$ for $a, b \in R$.

(2) $\implies$ (1). Let condition (2) holds. Suppose $ab = 0$ for any $a, b \in R$. By condition (2), $ba \in T(R) \implies ba = (ba)^3 \implies ba = bababa = 0$, as $ab = 0$. Hence, $ab = 0$ implies $ba = 0$ for any $a, b \in R$. Thus R is a reversible ring. □

Corollary 2.2. *Let R be a reversible ring. If $ab \in T(R)$ for $a, b \in R$, then $ab = ba$.*

Proof. Let R be a reversible ring and $ab \in T(R)$ for $a, b \in R$, then by Theorem 2.1, we get $ba \in T(R)$. Now

$$\begin{aligned} ba = (ba)^3 = bababa = b(ab)(ab)a &= (ab)b(ab)a \\ &= abab(ba) \\ &= aba(ba)b \\ &= ababab \\ &= (ab)^3 \\ &= ab, \end{aligned}$$

as R is Abelian. So, $ab = ba$ for $a, b \in R$.

If ab and ba are not idempotent elements of the Abelian ring R, then the above relation is not true in general. As $ab \in T(R)$, then $(ab)^2$ is idempotent but ab may not be an idempotent element of R. □

The following Theorem 2.3 exhibits a relation between tripotent elements and zero-divisors in reduced rings.

Theorem 2.3. *For a ring R the following conditions are equivalent:*

(1) *R is reduced*

(2) $a^2 \in T(R)$ *implies* $a = a^5$, *where* a *is unit in* R.

Proof. (1) $\implies$ (2). Let R be a reduced ring and $a^2 \in T(R)$. Then

$$(a^2)^3 = a^2 \implies a^2(1 - a^4) = 0.$$

Since

$$\begin{aligned}(1 - a^4)^2 &= 1 - 2a^4 + (a^4)^2 \\ &= 1 - 2a^4 + a^4a^4 \\ &= 1 - 2a^4 + a^4a^6a^{-2} \\ &= 1 - 2a^4 + a^4a^2a^{-2},\end{aligned}$$

as $a^6 = a^2$ and a is unit in R. So, $(1-a^4)^2 = 1-2a^4+a^4 \implies (1-a^4)^2 = 1-a^4$. Now, $a^2(1 - a^4) = 0 \implies (a(1 - a^4))^2 = 0$. Since R is a reduced ring. Therefore $a(1 - a^4) = 0 \implies a - a^5 = 0 \implies a = a^5$ for $a \in R$.

(2) $\implies$ (1). Let condition (2) holds. Suppose $a^2 = 0$ for any $a \in R$. So by condition (2), we get $a = a^5 = (a^2)^2a = 0$. Thus $a^2 = 0$ implies $a = 0$ for any $a \in R$. This shows that R is a reduced ring.

□

Remark 2.4. In the above proof of the Theorem 2.3, it is observed that, $1 - 2a^4 + a^4a^4 = 1 - 2a^4 + a^4a^6a^{-2}$ is not true in general, since every element of a ring is not necessarily invertible, even in reversible rings. For example, we consider the reduced ring $R = \mathbb{Z}_{10}$. Let $a = 2$, then $a^2 \in T(R)$, but a is not a unit element of $R = \mathbb{Z}_{10}$, and so it can not be written in the form $a^4 = a^6a^-2$.

Theorem 2.5. *Let* R *be a ring. If* R *is reduced ring of characteristics* 2, *then* $a^2 \in T(R)$ *implies* $a \in T(R)$ *for* $a \in R$.

Proof. Let R be a reduced ring of characteristics 2. So for any $a \in R$, we get $2a = 0$. Also, let $a^2 \in T(R)$, then

$$\begin{aligned}a^2 = (a^2)^3 &\implies a^2(1 - a^4) = 0 \\ &\implies a^2(1 - a^2)(1 + a^2) = 0 \\ &\implies a^2(1 - a^2)(1 - a^2 + 2a^2) = 0 \\ &\implies a^2(1 - a^2)(1 - a^2 + 2a.a) = 0 \\ &\implies a^2(1 - a^2)(1 - a^2 + 0) = 0,\end{aligned}$$

as $Cha(R) = 2$. $\implies a^2(1 - a^2)^2 = 0 \implies (a(1 - a^2))^2 = 0$. Since R is a reduced ring, therefore $a(1 - a^2) = 0 \implies a = a^3$. Thus $a \in T(R)$.

□

Theorem 2.6. *For a ring R the following conditions are equivalent:*
(1) *R is symmetric*
(2) *$abc \in T(R)$ implies $acb = acb(cab)^2$*
(3) *$abc \in T(R)$ implies $acb = acb(bca)^2$*
(4) *$abc \in T(R)$ implies $acb = acb(abc)^2$; for $a, b, c \in R$.*

Proof. (1) $\implies$ (2). Let R be a symmetric ring and $abc \in T(R)$ for $a, b, c \in R$. Then

$$\begin{aligned} abc = (abc)^3 &\implies abc - (abcabcabc) = 0 \\ &\implies ab(c - (cabcabc)) = 0 \\ &\implies ab(1 - cabcab)c = 0. \end{aligned}$$

Since R is symmetric,

$$\begin{aligned} acb(1 - cabcab) = 0 &\implies acb - acbcabcab = 0 \\ &\implies acb = acb(cabcab) \\ &\implies acb = acb(cab)^2. \end{aligned}$$

(2) $\implies$ (1). Let condition (2) holds. First of all we show R is a reversible ring. Let $de = 0$ for $d, e \in R$. Then $ed = 1ed$, since $1 \in R$. Thus by condition (2), we get $ed = 1ed(e1d)^2 = 1ed(e1de1d) = 0$, as $de = 0$. Thus $de = 0 \implies ed = 0$ for $d, e \in R$. So R is a reversible ring.

Now we show that R is a symmetric ring. Suppose $abc = 0$ for $a, b, c \in R$. Then

$$acb = acb(cab)^2 = acb(cabcab) \implies acb = acb(c(abc)ab) = 0,$$

as $abc = 0$. Thus R is a symmetric ring.

Similarly, the equivalences of the conditions (2), (3) and (4) are easily shown by using Corollary 2.2, i.e, $abc = bca = cab$, whenever $abc \in T(R)$. □

According to [8], a ring R satisfies the condition that $abc \in I(R)$ implies $acb \in I(R)$ for $a, b, c \in R$, then R is symmetric by a similar method to the proof of Theorem 2.6., in case of idempotent element. Whether symmetric rings always satisfy this condition.

