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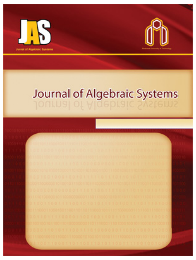

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SOME IDENTITIES INVOLVING ENDOMORPHISMS OF PRIME RINGS

A. BOUA

Abstract. In this paper, we will extend some results on the commutativity of quotient rings proved in [2] and [11]. However, we will consider endomorphisms instead of derivations and generalized derivations, which is sufficient to obtain good results. We will also show that the "primality of the ideal" condition imposed in our theorems cannot be removed.

1. Introduction

Throughout this article, $\mathcal{R}$ will represent an associative ring with center $Z(\mathcal{R})$. For any $x, y \in \mathcal{R}$, the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anticommutator $xy + yx$. $\mathcal{R}$ is 2-torsion free if whenever $2x = 0$, with $x \in \mathcal{R}$ implies $x = 0.\mathcal{R}$ is prime if $a\mathcal{R}b = \{0\}$ implies $a = 0$ or $b = 0$. Over the last 30 years, several authors have studied the relationship between the commutativity of the ring $\mathcal{R}$ and certain special types of mappings on $\mathcal{R}$. Over the last decades, several authors have subsequently refined and extended these results in different directions ([2], [4], [3], [5], [6], [8], [9], [10], [11], and [12], where further references can be found). The first result in this direction is due to Divinsky [6], who proved that the simple Artinian ring must be commutative if it has a commuting nontrivial automorphism. Two years later, Posner [10] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. In the last two decades, a lot of work has been done on commutativity preserving mappings (for reference see [4]). In 1994, Bell and Daif [5] studied the commutativity of prime and semiprime rings admitting derivations and endomorphisms which are SCP on their particular subset. In particular, they proved that if a prime ring $\mathcal{R}$ admits a non-identical endomorphism which is SCP on an essential right ideal of $\mathcal{R}$, then $\mathcal{R}$ is commutative. Two years after the publication of this paper, the authors Deng and Ashraf published a good paper on this topic, in which they initiated the study of a more general concept than SCP mappings, that

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is, They considered the situation $[T(x),T(y)]-[x,y]\in Z(\mathcal{R})$ for all $x,y\in U$, where T is a non-identical endomorphism of $\mathcal{R}$ and U is an essential right ideal of $\mathcal{R}$. Inspired by these works, our goal in the current paper is to investigate the commutativity of a quotient ring $\mathcal{R}/P$, (where $\mathcal{R}$ is an arbitrary ring and P is a prime ideal of $\mathcal{R}$) with endomorphisms of $\mathcal{R}$ that satisfy certain algebraic identities without assuming that the ring $\mathcal{R}$ is prime (or semiprime).

2. Main results

We start with the first two lemmas that we use to prove our theorems, which are listed below:

Lemma 2.1. [1, Theorem 2.9(i), Theorem 2.10(i)]. *Let $\mathcal{R}$ be a prime ring.*

(i) *If $[x,y]\in Z(\mathcal{R})$ for all $x,y\in\mathcal{R}$, then $\mathcal{R}$ is a commutative ring.*

(ii) *If $\mathcal{R}$ is a 2-torsion free and $x\circ y\in Z(\mathcal{R})$ for all $x,y\in\mathcal{R}$, then $\mathcal{R}$ is a commutative ring.*

Lemma 2.2. [7, Lemma 1.1.9]. *Let $\mathcal{R}$ be a 2-torsion free semiprime ring. If $y\in\mathcal{R}$ and $[[x,y],y]=0$ for all $x\in\mathcal{R}$, then $y\in Z(\mathcal{R})$.*

In [2, Theorem 2], Bell and Daif proved that a prime ring $\mathcal{R}$ must be commutative if it admits a non-identity endomorphism T which is SCP on a nonzero right ideal U of $\mathcal{R}$. Our goal in the following theorem is to generalize this result to the quotient ring center $\mathcal{R}/P$, where P is a prime ideal of the ring $\mathcal{R}$. Specifically, we will prove the following result:

