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$w-$FILTERS OF ALMOST DISTRIBUTIVE LATTICES

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Abstract. The notion of $w-$filters is introduced in an Almost Distributive Lattice (ADL) and properties are investigated. A necessary and sufficient condition is derived for a maximal filter of an ADL to become a $w-$filter which leads to a characterization of a quasi-complemented ADL. Also, $w-$filters of an ADL are characterized in terms of minimal prime $D-$filters.

1. Introduction

The notion of an Almost Distributive Lattice(ADL) was first introduced by Swamy U.M. and Rao G.C. in their work [10]. This novel concept serves as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. The ADL framework introduced in the paper introduces the notion of an ideal within an ADL, analogous to the concept in distributive lattices. Notably, the authors established that the collection of principal ideals within an ADL constitutes a distributive lattice structure. This provided a path for the expansion of various lattice theory concepts and notions to the class of ADLs. In [5], the concept of quasi-complemented Almost Distributive Lattice was introduced and certain properties of quasi-complemented ADLs were derived. In [3], the concept of $D-$filters is introduced recently by Rafi, et.al. and studied their properties. The notion of $\omega-$filters in lattices was introduced and studied their properties in [8] by Sambasiva Rao, et.al. In the study this paper, a concept called "$w-$filters" is introduced in Almost Distributive Lattices(ADLs) and their properties are investigated. For every maximal filter in an ADL to become a $w-$filter, a set of equivalent conditions must be satisfied. These conditions are established and help to characterize a quasi-complemented ADL. In addition to characterizing w-filters, sufficient conditions are derived to identify when a proper $D-$filter of an ADL becomes

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a $w-$filter. Finally, the $w-$filters of an ADL can be characterized using minimal prime $D-$filters.

2. Preliminaries

In this section, we recall certain definitions and important results from [4] and [10], those will be required in the text of the paper.

Definition 2.1. [10] An algebra $R = (R, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL), if it satisfies the following conditions:

(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(3) $(a \vee b) \wedge b = b$
(4) $(a \vee b) \wedge a = a$
(5) $a \vee (a \wedge b) = a$
(6) $0 \wedge a = 0$
(7) $a \vee 0 = a$, for all $a, b, c \in R$.

Example 2.2. Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL.

If $(R, \vee, \wedge, 0)$ is an ADL, for any $a, b \in R$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on R.

Theorem 2.3. [10] *If $(R, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in R$, we have the following:*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$
(2) $a \vee b = b \Leftrightarrow a \wedge b = a$
(3) $\wedge$ *is associative in* R
(4) $a \wedge b \wedge c = b \wedge a \wedge c$
(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(6) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(7) $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$
(8) $a \wedge a = a$ *and* $a \vee a = a$.

It can be observed that an ADL R satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL R a distributive lattice.

As usual, an element $m \in R$ is called maximal if it is a maximal element in the partially ordered set $(R, \leq)$. That is, for any $a \in R$, $m \leq a \Rightarrow m = a$.

As in distributive lattices [1, 2], a non-empty subset I of an ADL R is called an ideal of R if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in R$. Also, a non-empty subset F of R is said to be a filter of R if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in R$.

The set $\mathfrak{I}(R)$ of all ideals of R is a bounded distributive lattice with least element $\{0\}$ and greatest element R under set inclusion in which, for any $I, J \in \mathfrak{I}(R)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal(filter) P of R is called a prime ideal(filter) if, for any $x, y \in R$, $x \wedge y \in P(x \vee y \in P) \Rightarrow x \in P$ or $y \in P$. A proper ideal(filter) M of R is said to be maximal if it is not properly contained in any proper ideal(filter) of R. It can be observed that every maximal ideal(filter) of R is a prime ideal(filter). Every proper ideal(filter) of R is contained in a maximal ideal(filter). For any subset S of R the smallest ideal containing S is given by

$$(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in R \text{ and } n \in \mathbb{N}\}.$$

If $S = \{s\}$, we write $(s]$ instead of $(S]$ and such an ideal is called the principal ideal of R. Similarly, for any $S \subseteq R$,

$$[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in R \text{ and } n \in \mathbb{N}\}.$$

If $S = \{s\}$, we write $[s)$ instead of $[S)$ and such a filter is called the principal filter of R.

For any $a, b \in R$, it can be verified that $(a] \vee (b] = (a \vee b]$ and $(a] \cap (b] = (a \wedge b]$. Hence the set $(\mathfrak{I}^{PI}(R), \vee, \cap)$ of all principal ideals of R is a sublattice of the distributive lattice $(\mathfrak{I}(R), \vee, \cap)$ of all ideals of R. Also, we have the set $(\mathfrak{F}(R), \vee, \cap)$ of all filters of R is a bounded distributive lattice.

Theorem 2.4. [6] *Let R be an ADL with maximal elements. Then P is a prime ideal of R if and only if $R \setminus P$ is a prime filter of R.*

It is known that, for any $x, y \in R$ with $x \leq y$, the interval $[x,\ y]$ is a bounded distributive lattice. Now, an ADL R is said to be relatively complemented if, for any $x, y \in R$ with $x \leq y$, the interval $[x,\ y]$ is a complemented distributive lattice.

Theorem 2.5. [9] *An ADL R with maximal elements is relatively complemented if and only if $B(R) = R$, where*

$$B(R) = \{x \in R | \textit{there exists } y \in R \textit{ such that } x \wedge y = 0 \\ \textit{and } x \vee y \textit{ is maximal}\}.$$

Definition 2.6. [7] For any nonempty subset A of an ADL R, define $A^* = \{x \in R | a \wedge x = 0 \text{ for all } a \in A\}$. Here A^* is called the annihilator of A in R.