The following problem given by Jung et al. [8] has been verified with a suitable example in the next part.

Question. Does a symmetric ring R satisfy the condition that $abc \in I(R)$ implies $acb \in I(R)$ for any $a, b, c \in R$?

Solution: If R is a reduced ring then R is symmetric by [1, Theorem I.3] and satisfies the condition if $abc \in I(R)$ implies $acb \in I(R)$ for $a, b, c \in R$. For example, consider $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, then $\mathbb{Z}_6$ is a reduced ring and hence symmetric, and satisfies $1.2.5 = 10 \cong 4 \in \mathbb{Z}_6$. This implies

$$(1.2.5)^2 \cong 4^2 = 16 \cong 4 \in I(R)$$

and $(1.5.2)^2 \cong 4^2 = 16 \cong 4 \in I(R)$. Thus $1.2.5 \in I(R)$ implies that $1.5.2 \in I(R)$, where $I(R)$ stands for set of all idempotent elements of R.

Remark 2.7. For a non reduced ring the question is still open.

Remark 2.8. The above problem also satisfies for $T(R)$, the set of tripotent elements. Thus, if R is a reduced ring, then R is symmetric and satisfies the condition that $abc \in T(R)$ implies that $acb \in T(R)$. So, in $\mathbb{Z}_6$, $1.2.5 \in T(R)$ implies $1.5.2 \in T(R)$.

Theorem 2.9. *For a ring R the following conditions are equivalent:*
1. *R is IFP.*
2. *For $a, b \in R$, $ab \in T(R)$ implies $arb = arb(ab)^2$ for all $r \in R$.*

Proof. (1) $\implies$ (2). Let R be a IFP ring. Suppose $ab \in T(R)$ for $a, b \in R$. Then $ab(1 - (ab)^2) = 0$. Since R is IFP so for all $r \in R$, we get $arb(1 - (ab)^2) = 0 \implies arb = arb(ab)^2$ for all $r \in R$.

(2) $\implies$ (1). Let $ab = 0$ for $a, b \in R$. Then for all $r \in R$,

$$arb = arb(ab)^2 = 0.$$

Thus $ab = 0$ implies $arb = 0$ for all $r \in R$. Hence R is a IFP ring.

□

Theorem 2.10. *Let R be a ring for any $a, b \in R$ such that $ab \in T(R)$ implies $arb \in T(R)$ for all $r \in R$. Then R is IFP.*

Proof. Let $ab = 0$ for any $a, b \in R$ and $arb \in T(R)$ for all $r \in R$. Now,

$$\begin{aligned}(barbar)^3 &= (barbar.barbar.barbar)\\ &= b(arb.arb.arb.arb.arb).ar\\ &= b((arb)^3 arbarb)ar\\ &= b(arb)^3 ar\\ &= barbar,\end{aligned}$$

as $(arb)^3 = arb$. Thus, $(barbar)^3 = barbar$. So, $barbar \in T(R)$. Also $(barbar)^2 = barbarbarbar = b(arb)^3 ar = barbar$ is an idempotent and so,

$barbar$ is central in R. Since $arb \in T(R)$ for all $r \in R$, so

$$\begin{aligned} arb = (arb)^3 &= arb.arb.arb \\ &= (arb)^3.(arb)^3.(arb)^3 \\ &= (arb.arb.arb)(arb.arb.arb)(arb.arb.arb) \\ &= ar(barbar.barbar.barbar).b(arb.arb) \\ &= ar(barbar)^3.b(arb.arb) \\ &= ar(barbar)b(arb.arb) \end{aligned}$$

Hence $arb = a(barbar)rb(arb.arb) = (ab)arbarrb(arb.arb) = 0$, as $ab = 0$ and R is Abelian. Thus $ab = 0 \implies arb = 0$, for all $r \in R$. Hence R is IFP. □

Remark 2.11. It is noticed that $barbar \in T(R)$ is central if it is an idempotent element of R. Otherwise, the relation

$$arb = ar(barbar)b(arb.arb) \implies arb = a(barbar)rb(arb.arb)$$

is not true.

Corollary 2.12. *The converse of the Theorem 2.10 does not hold in general.*

Proof. Let R be a reduced ring. Then by [11, Proposition 1.2], $D_3(R)$ is IFP. Now we verify for tripotent element which is not idempotent. Let $a = (-t_{11} - t_{22} - t_{33}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $b = (t_{11} + t_{22} + t_{33}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then

$$ab = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Clearly $(ab)^3 = ab$. Thus $ab \in T(D_3(R))$. But, for $r = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we have

$$arb = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \notin T(D_3(R)).$$

□

3. Quasi Tri reversible rings

In this section we introduce quasi tri reversible ring and study some of its properties using the concept of tripotent elements of a ring. We begin with the following lemmas.

Lemma 3.1. *Let R be a reversible ring and suppose that $AB \in T(M_2(R))'$ for $A = \begin{pmatrix} a_1 & a_3 \\ 0 & a_2 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix} \in M_{2\times 2}(R)$. Then $(b_1a_3+b_3a_2)(b_1a_1+b_2a_2) = 0$ and*

$$\begin{pmatrix} b_1a_1 & (b_1a_3+b_3a_2)(b_1a_1+b_2a_2) \\ 0 & b_2a_2 \end{pmatrix}, \begin{pmatrix} b_1a_1 & (b_1a_3+b_3a_2)b_1a_1 \\ 0 & b_2a_2 \end{pmatrix},$$
$$\begin{pmatrix} b_1a_1 & (b_1a_3+b_3a_2)b_2a_2 \\ 0 & b_2a_2 \end{pmatrix}, \begin{pmatrix} b_1a_1 & -(b_1a_3+b_3a_2)b_1a_1 \\ 0 & b_2a_2 \end{pmatrix},$$
$$\begin{pmatrix} b_1a_1 & -(b_1a_3+b_3a_2)b_2a_2 \\ 0 & b_2a_2 \end{pmatrix} \in T(M_2(R))'.$$