Theorem 2.3. *Let $\mathcal{R}$ be a 2-torsion free prime ring, P a prime ideal of $\mathcal{R}$ and φ an endomorphism of $\mathcal{R}$. Then the following assertions are equivalent:*

(i) $\overline{\varphi([x,y])-[x,y]}\in Z(\mathcal{R}/P)$ *for all $x,y\in\mathcal{R}$.*

(ii) $(\varphi-id_{\mathcal{R}})(\mathcal{R})\subset P$ *or $\mathcal{R}/P$ is commutative.*

Proof. It is clear that $(ii)\Rightarrow(i)$. Thus, it is sufficient to demonstrate the implication $(i)\Rightarrow(ii)$.
$(i)\Rightarrow(ii)$ Suppose that

$$\overline{\varphi([x,y])-[x,y]}\in Z(\mathcal{R}/P) \text{ for all } x,y\in\mathcal{R}. \tag{2.1}$$

If $Z(\mathcal{R}/P)=\{0\}$, then

$$\varphi([x,y])-[x,y]\in P \text{ for all } x,y\in\mathcal{R}. \tag{2.2}$$

Taking xy in place of y in (2.2), we get

$$\begin{aligned}\overline{x[x,y]} &= \overline{[x,xy]}\\ &= \overline{\varphi([x,xy])}\\ &= \overline{\varphi(x)\varphi([x,y])}\\ &= \overline{\varphi(x)[x,y]} \text{ for all } x,y\in\mathcal{R},\end{aligned}$$

which easily gives us

$$(\varphi(x)-x)[x,y]\in P \text{ for all } x,y\in\mathcal{R}. \tag{2.3}$$

Replacing y by yt in (2.3), we obtain $(\varphi(x)-x)\mathcal{R}([x,t])\in P$ for all $x,t\in\mathcal{R}$. Since P is prime, we obtain $\varphi(x)-x\in P$ or $[x,t]\in P$ for all $x,t\in\mathcal{R}$. In this case, we can construct the following two sets:

$$\mathcal{T}=\{x\in\mathcal{R}\mid\varphi(x)-x\in P\} \text{ and } \mathcal{S}=\{x\in\mathcal{R}\mid\overline{x}\in Z(\mathcal{R}/P)\}.$$

It is clear that $\mathcal{T}$ and $\mathcal{S}$ are two subgroups of $\mathcal{R}$, but the union of two subgroups of $\mathcal{R}$ is at most a subgroup unless one is contained in the other. So, we definitely have $\mathcal{R}=\mathcal{T}$ or $\mathcal{R}=\mathcal{S}$. Then, we have $\varphi(x)-x\in P$ for all $x\in\mathcal{R}$ or $\overline{u}\in Z(\mathcal{R}/P)$ for all $u\in\mathcal{R}$, which forces that $(\varphi-id_{\mathcal{R}})(\mathcal{R})\subset P$ or $\mathcal{R}/P$ is commutative.

Now, suppose that $Z(\mathcal{R}/P)\neq\{0\}$ and

$$\overline{\varphi([x,y])-[x,y]}\in Z(\mathcal{R}/P) \text{ for all } x,y\in\mathcal{R}. \tag{2.4}$$

Replacing y by yx in (2.4), then

$$\overline{\varphi([x,y])\varphi(x)-[x,y]x}\in Z(\mathcal{R}/P) \text{ for all } x,y\in\mathcal{R}, \tag{2.5}$$

which implies that

$$\overline{\big(\varphi([x,y])-[x,y]\big)\varphi(x)+[x,y](\varphi(x)-x)}\in Z(\mathcal{R}/P) \text{ for all } x,y\in\mathcal{R}.$$

Using (2.4) and (2.5), we find

$$\Big(\varphi(x)[x,y](\varphi(x)-x)-[x,y](\varphi(x)-x)\varphi(x)\Big)\in P \text{ for all } x,y\in\mathcal{R}.$$