For any $a \in R$, we have $\{a\}^* = (a]^*$, where $(a]$ is the principal ideal generated by a. An element a of an ADL R is called dense element if $(a]^* = \{0\}$ and the set D of all dense elements in ADL is a filter if D is non-empty.

Definition 2.7. [5] An ADL R is said to be quasi-complemented if to each $a \in R$, there exists an element $b \in R$ such that $a \wedge b = 0$ and $a \vee b \in D$.

Definition 2.8. [3] A filter G of R is said to be a $D-$filter of R if $D \subseteq G$. An $D-$filter Q is said to be proper if $Q \subsetneq R$. A proper $D-$filter Q is said to be maximal if it is not properly contained in any proper $D-$filter of R. A proper $D-$filter Q of an ADL R is said to be a prime $D-$filter if Q is prime filter of R.

Definition 2.9. [3] A prime $D-$filter M of an ADL R containing a $D-$filter G is said to be a minimal prime $D-$filter belonging to G if there exists no prime $D-$filter N such that $G \subseteq N \subseteq M$.

Note that if we take $D = G$ in the above definition then we say that M is a minimal prime $D-$filter.

Definition 2.10. [3] For any nonempty subset S of R, define

$$(S, D) = \{a \in R \mid s \vee a \in D,\ for\ all\ s \in S\}.$$

We call this set as relative annihilator of S with respect to the filter D.

For $S = \{s\}$, we denote $(\{s\}, D)$ by (s, D).

Theorem 2.11. [3] *For any $x, y \in R$ we have the following:*
(1) $([x), D) = (x, D)$,
(2) $x \leq y \Rightarrow (x, D) \subseteq (y, D)$,
(3) $(x \wedge y, D) = (x, D) \cap (y, D)$,

(4) $((x \vee y, D), D) = ((x, D), D) \cap ((y, D), D)$,
(5) $(x, D) = R \Leftrightarrow x \in D$.

3. $w-$filters of an ADL

In this section, the concept of $w-$filters is introduced in an ADL and their properties are investigated. For every maximal filter in an ADL to become a $w-$filter, a set of equivalent conditions must be satisfied. These conditions are established and help to characterize a quasi-complemented ADL. In addition to characterizing w-filters, sufficient conditions are derived to identify when a proper $D-$filter of an ADL becomes a $w-$filter. Finally, the $w-$filters of an ADL can be characterized using minimal prime $D-$filters.

Proposition 3.1. *For any prime filter M of a quasi-complemented ADL R with maximal element m, the following are equivalent:*
(1) $D \subseteq M$,
(2) *for any* $a \in R,\ a \in M$ *if and only if* $(a, D) \nsubseteq M$,
(3) *for any* $a, b \in R$ *with* $(a, D) = (b, D), a \in M$ *implies that* $b \in M$,
(4) $D \cap (R \setminus M) = \emptyset$.

Proof. (1) $\Rightarrow$ (2): Suppose $a \in M$. Since R is quasi-complemented, there exists $b \in R$ such that $a \wedge b = 0$ and $a \vee b \in D$. Then $b \in (a, D)$. Clearly, we have $b \notin M$ and hence $(a, D) \nsubseteq M$. Conversely, assume that $(a, D) \nsubseteq M$. Then there exists $b \in R$ such that $b \in (a, D)$ and $b \notin M$. Clearly, $a \vee b \in D \subseteq M$. Since M is prime and $b \notin M$, we get $a \in M$.

(2) $\Rightarrow$ (3): Let $a, b \in R$ with $(a, D) = (b, D)$. Suppose $a \in M$. By our assumption, we get $(a, D) \nsubseteq M$ and hence $(b, D) \nsubseteq M$. Therefore $b \in M$.

(3) $\Rightarrow$ (4): Let $a \in R$. If $a \in D \cap (R \setminus M)$. Then $(a, D) = R$ and $a \notin M$. Therefore $(a, D) = R = (m, D)$. Since $m \in M$, by our assumption, we get $a \in M$, which is a contradiction. Hence $D \cap (R \setminus P) = \emptyset$.

(4) $\Rightarrow$ (1): It is clear. $\square$

Theorem 3.2. *Let a' be the quasi-complement of a in an ADL R. Then every prime $D-$filter contain exactly one of a or a'.*

Proof. Since a' be the quasi-complement of a, we have $a \wedge a' = 0$ and $a \vee a' \in D$. Let M be a prime $D-$filter of L. Clearly, $a \vee a' \in D \subseteq M$. Since M is prime, we get $a \in M$ or $a' \in M$. Suppose $a \in M$ and $a' \in M$. Then $0 = a \wedge a' \in M$, which is a contradiction. Hence M must contain exactly one of a or a'. $\square$

Proposition 3.3. *Let R be a quasi-complemented ADL. Then the following conditions are equivalent:*

(1) *R is a relatively complemented ADL,*
(2) *every prime filter contains exactly one of a or a', where a' is the quasi-complement of a in R,*
(3) *every prime filter is a $D-$filter,*
(4) *every minimal prime filter is a $D-$filter.*

Proof. (1) $\Rightarrow$ (2): Let M be a prime filter of R and $a \in M$. By our assumption, there exists an element $a' \in R$ such that $a \wedge a' = 0$ and $a \vee a'$ is a maximal element. Since $a \vee a'$ is maximal, we get $a \vee a' \in M$. Since M is prime, we get $a \in M$ or $a' \in M$. Since $a \wedge a' = 0$, we get M contain exactly one of a or a'.