Proof. Since $AB \in T(M_2(R))'$, so $0 \neq AB \in T(M_2(R))$, we have that $0 \neq a_1b_1 \in T(R)$ or $0 \neq a_2b_2 \in T(R)$. By Corollary 2.2, we get $a_1b_1 = b_1a_1$ and $a_2b_2 = b_2a_2$, and let $a_1b_1 = t$ and $a_2b_2 = s$. From $(AB)^3 = AB$ yields

$$\begin{pmatrix} (a_1b_1)^3 & (a_1b_1)^2(a_1b_3+a_3b_2) + \{a_1b_1(a_1b_3+a_3b_2) + (a_1b_3+a_3b_2)a_2b_2\}a_2b_2 \\ 0 & (a_2b_2)^3 \end{pmatrix}$$
$$= \begin{pmatrix} a_1b_1 & a_1b_3+a_3b_2 \\ 0 & a_2b_2 \end{pmatrix}.$$

Thus we get

$$(a_1b_1)^3 = a_1b_1 \implies t^3 = t \implies t(1-t^2) = 0 \implies 1-t^2 = 0,$$

as $t \neq 0$ so we consider $t = 1$ or $t = -1$, since both 1 and -1 are tripotents. Similarly, $(a_2b_2)^3 = a_2b_2$ and hence $1 - s^2 = 0$, as $s \neq 0$ and $s = 1$ or $s = -1$, (*). Also, from

$$(a_1b_1)^2(a_1b_3+a_3b_2) + \{a_1b_1(a_1b_3+a_3b_2) + (a_1b_3+a_3b_2)a_2b_2\}a_2b_2$$
$$= a_1b_3 + a_3b_2$$

yields

$$t^2(a_1b_3+a_3b_2) + t(a_1b_3+a_3b_2)s + (a_1b_3+a_3b_2)s^2 = a_1b_3+a_3b_2$$
$$\implies t(a_1b_3+a_3b_2)s + (a_1b_3+a_3b_2) = (a_1b_3+a_3b_2)(1-s^2)$$
$$\implies +a_1b_1a_3b_2a_2b_2 + a_1b_3 + a_3b_2 = 0, \quad (**).$$

Multiplying b_1 on the left and a_2 on the right of equality (**) we get

$$\begin{aligned} & b_1a_1b_1a_1b_3a_2b_2a_2 + b_1a_1b_1a_3b_2a_2b_2a_2 + b_1a_1b_3a_2 + b_1a_3b_2a_2 = 0 \\ \Longrightarrow\ & (b_1a_1)^2b_3a_2b_2a_2 + b_1a_1b_1a_3(b_2a_2)^2 + b_1a_1b_3a_2 + b_1a_3b_2a_2 = 0 \\ \Longrightarrow\ & b_3a_2b_2a_2 + b_1a_1b_1a_3 + b_1a_1b_3a_2 + b_1a_3b_2a_2 = 0 \\ \Longrightarrow\ & (b_1a_3 + b_3a_2)(b_1a_1 + b_2a_2) = 0 \\ \Longrightarrow\ & (b_1a_3 + b_3a_2)(t + s) = 0. \end{aligned}$$

This result gives us

$$\begin{pmatrix} b_1a_1 & (b_1a_3 + b_3a_2)(b_1a_1 + b_2a_2) \\ 0 & b_2a_2 \end{pmatrix}, \begin{pmatrix} b_1a_1 & (b_1a_3 + b_3a_2)b_1a_1 \\ 0 & b_2a_2 \end{pmatrix},$$
$$\begin{pmatrix} b_1a_1 & (b_1a_3 + b_3a_2)b_2a_2 \\ 0 & b_2a_2 \end{pmatrix}, \begin{pmatrix} b_1a_1 & -(b_1a_3 + b_3a_2)b_1a_1 \\ 0 & b_2a_2 \end{pmatrix},$$
$$\begin{pmatrix} b_1a_1 & -(b_1a_3 + b_3a_2)b_2a_2 \\ 0 & b_2a_2 \end{pmatrix} \in T(M_2(R))'.$$

□

Remark 3.2. It is noticed that from (*) i.e., $1 - t^2 = 0$ or, $(1 - s^2 = 0)$ are also hold other than $t = 1, -1$ or, $(s = 1, -1)$. For example, $3 \in T(\mathbb{Z}_8)$ and $1 - 3^2 = 0$ but $3 \neq 1, -1$.

Lemma 3.3. *Let R be a ring with $T(R) = \{0, 1, -1\}$. If $AB \in T(D_2(R))'$ for $A, B \in D_2(R)$, then $AB = I_2 = BA$ or $AB = -I_2 = BA$.*

Proof. (1) Since $A, B \in D_2(R)$, so let $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$. Suppose $AB \in T(D_2(R))'$ and let $AB = I_2$. This implies that, $\begin{pmatrix} ab & 0 \\ 0 & ab \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This gives $ab = 1$. So either, $a = 1$, $b = 1$ or $a = -1$, $b = -1$. Thus in each case, $BA = \begin{pmatrix} ba & 0 \\ 0 & ba \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$. Hence $AB = I_2 = BA$.

(Another proof) Let R be a ring with $T(R) = \{0, 1, -1\}$. Then R is Abelian. So by the help of [6, Lemma 2], we get $T(D_2(R)) = \{0, I_2, -I_2\}$. Hence $D_2(R)$ is also Abelian. Suppose $AB \in T(D_2(R))'$ for $A, B \in D_2(R)$ and if $AB = I_2$, then $BA = I_2$, as $D_2(R)$ is directly finite. Thus $AB = I_2 = BA$.

(2) Since $A, B \in D_2(R)$, so let $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$. Suppose $AB \in T(D_2(R))'$ and let $AB = -I_2$. This implies that

$$\begin{pmatrix} ab & 0 \\ 0 & ab \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This gives $ab = -1$. So either $a = 1$, $b = -1$ or $a = -1$, $b = 1$. Thus in each case, $BA = \begin{pmatrix} ba & 0 \\ 0 & ba \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$. Hence $AB = -I_2 = BA$. □

Here we introduce the quasi tri reversible ring using non-zero tripotent elements of a ring as follows.

Definition 3.4. A ring R is a quasi tri reversible if $ab \in T(R)'$ for $a, b \in R$ implies that $ba \in T(R)$, where $T(R)'$ is the set of all non-zero tripotent elements of R.

Example 3.5. (1) Let $R = M_2(\mathbb{Z}_3)$ be an upper triangular matrix ring over field $\mathbb{Z}_3$. Then R is a quasi tri reversible ring.