Taking $[u,v]$ in place of x, we get

$$\begin{aligned}&\Big(\varphi([u,v])[[u,v],y](\varphi([u,v])-[u,v])\\ &\quad-[[u,v],y](\varphi([u,v])-[u,v])\varphi([u,v])\Big)\in P\end{aligned}$$

for all $u, v, y \in \mathcal{R}$.
Using our hypothesis, we arrive at

$$\big(\varphi([u,v]) - [u,v]\big)\mathcal{R}(\varphi([u,v])[[u,v],y] - [[u,v],y]\varphi([u,v])) \in P$$

Sine P is prime, we conclude that

$$\varphi([u,v]) - [u,v] \in P \text{ or } \varphi([u,v])[[u,v],y] - [[u,v],y]\varphi([u,v]) \in P \quad (2.6)$$

for all $u, v, y \in \mathcal{R}$.
Suppose there exist $m, n \in \mathcal{R}$ such that

$$\varphi([m,n])[[m,n],y] - [[m,n],y]\varphi([m,n]) \in P \text{ for all } y \in \mathcal{R}. \quad (2.7)$$

Using equation (2.1), we get

$$(\varphi([m,n]) - [m,n])[[m,n],y] - [[m,n],y](\varphi([m,n]) - [m,n]) \in P$$

for all $y \in \mathcal{R}$.
Developing the last expression and invoking (2.7), we can easily get

$$\Big[\big[y,[m,n]\big],[m,n]\Big] \in P \text{ for all } y \in \mathcal{R}. \quad (2.8)$$

Using Lemma 2.2, we conclude that $\overline{[m,n]} \in Z(\mathcal{R}/P)$. By (2.1) we have also $\overline{\varphi([m,n])} \in Z(\mathcal{R}/P)$. Replacing y by $y[m,n]$ in (2.1), we get

$$\overline{(\varphi([x,y]) - [x,y])\varphi([m,n]) + [x,y](\varphi([m,n]) - [m,n])} \in Z(\mathcal{R}/P).$$

Using again (2.1) and the primeness of $\mathcal{R}/P$, we arrive at

$$\overline{[x,y]} \in Z(\mathcal{R}/P)$$

for all $x, y \in \mathcal{R}$ or $\varphi([m,n]) - [m,n] \in P$. Then (2.6) becomes $\overline{[x,y]} \in Z(\mathcal{R}/P)$ for all $x, y \in \mathcal{R}$ or $\varphi([m,n]) - [m,n] \in P$ for all $m, n \in \mathcal{R}$. Using the same proof after equation (2.2) and Lemma 2.1(i), we conclude that $(\varphi - id_{\mathcal{R}})(\mathcal{R}) \subset P$ or $\mathcal{R}/P$ is commutative. □

The following result is a direct consequence of our theorem:

Corollary 2.4. *Let $\mathcal{R}$ be a 2-torsion free prime ring and φ an endomorphism of $\mathcal{R}$. Then the following assertions are equivalent:*
(i) $\varphi([x,y]) - [x,y] \in Z(\mathcal{R})$ *for all* $x, y \in \mathcal{R}$.
(ii) $\varphi = id_{\mathcal{R}}$ *or* $\mathcal{R}$ *is commutative.*

The previous theorem remains valid if we replace the Lie product by the Jordan product, and we get the following result:

Theorem 2.5. *Let $\mathcal{R}$ be a 2-torsion free prime ring, P a prime ideal of $\mathcal{R}$ and φ an endomorphism of $\mathcal{R}$. Then the following assertions are equivalent:*

(i) $\overline{\varphi(x \circ y) - x \circ y} \in Z(\mathcal{R}/P)$ *for all* $x, y \in \mathcal{R}$.

(ii) $(\varphi - id_{\mathcal{R}})(\mathcal{R}) \subset P$ *or* $\mathcal{R}/P$ *is commutative.*

Proof. $(i) \Rightarrow (ii)$ Assume that

$$\overline{\varphi(x \circ y) - x \circ y} \in Z(\mathcal{R}/P) \text{ for all } x, y \in \mathcal{R}. \tag{2.9}$$

If $Z(\mathcal{R}/P) = \{0\}$, then

$$\varphi(x \circ y) - x \circ y \in P \text{ for all } x, y \in \mathcal{R}. \tag{2.10}$$