(2) $\Rightarrow$ (3): Let M be a prime filter of R. Let $a \in D$. Since R is quasi-complemented, we get $a' \in (a)^* = \{0\}$. Hence $a' = 0 \notin M$. By the condition (2), we get $a \in M$. Thus $D \subseteq M$. Therefore M is a $D-$filter of R.

(3) $\Rightarrow$ (4): It is clear.

(4) $\Rightarrow$ (1): Let $a \in R$. Suppose $a \vee a'$ is not a maximal element of R. Then there exists a maximal ideal M of R such that $a \vee a' \in M$. Clearly, $R \setminus M$ is a minimal prime filter such that $a \vee a' \notin R \setminus M$. By our assumption, we get $R \setminus M$ is $D-$filter and $a \vee a' \in R \setminus M$. It gives $R \setminus M$ must contains exactly one of a or a', which is a contradiction. Therefore $a \vee a'$ is a maximal element and hence R is a relatively complemented ADL. $\square$

Theorem 3.4. *For any proper filter M of a quasi-complemented ADL R, M is maximal if and only if M is a prime $D-$filter.*

Proof. Let M be any proper filter of R. Assume that M is a maximal filter of R. Clearly, M is prime. Let $a \in D$. Then $(a)^* = \{0\}$. Suppose $a \notin M$. Then $M \vee [a) = R$. There exist $s \in M$ and $t \in [a)$ such that $0 = s \wedge t$. Therefore $s \wedge a = 0$, which gives $s \in (a)^*$. Since $(a)^* = \{0\}$, we get $s = 0$. Therefore $0 \in M$, which leads $M = R$, which is a contradiction. Hence $a \in M$. Thus $D \subseteq M$. Therefore M is a prime $D-$filter of R. Conversely, assume that M is a prime $D-$filter of R. Suppose M is not maximal. Then there exists a proper filter N of R such that $M \subsetneq N$. Choose $a \in N \setminus M$. Since R is quasi-complemented, there exists $a' \in R$ such that $a \wedge a' = 0$ and $a \vee a' \in D \subseteq M$. Since M is prime and $a \notin M$, we get $a' \in M \subset N$. Then $0 = a \wedge a' \in N$, which is a contradiction. Therefore M is maximal. $\square$

In a quasi-complemented Almost Distributive Lattice (ADL), the class of all maximal filters and the class of all prime D-filters of R are equivalent. Additionally, since every prime D-filter is maximal, we can deduce that every prime D-filter is minimal in a quasi-complemented ADL. Consequently, the sets of maximal filters, prime D-filters, and minimal prime D-filters are all identical in a quasi-complemented ADL.

Definition 3.5. For any ideal I of an ADL R, define

$$w(I) = \{a \in R \mid a \vee s \in D, \text{ for some } s \in I\}.$$

Clearly, $w(I) = \bigcup_{a \in I} (a, D)$.

Proposition 3.6. *Let I be an ideal of an ADL R. Then $w(I)$ is a $D-$filter of R.*

Proof. Clearly, $D \subseteq w(I)$. Let $a, b \in w(I)$. Then there exist $s, t \in I$ such that $a \vee s \in D$ and $b \vee t \in D$. Since D is a filter of R, we get $s \vee t \vee a \in D$ and $s \vee t \vee b \in D$. Then $(s \vee t \vee a) \wedge (s \vee t \vee b) \in D$ and hence $(s \vee t) \vee (a \wedge b) \in D$. Therefore $(a \wedge b) \vee (s \vee t) \in D$. Since $s, t \in I$, we get $s \vee t \in I$ and hence $a \wedge b \in w(I)$. Let $a \in w(I)$. Then there exists $s \in I$ such that $a \vee s \in D$. Let $r \in R$. Since D is a filter of R, we get $(r \vee a) \vee s \in D$ and hence $r \vee a \in w(I)$. Therefore $w(I)$ is a $D-$filter of R. □

Lemma 3.7. *Let I, J be two ideals of an ADL R. Then we have the following:*
(1) $I \cap w(I) \neq \emptyset \Leftrightarrow w(I) = R$,
(2) $I \subseteq J \Rightarrow w(I) \subseteq w(J)$,
(3) $w(I) \cap w(J) = w(I \cap J)$.

Proof. (1). Assume that $I \cap w(I) \neq \emptyset$. Then choose an element $a \in I \cap w(I)$. Then $a \in I$ and $a \in w(I)$. Since $a \in w(I)$, there exists $s \in I$ such that $a \vee s \in D$. By Theorem 2.11(5), we get $(a \vee s, D) = R$. Since $a \in I$ and $s \in I$, we get $a \vee s \in I$. Hence $w(I) = \bigcup_{a \in I} (a, D) = R$. Conversely, assume that $w(I) = R$. Then $0 \in w(I)$ and hence $0 \in I \cap w(I)$. Thus $I \cap w(I) \neq \emptyset$.

(2). Assume $I \subseteq J$. Let $a \in w(I)$. Then there exists $s \in I$ such that $a \vee s \in D$. Since $I \subseteq J$, we get $s \in J$ and hence $a \in w(J)$. Thus $w(I) \subseteq w(J)$.