(2) The upper triangular matrix ring, $M_2(R)$ is a quasi tri reversible ring by Theorem 3.8 to follow.

Remark 3.6. By Lemma 3.3, every ring with $T(R) = \{0, 1, -1\}$ is quasi tri reversible. Reversible rings are quasi tri reversible by Theorem 2.1, but not conversely by Theorem 3.8 to follow. Also we recall that reversible rings are Abelian. But quasi tri reversible rings need not be Abelian by Theorem 3.8 to follow.

Lemma 3.7. (1) *A ring R is quasi tri reversible if and only if $ab \in T(R)'$ for $a, b \in R$ implies $ba \in T(R)'$.*

(2) *The class of quasi tri reversible rings are closed under subrings (with or without identity).*

Proof. (1) Let R be a quasi tri reversible ring and suppose that $ab \in T(R)'$ for $a, b \in R$. Then by definition, $ba \in T(R)$. If possible let $ba = 0$. Since $ab \in T(R)'$, then $(ab)^3 = ab$, and so $ab = a(ba)bab = 0$, which contradicts that $ab \neq 0$. Thus $ba \in T(R)'$.

(2) Let R be a quasi tri reversible ring and S be a subring (possibly without identity) of R. Suppose $ab \in T(S)'$ for $a, b \in S$. We have noted that $T(S) = T(R) \cap S$ and $T(S)' = T(R)' \cap S$. Since R is quasi tri reversible, $ba \in T(R)$. But $ba \in S$, and $ba \in T(S)$ follows. □

Following G. Marks [14], a ring R is called NI if $N^*(R) = N(R)$. It is obvious that a ring R is NI if and only if $N(R)$ forms an ideal of R if and only if $R/N^*(R)$ is a reduced ring. By [9, Proposition 2.7(1)], a ring R is

NI, then R is directly finite. We use freely the fact that reversible rings are easily shown to be NI.

Theorem 3.8. *A ring R is a reversible with $T(R) = \{0, 1, -1\}$ if and only if $M_2(R)$ is a quasi tri reversible ring.*

Proof. Let R be a reversible ring with $T(R) = \{0, 1, -1\}$. Through a simple computation, we get

$$T(M_2(R)) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & s \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} -1 & u \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & v \\ 0 & -1 \end{pmatrix} | r, s, u, v \in R \right\},$$

since $T(R) = \{0, 1, -1\}$. Suppose that $AB \in T(M_2(R))'$ for

$$A = \begin{pmatrix} a_1 & a_3 \\ 0 & a_2 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix} \in M_2(R).$$

Then $AB = \begin{pmatrix} a_1b_1 & a_1b_3 + a_3b_2 \\ 0 & a_2b_2 \end{pmatrix}$. Now consider the following four cases:

Case 1: Let $a_1b_1 = 1$ and $a_2b_2 = 1$. So by Corollary 2.2., we get $b_1a_1 = 1$ and $b_2a_2 = 1$. If $AB = I_2$, then $a_1b_3 + a_3b_2 = 0$. Thus $BA = I_2$, as $M_2(R)$ is NI and so directly finite. Otherwise,

$$\begin{aligned} BA &= \begin{pmatrix} b_1a_1 & b_1a_3 + b_3a_2 \\ 0 & b_2a_2 \end{pmatrix} \\ &= \begin{pmatrix} b_1a_1 & (b_1a_3 + b_3a_2)1 \\ 0 & b_2a_2 \end{pmatrix} \\ &= \begin{pmatrix} b_1a_1 & (b_1a_3 + b_3a_2)b_1a_1 \\ 0 & b_2a_2 \end{pmatrix} \\ &\in T(M_2(R))', \end{aligned}$$

by Lemma 3.1. Thus $M_2(R)$ is a quasi tri reversible ring.

Case 2: Let $a_1b_1 = -1$ and $a_2b_2 = -1$, then $b_1a_1 = -1$ and $b_2a_2 = -1$, by Corollary 2.2. This implies

$$\begin{aligned} BA &= \begin{pmatrix} -1 & b_1a_3 + b_3a_2 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} b_1a_1 & (-1)(b_1a_3 + b_3a_2)(-1) \\ 0 & b_2a_2 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} b_1a_1 & -(b_1a_3 + b_3a_2)b_1a_1 \\ 0 & b_2a_2 \end{pmatrix}$$
$$\in T(M_2(R))',$$

by Lemma 3.1. Thus $M_2(R)$ is a quasi tri reversible ring.

Case 3: Firstly let $a_1b_1 = 1$ and $a_2b_2 = 0$, then $b_1a_1 = 1$ and $b_2a_2 = 0$, by Corollary 2.2. This gives,

$$BA = \begin{pmatrix} 1 & b_1a_3 + b_3a_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b_1a_1 & (b_1a_3 + b_3a_2)b_1a_1 \\ 0 & b_2a_2 \end{pmatrix} \in T(M_2(R))',$$

by Lemma 3.1. Secondly let $a_1b_1 = 0$ and $a_2b_2 = 1$, then $b_1a_1 = 0$ and $b_2a_2 = 1$, by Corollary 2.2. This gives

$$BA = \begin{pmatrix} 0 & b_1a_3 + b_3a_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b_1a_1 & (b_1a_3 + b_3a_2)b_2a_2 \\ 0 & b_2a_2 \end{pmatrix} \in T(M_2(R))',$$

by Lemma 3.1. Thus $M_2(R)$ is a quasi tri reversible ring.

Case 4: Firstly let $a_1b_1 = -1$ and $a_2b_2 = 0$, then $b_1a_1 = -1$ and $b_2a_2 = 0$, by Corollary 2.2. This gives

$$BA = \begin{pmatrix} -1 & b_1a_3 + b_3a_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b_1a_1 & -(b_1a_3 + b_3a_2)b_1a_1 \\ 0 & b_2a_2 \end{pmatrix} \in T(M_2(R))',$$

by Lemma 3.1. Secondly let $a_1b_1 = 0$ and $a_2b_2 = -1$, then $b_1a_1 = 0$ and $b_2a_2 = -1$, by Corollary 2.2. This gives

$$BA = \begin{pmatrix} 0 & b_1a_3 + b_3a_2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} b_1a_1 & -(b_1a_3 + b_3a_2)b_2a_2 \\ 0 & b_2a_2 \end{pmatrix} \in T(M_2(R))',$$

by Lemma 3.1. Finally, if $AB \in T(M_2(R))'$, then $BA \in T(M_2(R))'$. This shows that $M_2(R)$ is a quasi tri reversible ring.