Taking xy in place of y in (2.10), we get

$$\begin{aligned} \overline{x(x \circ y)} &= \overline{x \circ xy} \\ &= \overline{\varphi(x \circ xy)} \\ &= \overline{\varphi(x)\varphi(x \circ y)} \\ &= \overline{\varphi(x)(x \circ y)} \text{ for all } x, y \in \mathcal{R}, \end{aligned}$$

which implies that

$$(\varphi(x) - x)(x \circ y) \in P \text{ for all } x, y \in \mathcal{R}. \tag{2.11}$$

Replacing y by yt in (2.11), we obtain $(\varphi(x) - x)\mathcal{R}([x, t]) \in P$ for all $x, t \in \mathcal{R}$. Using the same proof after the equation (2.3), we can easily arrive at $(\varphi - id_{\mathcal{R}})(\mathcal{R}) \subset P$ or $\mathcal{R}/P$ is commutative.

Now, suppose that $Z(\mathcal{R}/P) \neq \{0\}$ and

$$\overline{\varphi(x \circ y) - x \circ y} \in Z(\mathcal{R}/P) \text{ for all } x, y \in \mathcal{R}. \tag{2.12}$$

Replacing y by yx in (2.12), then

$$\overline{\varphi(x \circ y)\varphi(x) - (x \circ y)x} \in Z(\mathcal{R}/P) \text{ for all } x, y \in \mathcal{R}, \tag{2.13}$$

equivalently,

$$\overline{\big(\varphi(x \circ y) - (x \circ y)\big)\varphi(x) + (x \circ y)(\varphi(x) - x)} \in Z(\mathcal{R}/P)$$

for all $x, y \in \mathcal{R}$.

Using (2.12), then the last expression implies that

$$\varphi(x)(x \circ y)(\varphi(x) - x) - (x \circ y)(\varphi(x) - x)\varphi(x) \in P \text{ for all } x, y \in \mathcal{R}.$$

Taking $u \circ v$ in place of x, we get

$$\varphi(u \circ v)((u \circ v) \circ y)(\varphi(u \circ v) - u \circ v) - ((u \circ v) \circ y)(\varphi(u \circ v) - u \circ v)\varphi(u \circ v) \in P$$

for all $u, v, y \in \mathcal{R}$. Using our hypothesis, we arrive at

$$\big(\varphi(u\circ v)-u\circ v\big)\mathcal{R}\big(\varphi(u\circ v)((u\circ v)\circ y)-((u\circ v)\circ y)\varphi(u\circ v)\big)\in P$$

Sine P is prime, we conclude that

$$\varphi(u\circ v)-u\circ v\in P \text{ or } \varphi(u\circ v)((u\circ v)\circ y)-((u\circ v)\circ y)\varphi(u\circ v)\in P \quad (2.14)$$

for all $u, v, y \in \mathcal{R}$.
Suppose there exist $m, n \in \mathcal{R}$ such that

$$\varphi(m\circ n)\big((m\circ n)\circ y\big)-\big((m\circ n)\circ y\big)\varphi(m\circ n)\in P \text{ for all } y\in\mathcal{R}. \quad (2.15)$$

Using equation (2.9), we get

$$\begin{aligned}&(\varphi(m\circ n)-m\circ n)((m\circ n)\circ y)\\ &-((m\circ n)\circ y)(\varphi(m\circ n)-m\circ n)\in P\end{aligned}$$

for all $y \in \mathcal{R}$.
Developing the last expression and invoking (2.15), we can easily get

$$\Big[(m\circ n)\circ y, m\circ n\Big]\in P \text{ for all } y\in\mathcal{R}. \quad (2.16)$$

By a simple calculation this equation gives easily $\overline{(m\circ n)^2} \in Z(\mathcal{R}/P)$. Using (2.9), for $x = y$ we can easily arrive at $\overline{\varphi(x^2)-x^2} \in Z(\mathcal{R}/P)$ for all $x \in \mathcal{R}$, then $\overline{\varphi((m\circ n)^2)-(m\circ n)^2} \in Z(\mathcal{R}/P)$, which obviously gives $\overline{\varphi((m\circ n)^2)} \in Z(\mathcal{R}/P)$. Replacing y by $y(m\circ n)^2$ in (2.9), we get

$$\overline{\varphi(x\circ y)\varphi((m\circ n)^2)-(x\circ y)(m\circ n)^2}\in Z(\mathcal{R}/P) \text{ for all } x,y\in\mathcal{R}. \quad (2.17)$$