(3). Clearly, $w(I \cap J) \subseteq w(I) \cap w(J)$. Let $a \in w(I) \cap w(J)$. Then there exist $s \in I$ and $t \in J$ such that $a \vee s \in D$ and $a \vee t \in D$. Since $s \in I$ and $t \in J$, we get $s \wedge t \in I \cap J$ and hence $a \vee (s \wedge t) = (a \vee s) \wedge (a \vee t) \in D$. Therefore $a \in w(I \cap J)$. Hence $w(I) \cap w(J) \subseteq w(I \cap J)$. □

Proposition 3.8. *If I, J are two ideals of an ADL R with $w(I) \cap J = \emptyset$, then there exists a prime $D-$filter M such that $w(I) \subseteq M$ and $M \cap J = \emptyset$.*

Proof. Let I and J be two ideals of an ADL R such that $w(I) \cap J = \emptyset$. Then there exists a prime ideal P such that $J \subseteq P$ and $w(I) \cap P = \emptyset$. Since

$w(I) \cap P = \emptyset$, we get $D \subseteq w(I) \subseteq R \setminus P$. Since $R \setminus P$ is a prime filter of R, we get $R \setminus P = M$ is a prime $D-$filter of R containing $w(I)$. □

The definition of $w-$filter in an ADL is now as follows.

Definition 3.9. A $D-$filter G of an ADL R is said to be a $w-$filter if $G = w(I)$, for some ideal I of R such that $I \cap D = \emptyset$.

From the above definition, it is easy to verify that $w(\{0\}) = D$. Hence D is proper and the smallest $w-$filter of R.

Example 3.10. Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and define $\vee,\ \wedge$ on R as follows:

$\wedge$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	1	2	3	4	5	6	7
3	0	3	3	3	0	0	3	0
4	0	4	5	0	4	5	7	7
5	0	4	5	0	4	5	7	7
6	0	6	6	3	7	7	6	7
7	0	7	7	0	7	7	7	7

$\vee$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	1	2	3	1	2	6	6
4	4	1	1	1	4	4	1	4
5	5	2	2	2	5	5	2	5
6	6	1	2	6	1	2	6	6
7	7	1	2	6	4	5	6	7

Then $(R, \vee, \wedge)$ is an ADL. Clearly, $D = \{1, 2, 6\}$ and $G = \{1, 2, 3, 6\}$ are filters of R satisfying $D \subseteq G$. Therefore G is a $D-$filter of R. Consider an ideal $I = \{0, 7\}$. Then clearly, $w(I) = \{1, 2, 3, 6\} = G$. Hence G is a $w-$filer of R.

Proposition 3.11. *For any $a \notin D$ in an ADL R. we have (a, D) is a $w-$filter of R.*

Proof. Let $a \notin D$. Clearly, we have $(a] \cap D = \emptyset$. Let $s \in (a, D)$. Then $s \vee a \in D$. Since $a \in (a]$, we get $s \in w((a])$ and hence $(a, D) \subseteq w((a])$. Let $s \in w((a])$. Then there exists $b \in (a]$ such that $s \vee b \in D$ and hence $s \vee a \in D$. It follows $s \in (a, D)$. Therefore $w((a]) \subseteq (a, D)$ and hence $(a, D) = w((a])$. Thus (a, D) is a $w-$filter of R. □

Theorem 3.12. *Let M be a prime $D-$filter of an ADL R with $(M, D) \neq D$. Then M is a $w-$filter.*

Proof. Assume that $(M, D) \neq D$. Since $D \subseteq (M, D)$, we get $(M, D) \nsubseteq D$. Then there exists $a \in (M, D)$ such that $a \notin D$. Clearly, $(a] \cap D = \emptyset$ and $a \notin M$.

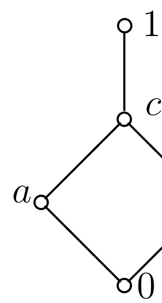
Then $M \subseteq ((M,D),D) \subseteq (a,D)$. Therefore $M \subseteq (a,D)$. Let $s \in (a,D)$. Then $s \vee a \in D \subseteq M$. Since $a \notin M$, we have $s \in M$. Then $(a,D) \subseteq M$. Therefore $M = (a,D) = w((a])$ and hence M is a $w-$filter of R. □

Theorem 3.13. *Every minimal prime $D-$filter of an ADL R is a $w-$filter.*

Proof. Let M be a minimal prime $D-$filter of R. Then $R \setminus M$ is a prime ideal of R such that $D \cap (R \setminus M) = \emptyset$. Now prove that $M = w(R \setminus M)$. Let $a \in M$. Since M is minimal, there exists $b \in R \setminus M$ such that $a \vee b \in D$. Hence $a \in w(R \setminus M)$. Therefore $M \subseteq w(R \setminus M)$. Let $a \in w(R \setminus M)$. Then there exists $s \in R \setminus M$ such that $a \vee s \in D \subseteq M$. Since M is prime and $s \notin M$, we get $a \in M$. Therefore $w(R \setminus M) \subseteq M$ and hence $M = w(R \setminus M)$. Thus M is a $w-$filter of R. □

We now turn our intension towards the converse of the above theorem. In general, every $w-$filter of an ADL need not be a minimal prime $D-$filter. In fact it need not even be a prime $D-$filter. It can be observed in the following example:

Example 3.14. Consider a distributive lattice $L = \{0, a, b, c, 1\}$ and discrete ADL $A = \{0', a'\}$.



Clearly,

$$R = A \times L = \{(0',0),(0',a),(0',b),(0',c),(0',1),(a',0),\\ (a',a),(a',b),(a',c),(a',1)\}$$

is an ADL with zero element $(0,0')$. Clearly, $D = \{(a',c),(a',1)\}$ is a dense set. Consider a $D-$filter

$$G = \{(a',c),(a',1),(a',b)\}$$

and an ideal $I = \{(0',0),(0',a)\}$. Clearly, we have $w(I) = G$ and hence G is a $w-$filter. But G is not prime, because $(a',a) \vee (0',b) = (a',c) \in G$, but $(a',a) \notin G$ and $(0',b) \notin G$.