Conversely, let $M_2(R)$ be a quasi tri reversible ring. We have to show that R is a reversible ring. Let $ab = 0$ for $a, b \in R$. We consider two matrices $A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$ of $M_2(R)$. Then

$$0 \neq AB = \begin{pmatrix} ab & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \in T(M_2(R)).$$

Since $M_2(R)$ is a quasi tri reversible ring, so $BA = \begin{pmatrix} ba & 0 \\ 0 & -1 \end{pmatrix} \in T(M_2(R))'$. This implies that $(ba)^3 = ba \implies b(ab)aba = ba \implies 0 = ba$, as $ab = 0$. Therefore R is a reversible ring.

Next we assume on the contrary that there exists $t^3 = t$ with $t \notin \{0, 1, -1\}$. Let us consider two matrices $C = \begin{pmatrix} t & 1 \\ 0 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ of $M_2(R)$. Then $0 \neq CD = \begin{pmatrix} t & t \\ 0 & 0 \end{pmatrix} \in T(M_2(R))$. But $DC = \begin{pmatrix} t & 1 \\ 0 & 0 \end{pmatrix} \notin T(M_2(R))$, as $\begin{pmatrix} t & 1 \\ 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} t & t^2 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} t & 1 \\ 0 & 0 \end{pmatrix}$, which contradicts that $M_2(R)$ is a quasi tri reversible ring. Thus $T(R) = \{0, 1, -1\}$. □

Remark 3.9. The existence of reversible ring R, such that $T(R) = \{0, 1, -1\}$ and R is not reduced, which is evident by Theorem 3.8. For this, let R be a domain and $D_2(R)$ be the diagonal matrix. Then by [14, Proposition 1.6], we have observed that $D_2(R)$ is a reversible ring and $T(D_2(R)) = \{0, I_2, -I_2\}$, but $D_2(R)$ is not reduced, because it does not have a non-zero nilpotent element.

As a consequence of Theorem 3.8, we get the following results.

Corollary 3.10. (1) *If R is a domain, then $M_2(R)$ is a quasi tri reversible ring.*

(2) *Let R be a (quasi tri) reversible ring such that $T(R)$ contains $\{0, 1, -1\}$ properly, and $n \geq 2$. Then $Mat_n(R)$ and $M_n(R)$ need not be quasi tri reversible rings.*

Proof. (1) is an instant result of Theorem 3.8.

(2) By Theorem 3.8, we have $M_2(R)$ is not a quasi tri reversible ring and thus $Mat_n(R)$ and $M_n(R)$ need not be quasi tri reversible for $n \geq 2$, by Lemma 3.7(2). □

Considering Theorem 3.8 and Corollary 3.10, it is natural to raise a question. Whether $M_2(R)$ is a quasi tri reversible ring over a reduced ring R. The following Example 3.11 illuminates that the answer is negative.

Example 3.11. Let $R = \mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}$ be a domain. Then R is a reduced ring but not a domain, and
$T(R) = \{(0,0), (1,0), (0,1), (1,1), (-1,0), (0,-1), (1,-1), (-1,1), (-1,-1)\}$.
Then $M_2(R)$ is not quasi tri reversible, by Theorem 3.8. Because, for

$$A = \begin{pmatrix} (-1,0) & (1,1) \\ (0,0) & (1,0) \end{pmatrix}, B = \begin{pmatrix} (1,1) & (-1,1) \\ (0,0) & (-1,0) \end{pmatrix} \in M_2(R),$$

we have $AB \in T(M_2(R))$ but $BA \notin T(M_2(R))$. This shows that $M_2(R)$ is not quasi tri reversible. We have noted that the ring $R = \mathbb{Z} \times \mathbb{Z}$ is reversible.

The justification in the following Corollary 3.12, clarify Theorem 3.8.

Corollary 3.12. *Theorem 3.8 is not valid for $M_n(R)$, whenever $n \geq 3$.*

Proof. Let R be any ring and we consider for $n = 3$. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in M_3(R).$$

Then

$$AB = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \in T(M_3(R)),$$

as $(AB)^3 = AB$. But $BA = \begin{pmatrix} 0 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \notin T(M_3(R))$, as $(BA)^3 \neq BA$.

This argument can also be applicable for $n \geq 4$. Thus $M_n(R)$ cannot be quasi tri reversible, whenever $n \geq 3$. □

Next we extend [9, Theorem 1.8] by using tripotent elements, since every idempotents are also tripotents, so in the following Theorem 3.13, we observe another kind of quasi tri reversible rings in the class of simple Artinian rings.

Theorem 3.13. *The full matrix ring $Mat_2(\mathbb{Z}_2)$ is a quasi tri reversible ring, where $\mathbb{Z}_2$ is the ring of integers modulo 2.*

Proof. Let $R = Mat_2(\mathbb{Z}_2)$. So we have

$$T(R) = \left\{0, I_2, T_{11}, T_{22}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\},$$

by the help of [12, Lemma 2.3]. Let $AB \in T(R)'$ for

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in R.$$

We proceed our justification case by case. Suppose $AB = I_2$. Since R is Artinian, so R is directly finite. Thus $AB = I_2$ implies $BA = I_2$. So $BA \in T(R)$. Hence R is quasi tri reversible ring.
Let us assume that $AB = T_{11}$. Now proceeding by the same argument in [9, Theorem 1.8]. We have

$$\begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, $a_1b_1 + a_2b_3 = 1$, $a_1b_2 + a_2b_4 = 0$, $a_3b_1 + a_4b_3 = 0$ and $a_3b_2 + a_4b_4 = 0$. From $a_1b_1 + a_2b_3 = 1$, we have the cases of $(a_1b_1 = 1, a_2b_3 = 0)$ and $(a_1b_1 = 0, a_2b_3 = 1)$. We consider the case of $a_1b_1 = 1, a_2b_3 = 0$. Then $a_1 = 1, b_1 = 1$ and $a_2 = 0$ or $b_3 = 0$. Let $a_2 = 0$. Then from $a_1b_2 + a_2b_4 = 0$ we get $b_2 = 0$. So, $a_3b_2 + a_4b_4 = 0 \implies a_4b_4 = 0$. These results give us