This can be rewritten as

$$\begin{aligned}&\overline{(\varphi(x\circ y)-x\circ y)\varphi((m\circ n)^2)+(\varphi((m\circ n)^2-(m\circ n)^2)(x\circ y)}\\ &\in Z(\mathcal{R}/P)\end{aligned} \quad (2.18)$$

for all $x, y \in \mathcal{R}$.

Using (2.9) and the fact that $\overline{\varphi((m\circ n)^2)} \in Z(\mathcal{R}/P)$, then the primeness of $\mathcal{R}/P$ forces that $\varphi((m\circ n)^2)-(m\circ n)^2 \in P$ or $\overline{x\circ y} \in Z(\mathcal{R}/P)$ for all $x, y \in \mathcal{R}$. In the second case, we can easily prove the commutativity of $\mathcal{R}/P$ by Lemma 2.1(ii). The first case can be developed as follows

$$(\varphi(m\circ n)-m\circ n)(\varphi(m\circ n)+m\circ n)\in P.$$

Using (2.9), we conclude that $(\varphi(m\circ n)-m\circ n)\mathcal{R}(\varphi(m\circ n)+m\circ n)\in P$. From the primeness of $\mathcal{R}/P$, we find $\varphi(m\circ n)-m\circ n \in P$ or $\varphi(m\circ n)+m\circ n \in P$. Using (2.9) with the 2-torsion freeness of $\mathcal{R}$, we conclude that $\varphi(m \circ n) - m \circ n \in P$ or $m \circ n \in P$. The second case implies that $\varphi(m \circ n) \in P$, and replacing y by $y(m \circ n)$ in (2.9), we find that

$$\overline{(\varphi(x \circ y) - x \circ y)\varphi(m \circ n) - (x \circ y)(\varphi(m \circ n) - m \circ n)} \in Z(\mathcal{R}/P)$$

for all $x, y \in \mathcal{R}$.

Using (2.9) together with the primeness of $\mathcal{R}/P$, we get $\overline{x \circ y} \in Z(\mathcal{R}/P)$ for all $x, y \in \mathcal{R}$ or $\varphi(m \circ n) - m \circ n \in P$. So in any case, we have either $\varphi(u \circ v) - u \circ v \in P$ for all $u, v \in \mathcal{R}$ or $\overline{x \circ y} \in Z(\mathcal{R}/P)$ for all $x, y \in \mathcal{R}$. Using the same proof after equation (2.10) and Lemma 2.1(ii), we conclude that $(\varphi - id_{\mathcal{R}})(\mathcal{R}) \subset P$ or $\mathcal{R}/P$ is commutative. □

The following result is a direct consequence of our theorem.

Corollary 2.6. *Let $\mathcal{R}$ be a 2-torsion free prime ring, φ an endomorphism of $\mathcal{R}$. Then the following assertions are equivalent:*

(i) *$\varphi(x \circ y) - x \circ y \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$.*
(ii) *$\varphi = id_{\mathcal{R}}$ or $\mathcal{R}$ is commutative.*

The following example proves that we cannot remove the condition "P is prime" of Theorem 2.3 and Theorem 2.5.

Example 2.7. Let $\mathcal{R} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$, $P = \{0\}$ and $\varphi : \mathcal{R} \longrightarrow \mathcal{R}$ defined by: $\varphi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 2a + b - 2c \\ 0 & c \end{pmatrix}$. It is clear that $\mathcal{R}$ is a 2-torsion free ring, which is not prime. φ is an endomorphism of $\mathcal{R}$, which verifies the hypotheses of Theorem 2.3 and Theorem 2.5. But $\mathcal{R}$ is not commutative. Therefore, the hypothesis that P is a prime ideal imposed in our theorems and corollaries is crucial.

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