Though every $w-$filter need not be a prime $D-$filter, we derive a necessary and sufficient condition for a $w-$filter of an ADL to become a prime $D-$filter.

Theorem 3.15. *A proper $w-$filter G of an ADL R is a prime D-filter if and only if G contains a prime $D-$filter.*

Proof. Let G be a proper $w-$filter of R. Assume that G is a prime $D-$filter of R. Clearly, G contains a prime $D-$filter G. Conversely, assume that G contains a prime $D-$filter, say M. Since $D \subseteq M \subseteq G, G$ is a $D-$filter of R. Since G is a $w-$filter, we get $G = w(I)$, for some ideal I of R with $I \cap D = \emptyset$. Let $s, t \in R$ such that $s \notin G$ and $t \notin G$. Since $M \subseteq G$, we get $s \notin M$ and $t \notin M$. Since M is prime, we get $s \vee t \notin M$. Therefore $(s \vee t, D) \subseteq M \subseteq G = w(I)$. Suppose $s \vee t \in G = w(I)$. Then there exists $x \in I$ such that $s \vee t \vee x \in D$. It follows $x \in (s \vee t, D) \subseteq w(I)$. Therefore $x \in I \cap w(I)$ and hence $I \cap w(I) \neq \emptyset$. By Lemma 3.7(1), $G = w(I) = R$, which is a contradiction. Hence $s \vee t \notin G$. Thus G is a prime $D-$filter of R. □

It is observed that every minimal prime $D-$filter is a prime $w-$filter of R. Now we established the equivalency between prime $w-$filters and minimal prime $D-$filters of an ADL.

Theorem 3.16. *Every prime $w-$filter of an ADL R is a minimal prime $D-$filter.*

Proof. Let M be a prime $w-$filter of R. Then $M = w(I)$, for some ideal I of R with $I \cap D = \emptyset$. Let $a \in M = w(I)$. Then there exists $b \in I$ such that $a \vee b \in D$. Suppose $b \in M$. Then $b \in I \cap w(I)$. That implies $I \cap w(I) \neq \emptyset$. By Lemma 3.7(1), $M = w(I) = R$ which is a contradiction. Therefore $b \notin M$ and hence M is a minimal prime $D-$filter. □

Theorem 3.17. *In an ADL R, the following are equivalent:*
(1) *R is quasi-complemented,*
(2) *every prime $D-$filter is a $w-$filter,*
(3) *every prime $D-$filter is minimal,*
(4) *every maximal filter is a minimal prime $D-$filter,*
(5) *every maximal filter is a $w-$filter.*

Proof. (1) $\Rightarrow$ (2): Let M be a prime $D-$filter of R. Then $R \setminus M$ is a prime ideal of R such that $(R \setminus M) \cap D = \emptyset$. Now prove that $M = w(R \setminus M)$. Let $a \in M$. Since R is quasi-complemented, there exists $b \in R$ such that $a \wedge b = 0$ and $a \vee b \in D$. Clearly, $b \notin M$, which gives that $b \in R \setminus M$. Since $a \vee b \in D$, we get $a \in w(R \setminus M)$. Therefore $M \subseteq w(R \setminus M)$. Let $a \in w(R \setminus M)$. Then there exists $b \in R \setminus M$ such that $a \vee b \in D$. Since $a \vee b \in D \subseteq M$ and $b \notin M$, we get $a \in M$. Therefore $w(R \setminus M) \subseteq M$. Hence M is a $w-$filter of R.

(2) $\Rightarrow$ (3): Let M be a prime $D-$filter of R. By our assumption, P is a prime $w-$filter. By Theorem 3.16, we have P is minimal.

(3) $\Rightarrow$ (4): It is clear.

(4) $\Rightarrow$ (5): It is clear.

(5) $\Rightarrow$ (1): Let $a \in R$. Suppose $0 \notin [a) \vee (a, D)$. Then there exists a maximal filter M such that $[a) \vee (a, D) \subseteq M$. Therefore $a \in M$ and $(a, D) \subseteq M$. By the assumption, M is a $w-$filter. Since M is prime, by Theorem 3.16, M is minimal prime $D-$filter. Then $a \notin M$, which is a contradiction. Hence $0 \in [a) \vee (a, D)$. There exists $s \in (a, D)$ such that $a \wedge s = 0$. Since $s \in (a, D)$, we get $s \vee a \in D$. Thus R is quasi-complemented. ☐

We conclude this paper with a characterization theorem of $w-$filters in terms of minimal prime $D-$filters. For this, we first need the following results.

Lemma 3.18. *Let I be an ideal of an ADL R such that $I \cap D = \emptyset$. If M is a minimal prime $D-$filter containing $w(I)$, then $I \cap M = \emptyset$.*

Proof. Let M be a minimal prime $D-$filter of R with $w(I) \subseteq M$. Suppose $a \in I \cap M$. Then $a \in M$ and $a \in I$. Since M is minimal and $a \in M$, there exists $b \notin M$ such that $a \vee b \in w(I)$. Then there exists $x \in I$ such that $(a \vee b) \vee x \in D$. Hence $b \vee (a \vee x) \in D$ and $a \vee x \in I$. Therefore $b \in w(I) \subseteq M$, which is a contradiction. Thus $I \cap M = \emptyset$. ☐

Lemma 3.19. *Every minimal prime $D-$filter of an ADL R containing a $w-$filter is a minimal prime $D-$filter in R.*

Proof. Let G be a $w-$filter of R. Then $G = w(I)$, for some ideal I of R such that $I \cap D = \emptyset$. Let M be a minimal prime $D-$filter containing $G = w(I)$. By the above lemma, $I \cap M = \emptyset$. Let $a \in M$. Then there exists $b \notin M$ such that $a \vee b \in w(I)$. There exists $x \in I$ such that $(a \vee b) \vee x \in D$. Therefore $a \vee (b \vee x) \in D \subseteq M$ and $b \vee x \notin M$. Thus M is a minimal prime $D-$filter of R. ☐

Now, $w-$filters are characterized in terms of minimal prime $D-$filters.