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_3 + b_4a_3 & 0 \end{pmatrix} \in T(R),$$

as

$$\begin{pmatrix} 1 & 0 \\ b_3 + b_4a_3 & 0 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ b_3 + b_4a_3 & 0 \end{pmatrix}.$$

Let $b_3 = 0$. Then from $a_3b_1 + a_4b_3 = 0$ we get $a_3 = 0$. Since $a_3b_2 + a_4b_4 = 0$, we have $a_4b_4 = 0$. These results give us

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 1 & a_2 + b_2a_4 \\ 0 & 0 \end{pmatrix} \in T(R).$$

Now we consider the case of $a_1b_1 = 0, a_2b_3 = 1$. Then $a_2 = 1, b_3 = 1$ and $a_1 = 0$ or $b_1 = 0$. Let $a_1 = 0$. Then from $a_1b_2 + a_2b_4 = 0$ we get $b_4 = 0$. So $a_3b_2 + a_4b_4 = 0 \implies a_3b_2 = 0$. These results provide us

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 0 & b_1 + b_2a_4 \\ 0 & 1 \end{pmatrix} \in T(R).$$

Let $b_1 = 0$. Then from $a_3b_1 + a_4b_3 = 0$, we get $a_4 = 0$. So $a_3b_2 + a_4b_4 = 0$ implies $a_3b_2 = 0$. These results give us

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a_1 + b_4a_3 & 1 \end{pmatrix} \in T(R).$$

Thus $BA \in T(R)$ in any case when $AB = T_{11}$. This implies R is a quasi tri reversible ring.

Thus, by similar justification for $AB = T_{22}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, respectively, we have $BA \in T(R)$. Hence, $R = Mat_2(\mathbb{Z}_2)$ is a quasi tri reversible ring. □

Remark 3.14. Let $\mathbb{Z}_3 = \{0, 1, -1\}$. Following Theorem 3.13, if we consider $R = Mat_2(\mathbb{Z}_3)$, then

$$T(R) = \left\{0, I_2, -I_2, T_{11}, -T_{11}, T_{22}, -T_{22}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \right.$$

$$\begin{pmatrix}-1 & -1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\ -1 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\ -1 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix},$$
$$\begin{pmatrix}-1 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}-1 & 0\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 0 & -1\end{pmatrix}\Bigg\}.$$

But R is not quasi tri reversible ring, where $\mathbb{Z}_3$ is the ring of integers modulo 3. Let $A = \begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}$ and $B = \begin{pmatrix}b_1 & b_2\\ b_3 & b_4\end{pmatrix}$ with $AB = \begin{pmatrix}-1 & 1\\ 0 & 0\end{pmatrix} \in T(R)'$. Then $a_1b_1 + a_2b_3 = -1$, $a_1b_2 + a_2b_4 = 1$, $a_3b_1 + a_4b_3 = 0$ and $a_3b_2 + a_4b_4 = 0$. From $a_1b_1 + a_2b_3 = -1$, $a_1b_2 + a_2b_4 = 1$, we have the cases of

$$\begin{aligned}
&(a_1b_1 = -1, a_2b_3 = 0, a_1b_2 = 1, a_2b_4 = 0),\\
&(a_1b_1 = -1, a_2b_3 = 0, a_1b_2 = 0, a_2b_4 = 1),\\
&(a_1b_1 = 0, a_2b_3 = -1, a_1b_2 = 1, a_2b_4 = 0),\\
&(a_1b_1 = 0, a_2b_3 = -1, a_1b_2 = 0, a_2b_4 = 1).
\end{aligned}$$

We consider the case, $a_1b_1 = -1, a_2b_3 = 0, a_1b_2 = 1, a_2b_4 = 0$. Then $a_1 = b_2 = 1$ and $b = -1$ or $a_1 = b_2 = -1$ and $b = 1$. From $a_2b_3 = 0$ and $a_2b_4 = 0$ we have the cases, $a_2 = 0$ or $b_3 = 0$ and $a_2 = 0$ or $b_4 = 0$.
Let $a_1 = b_2 = 1$, $b = -1$ and $a_2 = 0$. From $a_3b_1 + a_4b_3 = 0$ and $a_3b_2 + a_4b_4 = 0$ we get $-a_3 + a_4b_3 = 0$ and $a_3 + a_4b_4 = 0$. If $a_3 = -1$, then $a_4 = -1$, $b_3 = 1$ and $b_4 = -1$ (or, $a_4 = 1$, $b_3 = -1$ and $b_4 = 1$). These results give us

$$\begin{aligned}
BA &= \begin{pmatrix}b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4\\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4\end{pmatrix}\\
&= \begin{pmatrix}(-1)(1) + (1)(-1) & (1)(-1)\\ 1(1) + (-1)(-1) & (-1)(-1)\end{pmatrix}\\
&= \begin{pmatrix}-2 & -1\\ 2 & 1\end{pmatrix}\\
&= \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\\
&\notin T(R).
\end{aligned}$$

So, R is not quasi tri reversible ring.

We have noted that, $Mat_2(\mathbb{Z}_2)$ is simple Artinian. So, we considering Theorem 3.13, one may ask naturally whether semisimple Artinian rings are quasi tri reversible. The following Example shows that the answer is negative.

Example 3.15. (1) Let F be a division ring and $R = F \times F$. Since $Mat_2(R)$ is isomorphic to $Mat_2(F) \times Mat_2(F)$, $Mat_2(R)$ is semisimple Artinian. Also we consider $M_2(R)$, then $M_2(R)$ is not quasi tri reversible (see Example 3.11). So, $Mat_2(R)$ is not quasi tri reversible by Lemma 3.7(2).

(2) Let $F = \mathbb{Z}_2$ in (1). Then $Mat_2(\mathbb{Z}_2)$ is quasi tri reversible by Theorem 3.13. But $Mat_2(\mathbb{Z}_2) \times Mat_2(\mathbb{Z}_2)$ is not quasi tri reversible by the part (1). Thus the class of quasi tri reversible rings is not closed under direct products.

4. On a property of Abelian rings

In this section we study the structure of Abelian rings and NI rings with the concept of tripotent elements. We begin with the following equivalent conditions to the fact that in any ring R, $ab \in T(R)$ for $a, b \in R$ implies $(ba)^3 \in T(R)$.

Theorem 4.1. *For a ring R the following conditions are equivalent:*

(1) *R is Abelian.*
(2) *If $ab \in T(R)'$ for $a, b \in R$, then $(ba)^3 = ab$.*
(3) *If $ab \in T(R)$ for $a, b \in R$, then $(ba)^3 = ab$.*

Proof. (1) $\Longrightarrow$ (2). Let R be an Abelian ring. We assume that $ab \in T(R)'$ for $a, b \in R$. Then $(ba)^3 \in T(R)$. Since, $ab \in T(R)'$, so $(ab)^2$ is always idempotent and hence it is central in R. Thus we have

$$(ba)^3 = bababa = b(ab)(ab)a = b(ab)^2a = (ab)^2(ba) = (ab)^3 = ab,$$

as R is Abelian.

(2) $\Longrightarrow$ (1). Let $t \in T(R)'$, then t^2 is always idempotent. To prove R is Abelian, it is enough to show that t^2 is semicentral in R. We assume on the contrary that there exists $a \in R$ such that $t^2a(1 - t^2) \neq 0$. Then $t^2 + t^2a(1 - t^2) \in T(R)'$. Otherwise, if $0 = t^2 + t^2a(1 - t^2)$, then

$$0 = (t^2 + t^2a(1 - t^2))t^2 = t^4 + t^2at^2 - t^2at^4 = t^2 + t^2at^2 - t^2at^2 = t^2,$$

contradicts $t \neq 0$. Thus $t^2 + t^2a(1 - t^2) \in T(R)'$ and clearly

$$t^2 + t^2a(1 - t^2) \in T(R).$$

Now, $t^2 = t^2 + at^2 - at^2 = t^2 + at^2 - at^4 = (1 + a(1 - t^2))t^2$. Since every idempotent is also tripotent, then

$$t^2 = (t^2)^3 = ((1 + a(1 - t^2))t^2)^3 = t^2(1 + a(1 - t^2)),$$

using condition (2) (i.e. $(ba)^3 = ab$). This implies, $t^2 = t^2 + t^2a(1-t^2)$, permitting $t^2a(1-t^2) = 0$, which contradicts $t^2a(1-t^2) \neq 0$.
Therefore t^2 is semicentral.

(2) $\Longleftrightarrow$ (3) is obvious. □

Remark 4.2. In the proof of Theorem 4.1, it is observed that $ab \in T(R)'$ implies $ba \neq 0$. It is easy to prove the reversible rings are Abelian. This fact is also obtained from Corollary 2.2 and Theorem 4.1. But Abelian ring need not be reversible. For example let $R = D_n(F)$ for $n \geq 3$ over any Abelian ring F. Then R is Abelian by [6, Lemma 2]. But R is not reversible by [11, Example 1.5]. Another example of Abelian ring that is not reversible, the polynomial ring $R_2[x]$ in Example 4.5 to follow is an Abelian but not reversible.

From Corollary 2.2 and Theorem 4.1, we can conclude the following corollary.

Corollary 4.3. *Let R be an Abelian ring but not reversible. Then there exists $a, b \in R$ such that $ab \in T(R)$, $(ba)^3 = ab$, and $ba \notin T(R)$.*

It is easy to prove the class of Abelian rings is closed under subrings and direct products. The notion of quasi tri reversible and Abelian rings are independent of each other as shown in following example.

Example 4.4. (1) Let R_1 be an Abelian and R_2 also be an Abelian ring but not reversible such that $R = R_1 \times R_2$. Let $t \in T(R_1)'$, then $-t \in T(R_1)'$ and $a, b \in R_2$ such that $ab = 0$ but $ba \neq 0$. Suppose $f = (t, a)$ and $g = (-t, b)$ be two elements in R. Then $fg = (-t^2, ab) = (-t^2, 0) \in T(R)'$, as

$$(fg)^3 = (-t^6, 0) = (-t^2, 0) = fg.$$

But $gf = (-t^2, ba)$ and

$$(gf)^3 = (-t^2, bababa) = (-t^2, b(ab)aba) = (-t^2, 0) \neq gf.$$

This implies that $gf \notin T(R)'$. Thus R is not a quasi tri reversible ring but R is Abelian.

(2) By Theorem 3.13, $Mat_2(\mathbb{Z}_2)$ is a quasi tri reversible ring but clearly it is non-Abelian.

By the help of argument in Example 4.4(1), we can deduce that the polynomial rings are not quasi tri reversible as shown in the following example.

Example 4.5. Let R_1 be a reduced ring and R_2 be the reversible ring. Let $R_3 = R_1 \times R_2$. Then R_3 is clearly reversible and hence quasi tri reversible by Theorem 2.1. Let $R_3[x]$ be a polynomial ring. Since R_2 is reversible and hence based on the justification in [11, Example 2.1], we get $R_2[x]$ is not reversible. Taking $f(x) = \sum_{i=0}^{m} \alpha_i x^i$, $g(x) = \sum_{j=0}^{n} \beta_j x^j \in R_2[x]$ such that $f(x)g(x) = 0$ but $g(x)f(x) \neq 0$, where we can let $m = n$ by using zero coefficients if necessary. Now, let $a(x) = (-1, \alpha_0) + \sum_{i=0}^{m}(0, \alpha_i)x^i$ and $b(x) = (1, \beta_0) + \sum_{j=0}^{m}(0, \beta_j)x^j$ in $R_3[x]$. Then

$$\begin{aligned} a(x)b(x) &= ((-1, \alpha_0) + \sum_{i=0}^{m}(0, \alpha_i)x^i)((1, \beta_0) + \sum_{j=0}^{m}(0, \beta_j)x^j) \\ &= (-1, \alpha_0\beta_0) + \sum_{j=0}^{m}(0, \alpha_0\beta_j)x^j + \sum_{i=0}^{m}(0, \alpha_i\beta_0)x^i + \sum_{i,j=0}^{m}(0, \alpha_j\beta_j)x^{i+j} \\ &= (-1, 0) \in T(R_3[x])', \end{aligned}$$

as $\alpha_i\beta_j = 0$, for all i, j. But

$$\begin{aligned} b(x)a(x) &= ((1, \beta_0) + \sum_{j=0}^{m}(0, \beta_j)x^j)((-1, \alpha_0) + \sum_{i=0}^{m}(0, \alpha_i)x^i) \\ &= (-1, 0) + c(x), \end{aligned}$$

for some $0 \neq c(x) = \sum_{k=1}^{l}(0, \gamma_k)x^k \in R_3[x]$, by the justification in [11, Example 2.1]. If possible let $b(x)a(x) \in T(R_3[x])$, then we get

$$\begin{aligned} (-1, 0) + c(x) = b(x)a(x) &= b(x)a(x)b(x)a(x)b(x)a(x) \\ &= b(x)[a(x)b(x)]^2 a(x) \\ &= b(x)(1, 0)a(x) \\ &= (1, 0)b(x)a(x) \\ &= (1, 0)[(-1, 0) + \sum_{k=1}^{l}(0, \gamma_k)x^k] \\ &= (-1, 0), \end{aligned}$$

a contradiction. Thus, $b(x)a(x) \notin T(R_3[x])$. Hence, $R_3[x]$ is not a quasi tri reversible ring.