Theorem 3.20. *Every $w-$filter of an ADL R is the intersection of all minimal prime $D-$filters containing it.*

Proof. Let G be a $w-$filter of R. Then $G = w(I)$, for some ideal I of R such that $I \cap D = \emptyset$. Let $H = \bigcap\{M | M \text{ is a minimal prime } D-\text{filter containing } G\}$. Clearly, $G \subseteq H$. Let $x \notin G = w(I)$. Then $x \vee s \notin D$, for all $s \in I$. Then there exists a minimal prime $D-$filter M such that $x \vee s \notin M$. It follows $x \notin M$ and $s \notin M$. Since M is prime, $(s, D) \subseteq M$, for all $s \in I$. Then $G = w(I) \subseteq M$.

Hence M is minimal such that $G \subseteq M$ and $x \notin M$. Therefore $x \notin H$, which leads $H \subseteq G$. Thus $G = H$. □

Theorem 3.21. *Let $\{G_\alpha\}_{\alpha\in\triangle}$ be a class of $w-$filters of an ADL R. Then $\bigcap\limits_{\alpha\in\triangle} G_\alpha$ is a $w-$filter of R.*

Proof. For each $\alpha \in \triangle$, let $G_\alpha = w(I_\alpha)$ where I_α is an ideal of R such that $I_\alpha \cap D = \emptyset$. Then $\{I_\alpha\}_{\alpha\in\triangle}$ will be an arbitrary family of ideals in R such that $I_\alpha \cap D = \emptyset$ for each $\alpha \in \triangle$. Hence $\bigcap\limits_{\alpha\in\triangle} I_\alpha$ is an ideal of R such that $\Big(\bigcap\limits_{\alpha\in\triangle} I_\alpha\Big) \cap D = \emptyset$. By Lemma 3.7(3), we get $\bigcap\limits_{\alpha\in\triangle} w(I_\alpha) = w\Big(\bigcap\limits_{\alpha\in\triangle} I_\alpha\Big)$. Therefore $\bigcap\limits_{\alpha\in\triangle} G_\alpha$ is a $w-$filter of R. □

Note that the class of all $w-$filters of an ADL is closed under set-intersection. In general, $w-$filters need not be closed under finite joins. However, in the following, we prove that the class $\mathfrak{F}_w(R)$ of all $w-$filters of an ADL R forms a complete lattice.

Theorem 3.22. *Let I, J be two ideals of an ADL R such that $I \cap D = J \cap D = \emptyset$. Then $w(I \vee J)$ is the smallest $w-$filter containing both $w(I)$ and $w(J)$.*

Proof. Let I, J be two ideals of R such that $I \cap D = J \cap D = \emptyset$. Clearly, $(I \vee J) \cap D = \emptyset$. By Lemma 3.7(2), we get $w(I) \subseteq w(I \vee J)$ and $w(J) \subseteq w(I \vee J)$. Suppose $w(I) \subseteq w(K)$ and $w(J) \subseteq w(K)$, for some ideal K of R with $K \cap D = \emptyset$. Let $a \in w(I \vee J)$. Then there exist $s \in I$ and $t \in J$ such that $a \vee (s \vee t) \in D$. Therefore $a \vee s \in w(J) \subseteq w(K)$. There exists $x \in K$ such that $a \vee s \vee x \in D$. Since $x \vee y \in K$, we get $a \in w(K)$. Therefore $w(I \vee J)$ is the supremum of $w(I)$ and $w(J)$. Consider this supremum by $w(I) \sqcup w(J)$. Thus $(\mathfrak{F}_w(R), \cap, \sqcup)$ forms a lattice. □

Corollary 3.23. *Let $\{w(I_\alpha)\}_{\alpha\in\triangle}$ be a class of $w-$filters of an ADL R where $I_\alpha \cap D = \emptyset$ for each $\alpha \in \triangle$. Then $\bigsqcup\limits_{\alpha\in\triangle} w(I_\alpha)$ is the smallest $w-$filter containing each $w(I_\alpha)$.*

It can be easily observed that the class of all $w-$filters of an ADL forms a complete lattice with respect to set inclusion $\subseteq$, in which for any $\{w(I_\alpha)\}_{\alpha\in\triangle}$ of $w-$filters, $inf\{w(I_\alpha)\}_{\alpha\in\triangle} = w(\bigcap\limits_{\alpha\in\triangle} I_\alpha)$ and

$$sup\{w(I_\alpha)\}_{\alpha\in\triangle} = w\Big(\bigvee_{\alpha\in\triangle} I_\alpha\Big).$$

Since the class of all ideals of an ADL forms a complete distributive lattice, the class $\mathfrak{F}_w(R)$ of all $w-$filters of an ADL R forms a complete distributive lattice. In general, the class $\mathfrak{F}_w(R)$ of all $w-$filters of an ADL R is not a sublattice of the filter lattice $\mathfrak{F}(R)$. However, in the following, we derive a set of equivalent conditions for $\mathfrak{F}_w(R)$ to become a sublattice of $\mathfrak{F}(R)$. For this, we first need the following result.