By the help of Theorem 4.1, we have the following results. Following Bell [2], a ring R is called IFP if $ab = 0$ for $a, b \in R$ implies $aRb = 0$. We have noted that reversible rings are IFP and IFP rings are Abelian. We use this fact comfortably.

Theorem 4.6. (1) *A ring R is reversible if and only if $ab \in T(R)$ for $a, b \in R$, then $bra = bra(ab)^2$ for all $r \in R$.*

(2) *A ring R is IFP if and only if for $a, b \in R$, $ab \in T(R)$ implies $arb = arb(ba)^4$ for all $r \in R$.*

Proof. (1) Let R be a reversible ring and $ab \in T(R)$ for $a, b \in R$. Then by Theorem 2.1, $ba \in T(R)$. So, $ba = (ba)^3$, and thus $ba(1-(ba)^2) = 0$. By Corollary 2.2, we get $ba(1-(ab)^2) = 0$. Since reversible rings are IFP, so R is IFP, then for all $r \in R$ we have $bra(1-(ab)^2) = 0 \implies bra = bra(ab)^2$.

Conversely, let $ab \in T(R)$ for $a, b \in R$ and $bra = bra(ab)^2$ for all $r \in R$. For $r = 1$, we get $ba = ba(ab)^2$. Now we assume that $ab = 0$, then $ba = ba(ab)^2 = 0$. This shows that R is a reversible ring.

(2) Let R be a IFP ring. Suppose $ab \in T(R)$ for $a, b \in R$. Then by Theorem 4.1, we get $ab = (ba)^3$. Since R is Abelian, so

$$ab = bababa = b(ab)ababa = (ab)babababa = ab(ba)^4$$

implies $ab(1-(ba)^4) = 0$. Since R is IFP, so for all $r \in R$ we get $arb(1-(ba)^4) = 0 \implies arb = arb(ba)^4$.

Conversely, let $ab = 0$ for $a, b \in R$. Then for all $r \in R$,

$$arb = arb(ba)^4 = arbb(ab)a = 0.$$

Thus $ab = 0$ implies $arb = 0$ for all $r \in R$. Hence R is a IFP ring. □

Remark 4.7. By the help of Theorem 4.6, we can show that reversible rings are IFP, through tripotent elements. Let R be a ring for $a, b \in R$ such that $ab \in T(R)$. Suppose R is reversible, then by Corollary 2.2, $ab = ba$; thus $arb = arb(ba)^2$ for all $r \in R$ by Theorem 4.6(1). This can be written as, $arb = arb(ba)^4$, hence R is IFP by Theorem 4.6(2).

However the concepts of IFP and quasi tri reversible rings are independent of each other as shown in the following example:

Example 4.8. (1) Let R_1 be any IFP ring and $R_2 = D_3(F)$, then by [11, Example 1.5] R_2 is IFP but not reversible when F is a reduced ring. Now, let $R = R_1 \times R_2$, then R is clearly IFP but not quasi tri reversible by Example 4.4(1).

(2) Let $R = Mat_2(\mathbb{Z}_2)$, then by Theorem 3.13, we have R is a quasi tri reversible ring. But R is not IFP, as R is non-Abelian.

According to Von Neumann [18], a ring R is regular provided that for every $x \in R$ there exists $y \in R$ such that $xyx = x$. We have noted that every regular ring R is semiprimitive (i.e. $J(R) = 0$). Next, we extend [4, Theorem 1.1] as, a ring R is regular if and only if every principal right(left) ideal of R is generated by a tripotent. Let R be a regular ring, then for any $x \in R$ there exists $y \in R$ such that $xyx = x$. Then $(xy)^3 = xyxyxy = xyxy = xy$. So $xy \in T(R)$ such that $xyR = xR$. Thus, every principal right(left) ideal of R is generated by a tripotent element. Again, if $x \in R$ there exists tripotent $t \in R$ such that $tR = xR$, then $t = xy$ for some $y \in R$ and $x = tx = xyx$. Moreover, we have the following equivalences by [4, Theorem 3.2] fact of semiprime NI rings being reduced. For a regular ring R the following conditions are equivalent:

(1) R is reduced, (2) R is reversible, (3) R is IFP, (4) R is NI, (5) R is Abelian.

However the quasi tri reversibility need not be equivalent to the reversibility when given rings are regular as follows.

Example 4.9. The ring $R = Mat_2(\mathbb{Z}_2)$ is regular by [4, Theorem 1.7] and quasi tri reversible by Theorem 3.13. But R is non-Abelian, hence not reversible.

According to Hoque and Saikia [5], a ring R is called weakly tri normal if for all $a, r \in R$ and $t \in R$ with $at = 0$, implies $Rtra$ is a nil left ideal of R. The weakly tri normal rings are directly finite. We have noted that, weakly reversible ring and Abelian rings are weakly tri normal rings.

Here we observe that notion of quasi tri reversible rings and weakly tri normal rings are independent of each other as shown in the following example:

Example 4.10. (1) The ring $M_n(R)$ is not quasi tri reversible ring when $n \geq 3$, by Corollary 3.12. But $M_n(R)$ is weakly tri normal ring for any $n \geq 1$ by [5, Corollary 2.12].

(2) By Theorem 3.13, the ring $R = Mat_2(\mathbb{Z}_2)$ is a quasi tri reversible ring. But $R = Mat_2(\mathbb{Z}_2)$ is not weakly tri normal ring as follows. Let $t = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \in T(R)$ and $a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ be any matrix in R. Then

$$at = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now for $r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in R$, we get

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \notin N(R)$. This shows that R is not a weakly tri normal ring.

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