Lemma 3.24. *Every proper $w-$filter is contained in a minimal prime $D-$filter.*

Proof. Let G be a proper $w-$filter of R. Then $G = w(I)$ for some ideal I of R with $I\cap D = \emptyset$. Hence $D \subseteq w(I) = G$. Clearly, $G\cap I = w(I)\cap I = \emptyset$. Consider, the set $\mathfrak{F} = \{H \mid H$ is an ideal of R such that $I \subseteq H$ and $G\cap H = \emptyset\}$. Clearly $I \in \mathfrak{F}$ and $\mathfrak{F}$ satisfies the Zorn's lemma. Let N be a maximal element of $\mathfrak{F}$. Then N is an ideal of R such that $I \subseteq N$ and $G \cap N = \emptyset$. Since $D \subseteq G$, we get $D \cap N = \emptyset$. Therefore N is an ideal which is maximal with respect to the property that $D \cap N = \emptyset$. Hence $R \setminus N$ is a minimal prime $D-$filter such that $G \subseteq R \setminus N$. □

Theorem 3.25. *In an ADL R, the following are equivalent:*
(1) $\mathfrak{F}_w(R)$ *is a sublattice of* $\mathfrak{F}(R)$,
(2) *for* $x, y \in R, x \vee y \in D$ *implies* $(x, D) \vee (y, D) = R$,
(3) *for* $x, y \in R, (x, D) \vee (y, D) = (x \vee y, D)$,
(4) *for* $I, J \in \mathfrak{I}(R), I \vee J = R$ *implies* $w(I) \vee w(J) = R$,
(5) *for* $I, J \in \mathfrak{I}(R), w(I) \vee w(J) = w(I \vee J)$.

Proof. (1) ⇒ (2): Let $x, y \in R$ with $x \vee y \in D$. Suppose

$$(x, D) \vee (y, D) \neq R.$$

Since (x, D) and (y, D) are $w-$filters of R, by hypothesis, we get $(x, D) \vee (y, D)$ is a proper $w-$filter of R. By Lemma 3.24, there exists a minimal prime $D-$filter M such that $(x, D) \vee (y, D) \subseteq M$. Hence $(x, D) \subseteq M$ and $(y, D) \subseteq M$. Since M is a minimal prime $D-$filter, we get $x \notin M$ and $y \notin M$. Since M is a prime filter, we get $x \vee y \notin M$, which is a contradiction to that $x \vee y \in D \subseteq M$. Therefore $(x, D) \vee (y, D) = R$.

(2) ⇒ (3): Let $x, y \in R$. Clearly $(x, D) \vee (y, D) \subseteq (x \vee y, D)$. Let $s \in (x \vee y, D)$. Then $s \vee (x \vee y) \in D$. It follows $(s \vee x) \vee (s \vee y) \in D$. By our assumption, we have $(s\vee x, D)\vee(s\vee y, D) = R$. Then $s \in (s\vee x, D)\vee(s\vee y, D)$. There exist $a \in (s \vee x, D)$ and $t \in (s \vee y, D)$ such that $s = a \wedge t$. Since

$a \in (s \vee x, D)$, we get $a \vee s \in (x, D)$. Similarly, we have $t \vee s \in (y, D)$. Clearly, $(s \vee a) \wedge (s \vee t) \in (x, D) \vee (y, D)$, which leads $s \vee (a \wedge t) \in (x, D) \vee (y, D)$. Since $s = a \wedge s$, we get $s \in (x, D) \vee (y, D)$. Therefore $(x \vee y, D) \subseteq (x, D) \vee (y, D)$ and hence $(x, D) \vee (y, D) = (x \vee y, D)$.

(3) $\Rightarrow$ (4): Let I, J be two ideals of R with $I \vee J = R$. Let x be a dense element of R. Then there exist $s \in I$ and $t \in J$ such that $x = s \vee t$. By our assumption, we get $R = (x, D) = (s \vee t, D) = (s, D) \vee (t, D) \subseteq w(I) \vee w(J)$. Hence $w(I) \vee w(J) = R$.

(4) $\Rightarrow$ (5): Let I, J be two ideals of R. Clearly, $w(I) \vee w(J) \subseteq w(I \vee J)$. Let $a \in w(I \vee J)$. Then there exists $s \in I \vee J$ such that $a \vee s \in D$. Since $s \in I \vee J$, there exist $x \in I$ and $y \in J$ such that $s = x \vee y$. Since $a \vee s \in D$, we get $a \vee (x \vee y) \in D$. Hence $((a \vee x) \vee (a \vee y)] = (D]$, which gives $(a \vee x] \vee (a \vee y] = R$. Therefore $w((a \vee x]) \vee w((a \vee y]) = R$ and hence $(a \vee x, D) \vee (a \vee y, D) = R$. Since $a \in R$, we have $a \in (a \vee x, D) \vee (a \vee y, D)$. Then there exist $s \in (a \vee x, D)$ and $t \in (a \vee y, D)$ such that $a = s \wedge t$. Since $s \in (a \vee x, D)$ and $t \in (a \vee y, D)$, we get $a \vee s \in (x, D)$ and $a \vee t \in (y, D)$. Then $(a \vee s) \wedge (a \vee t) \in (x, D) \vee (y, D)$, which leads $a \vee (s \wedge t) \in (x, D) \vee (y, D)$. Since $s \wedge t = a$, we get $a \in (x, D) \vee (y, D)$. Since $(x, D) \vee (y, D) \subseteq w(I) \vee w(J)$, we get $a \in w(I) \vee w(J)$. Therefore we get $w(I \vee J) \subseteq w(I) \vee w(J)$. Hence $w(I \vee J) = w(I) \vee w(J)$.

(5) $\Rightarrow$ (1): It is clear. □

Theorem 3.26. *Let $\mathfrak{F}_w(R)$ be a sublattice of $\mathfrak{F}(R)$. If $\{G_\alpha\}_{\alpha \in \triangle}$ be any class of $w-$filters of R, then $\bigvee\limits_{\alpha \in \triangle} G_\alpha$ is again a $w-$filter of R.*

Proof. For each $\alpha \in \triangle$, let $G_\alpha = w(I_\alpha)$ where I_α is an ideal of R such that $I_\alpha \cap D = \emptyset$. Then $\{I_\alpha\}_{\alpha \in \triangle}$ will be any class family of ideals of R with $I_\alpha \cap D = \emptyset$, for all $\alpha \in \triangle$. Clearly, $(\vee I_\alpha) \cap D = \emptyset$. Since $G_\alpha = w(I_\alpha) \subseteq w(\vee I_\alpha)$ for each $\alpha \in \triangle$, we get $\vee G_\alpha \subseteq w(\vee I_\alpha)$. Let $a \in w(\vee I_\alpha)$. Then there exists $s \in \vee I_\alpha$ such that $a \vee s \in D$. Then there exists a positive integer n such that $s = s_1 \vee s_2 \vee \cdots \vee s_n$ where $s_i \in I_{\alpha_i}$. By condition (4) of Theorem 3.25, we get

$$\begin{aligned} a \vee s \in D &\Rightarrow a \vee (s_1 \vee s_2 \vee \cdots \vee s_n) \in D \\ &\Rightarrow (a \vee s_1) \vee (a \vee s_2) \vee \cdots \vee (a \vee s_n) \in D \\ &\Rightarrow (a \vee s_1] \vee (a \vee s_2] \vee \cdots \vee (a \vee s_n] = R \\ &\Rightarrow w((a \vee s_1]) \vee w((a \vee s_2]) \vee \cdots \vee w((a \vee s_n]) = R \\ &\Rightarrow (a \vee s_1, D) \vee (a \vee s_2, D) \vee \cdots \vee (a \vee s_n, D) = R. \end{aligned}$$

Since $a \in R$ we get $a \in (a \vee s_1, D) \vee (a \vee s_2, D) \vee \cdots \vee (a \vee s_n, D)$. Then there exists $t_i \in (a \vee s_i, D)$ for $i = 1, 2, \cdots, n$ such that $a = t_1 \wedge t_2 \wedge \cdots \wedge t_n$. Now,

$$\begin{aligned} a &= a \vee a \\ &= a \vee (t_1 \wedge t_2 \wedge \cdots \wedge t_n) \\ &= (a \vee t_1) \wedge (a \vee t_2) \wedge \cdots \wedge (a \vee t_n) \in (s_1, D) \vee (s_2, D) \vee \cdots \vee (s_n, D) \\ &\subseteq w(I_1) \vee w(I_2) \vee \cdots \vee w(I_n) \\ &= G_1 \vee G_2 \vee \cdots \vee G_n \subseteq \vee G_\alpha. \end{aligned}$$

That implies $w(\vee I_\alpha) \subseteq \vee G_\alpha$. Thus $\vee G_\alpha$ is a $w-$filter of R. □

Theorem 3.27. *Let $\mathfrak{F}_w(R)$ be a sublattice of $\mathfrak{F}(R)$. For any $D-$filter G, there exists a unique $w-$filter contained in G.*

Proof. Let G be any $D-$filter of R. Consider $\mathfrak{M} = \{H \in \mathfrak{F}_w(L) \mid H \subseteq G\}$. Since D is the $w-$filter and $D \subseteq G$, we get $D \in \mathfrak{M}$. Clearly, $\mathfrak{M}$ satisfies the hypothesis of Zorn's Lemma. Then $\mathfrak{M}$ has a maximal element let it be N. It is enough to show that N is unique. Let Q be any maximal element of $\mathfrak{M}$ such that $N \subseteq Q$. Clearly, $N \vee Q \subseteq G$. By Theorem 3.25, $N \vee Q \in \mathfrak{M}$. Therefore $N = N \vee Q = Q$. Thus $\mathfrak{M}$ has a unique maximal element, which is the required $w-$filter contained in G. □

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w-FILTERS OF ALMOST DISTRIBUTIVE LATTICES

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w-فیلترهای مشبکه‌های تقریباً توزیع‌پذیر

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در این مقاله، مفهوم w-هافیلتر در یک مشبکه تقریباً توزیع‌پذیر (ADL) معرفی شده و برخی خواص آن بررسی شده‌اند. یک شرط لازم و کافی برای اینکه یک فیلتر ماکسیمال از ADL به یک w-فیلتر تبدیل شود، ارائه شده است که به مشخصه سازی یک ADL شبه-کامل‌شده منجر می‌شود. همچنین، w-فیلترهای یک ADL با توجه به D-فیلترهای اول مینیمال مشخصه سازی شده‌اند.

کلمات کلیدی: مشبکه تقریباً توزیع‌پذیر (ADL)، ADL شبه-کامل‌شده، ایده‌آل اول، D-فیلتر، w-فیلتر.